Numerical Algorithms for Dynamical Systems with Multiple Time Scales

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Multiple time scales are ubiquitous in dynamical models. They arise, for example, in the context of combustion, metabolic networks, neuroscience, lasers, locomotion. Dynamical systems theory employs geometric methods to study the qualitative properties of dynamical models. One aspect of the theory is the classification of bifurcations: qualitative changes in the limit sets of a system produced by varying parameters. The results of this research have been extraordinarily useful in providing "templates" for understanding dynamical instabilities observed in physical systems. The theory has also provided the foundations for numerical algorithms that detect, locate and continue bifurcations in dynamical models. When these methods are applied to systems with multiple time scales, they frequently fail. The presence of multiple time scales in a system can greatly distort the geometry found in systems with a single time scale.

This presentation describes research framed in the context of *geometric singular perturbation theory* for slow-fast systems of the form

$$\begin{aligned} \varepsilon \dot{x} &= f(x,y) \qquad x \in R^m \\ \dot{y} &= g(x,y) \qquad y \in R^n \end{aligned}$$

Investigations of representative examples, beginning with the seminal work of van der Pol in the 1920's, have revealed new dynamical phenomena that occur within this setting, including *relaxation oscillations, mixed mode oscillations* and *canards*. Cartwright and Littlewood discovered chaos in dissipative dynamical systems when studying the forced van der Pol system.

I shall describe new numerical methods for computing multiple time scale systems. While "stiff" solvers do a good job of computing trajectories near attracting slow manifolds, they yield seemingly paradoxical results when faced with more complex geometric features in multiple time scale systems. Visualization of key geometric objects in several examples will be presented.

One example that will be discussed is a slow manifold of saddle type. These manifolds are unstable in both forward and backward time directions: nearby initial values quickly diverge from the manifold on the fast time scale. Therefore, even a "perfect" solver will fail to compute slow manifolds of saddle type in the presence of round-off error in initial conditions. This is a dynamical instability rather than an instability of a numerical method solving initial value problems. Nonetheless, slow manifolds of saddle type arise as an essential component of problems such as computation of traveling wave profiles for parabolic PDE. Continuation methods have been able to compute solutions along these profiles, but even simple problems seem to go beyond their capability. New boundary value methods for computing these slow manifolds will be presented.

In addition to discovering new phenomena, this research is leading to a deeper understanding of classical work on multiple time scale systems. Three examples are the work of Cartwright and Littlewood, studies of mixed mode oscillations in models of chemical reactors and models of bursting oscillations of neurons. Some of the methods reduce the models to *hybrid* systems in which fast motion is represented as instantaneous events, but there are phenomena that are not captured in the reduced system and require more detailed analysis. Oscillations that occur near points called "folded nodes" are an example that will be discussed in the context of mixed mode oscillations and as one of several mechanisms underlying chaos in multiple time scale systems.

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