Time-periodic solutions of the Benjamin-Ono equation

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We present a spectrally accurate numerical method for finding non-trivial time-periodic solutions of non-linear partial differential equations. The method is based on minimizing a functional (of the initial condition and the period) that is positive unless the solution is periodic, in which case it is zero. We solve an adjoint PDE to compute the gradient of this functional with respect to the initial condition. We include additional terms in the functional to specify the free parameters, which, in the case of the Benjamin-Ono equation, are the mean, a spatial phase, a temporal phase and the real part of one of the Fourier modes at $t = 0$.

We use our method to study global paths of non-trivial time-periodic solutions connecting stationary and traveling waves of the Benjamin-Ono equation,

$$u_t = Hu_{xx} + uu_x, \qquad H = \text{Hilbert transform}, \qquad x \in \mathbb{R}/2\pi\mathbb{Z}.$$

As a starting guess for each path, we compute periodic solutions of the linearized problem by solving an infinite dimensional eigenvalue problem in closed form. We then use our numerical method to continue these solutions beyond the realm of linear theory until another traveling wave is reached (or until the solution blows up). By experimentation with data fitting, we identify the analytic form of these solutions in terms of the trajectories of the Fourier modes, which turn out to be power sums of a collection of N particles $\beta_j(t)$ in the unit disk of the complex plane:

$$\hat{u}_k(t) = \begin{cases} \sum_{j=1}^N 2\bar{\beta}_j(t)^k, & k < 0, \\ \alpha_0 & k = 0, \\ \sum_{j=1}^N 2\beta_j(t)^k, & k > 0. \end{cases}$$

We then prove that solutions of the form suggested by our numerical experiments exist and show how our representation fits into the hierarchy of previously known multi-periodic and multi-soliton solutions.

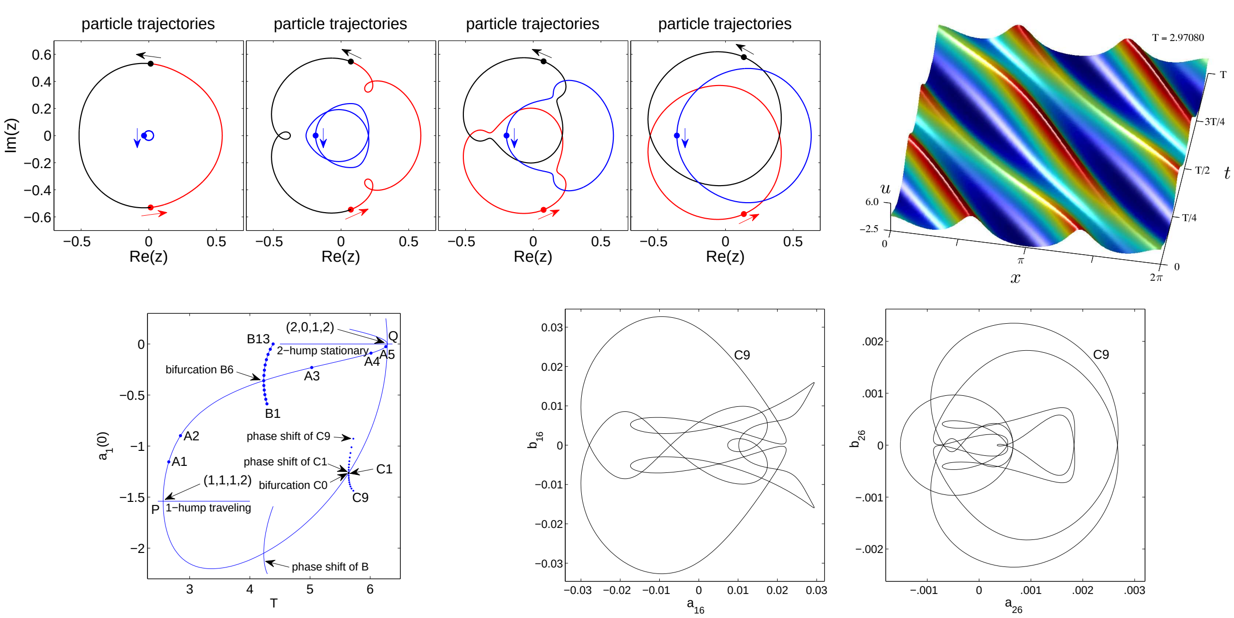

Figure 1: *Upper left:* Particle trajectories of four solutions on the path connecting a 2-hump left-traveling wave to a 3-hump left-traveling wave. *Upper right:* 3d plot of a solution on this path. *Lower left:* Interior bifurcations from the path connecting the one-hump right-traveling wave to the 2-hump stationary solution. *Lower right:* the trajectories of the Fourier modes in the complex plane can be very complicated.