A Different Perspective on Perspective Cuts

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We study mixed integer nonlinear programs (MINLP)s that are driven by a collection of indicator variables. "Turning off" an indicator variable forces some of the decision variables to assume fixed values, and "turning on" the indicator variable forces variable values to belong to a convex set. Many practical MINLPs contain integer variables of this type.

We first study a mixed integer set Q defined by a single separable quadratic constraint and a collection of variable upper and lower bound constraints:

$$Q = \Big\{ w \in \mathbb{R},\, x \in \mathbb{R}^n_+,\, z \in \mathbb{B}^n : \, w \geq \sum_{i=1}^{n} q_i x_i^2,\, u_i z_i \geq x_i \geq l_i z_i,\ i = 1, 2, \dots, n \Big\}.$$

A convex hull description of Q is derived in the original space of variables. A simpler expression for $\mathrm{conv}(Q)$ in a higher-dimensional space is also given.

We then extend this result to produce an explicit characterization of the convex hull of the union of a point and a bounded convex set defined by analytic functions. Specifically, let

$$\begin{aligned} W^0 &= \big\{ (x, z) \in \mathbb{R}^{n+1} \ : \ \ x = \bar{x},\ z = 0 \big\}, \text{ and} \\ W^1 &= \big\{ (x, z) \in \mathbb{R}^{n+1} \ : \ f_i(x) \leq 0 \text{ for } i \in I,\ u \geq x - \bar{x} \geq l,\ z = 1 \big\}, \end{aligned}$$

where $u, l \in \mathbb{R}^n_+$. An analytic expression for $\mathrm{conv}(W^0 \cup W^1)$ is derived.

Frangioni and Gentile (2006) studied the same sets and described a class of *perspective cuts* as first-order (outer)-approximations to $\mathrm{conv}(W^0 \cup W^1)$. We instead use perspective functions to obtain what we call the the *perspective formulation*. Our work thus provides a different perspective on the work of Frangioni and Gentile. Further, we show that for many classes of problems, the convex hull can be expressed via conic quadratic constraints, and thus relaxations in a branch-and-bound algorithm can be solved via second-order cone programming.

Finally, we apply our results to develop tight formulations of mixed integer nonlinear programs in which indicator variables play an important role. In particular, we present computational results for two applications – quadratic facility location and network design with congestion – that show the power of the reformulation technique. In some cases, instances are solved hundreds of times more effectively than previously reported in the literature.

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