Higher Order Compact Generalized Finite Difference Method

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Efficient solution of linear partial differential equations is still important in many areas of science and engineering, because it is either the main computational problem or appears as a subpart of a more complex task, for example in case of projection methods for Navier-Stokes equations. The method of finite differences was the first to be developed and is probably the best understood, but it suffers from inability to handle complex geometries and steeply rising costs of obtaining higher orders of accuracy. The last fifty years saw the emergence of two ideas for increasing the power of finite differences, addressing both of those issues. Generalized Finite Differences extends finite differences to arbitrary grids and point clouds while Higher Order Compact methods try to reduce the size of stencils required to obtain higher order convergence. Unfortunately, the two extensions seemed to be incompatible because the Higher Order Compact methodology requires prior knowledge of the finite difference stencil.

In our contribution we present a way of reconciling the two methods, leading to the Higher Order Compact Generalized Finite Difference method. Beginning with the fundamental ideas of finite differences, we construct a polynomial approximation to an unknown function as a cross-section of two subsets of the vector space of polynomials of a given order. We discuss the method of choosing neighbors on an arbitrary grid using linear programming and prove that our method prevents singularities of the linear systems used to determine finite difference stencils. In order to cure ill-conditioning of those systems we introduce a simple conditioning procedure which bounds the conditioning number and can be used to prove the order of accuracy of our method through a formal truncation error analysis. Finally, we discuss stability in terms of Gershgorin circle theorem and present ways of improving stability via small modifications of the method.

We conclude with a wide array of numerical tests where we apply our method to several linear partial differential equations—including the Laplace and Poisson equations, advection-diffusion equation, and time dependent diffusion equation—to prove numerically the claimed accuracy of our method and compare it with others.

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