Preprint ANL/MCS-P5404-0915

A HOMOGENEOUS MODEL FOR MONOTONE MIXED HORIZONTAL LINEAR COMPLEMENTARITY PROBLEMS

COSMIN G. PETRA* AND FLORIAN A. POTRA†

Abstract. We propose a homogeneous model for the class of mixed horizontal linear complementarity problems. The proposed homogeneous model is always solvable and provides the solution of the original problem if it exists, or a certificate of infeasibility otherwise. Our formulation preserves the sparsity of the original formulation and does not reduce to the homogeneous model of the equivalent standard linear complementarity problem. We study the properties of the model and show that interior-point methods can be used efficiently for the numerical solutions of the homogeneous problem. Numerical experiments show convincingly that it is much more efficient to use the proposed homogeneous model for the mixed horizontal linear complementarity problem than to use known homogeneous models for the equivalent standard linear complementarity problem.

Key words. mixed horizontal LCP, homogenization, interior-point method

AMS subject classifications. 15A15, 15A09, 15A23

1. Introduction. A nonlinear complementarity problem (NCP) over the non-negative orthant consists of finding vectors $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$ satisfying

$$xs = 0, \ s = f(x), \ x, s \geq 0, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given differentiable function. Here, xs is the componentwise product of x and s, $xs = [x_1 s_1 \,; x_2 s_2 \,; \ldots \,; x_n s_n]$, also known as the Hadamard product. The complementarity problem (1.1) is called *monotone* if $(u-v)^T(f(u)-f(v)) \geq 0$ for any $u, v \in \mathbb{R}^n_{++}$. In this paper we consider the more general notion of a mixed nonlinear complementarity problems (MNCP) of the form

$$xs = 0, \ F(x, s, y) = 0, \ x, s \geq 0, \tag{1.2}$$

where $F : \mathbb{R}^{2n+m} \to \mathbb{R}^{n+m}$ is a differentiable mapping. The term *mixed* indicates the presence of the *free variables* $y \in \mathbb{R}^m$ that are not subject to complementarity conditions and sign constraints. We say that a point (x, s, y) is feasible if it satisfies both the feasibility equations and the sign constraints. If f is an affine mapping then (1.1) becomes a linear complementarity problem in standard form (SLCP),

$$xs = 0, \ s = Mx + b, \ x, s \geq 0. \tag{1.3}$$

Similarly, by considering an affine mapping in (1.2) we obtain the mixed horizontal linear complementarity problem (MLCP)

$$xs = 0, \ Ax + Bs + Cy = b, \ x, s \geq 0. \tag{1.4}$$

*Mathematics and Computer Science Division, Argonne National Laboratory, 9700 S. Cass Avenue, Argonne, IL 60439, USA (`petra@mcs.anl.gov`).

†Department of Mathematics and Statistics, University of Maryland, Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250, USA (`potra@umbc.edu`).

Here $A, B \in \mathbb{R}^{(m+n)\times n}$, $C \in \mathbb{R}^{(m+n)\times m}$, and $b \in \mathbb{R}^{n+m}$. If no free variables are present, we obtain a horizontal linear complementarity problem (HLCP)

$$xs = 0,\ Qx + Rs = b,\ x, s \geq 0, \tag{1.5}$$

where $Q, R \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. MLCP (1.4) is said to be monotone if

$$Au + Bv + Cw = 0 \text{ implies } u^T v \geq 0, \text{for any } u, v \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^m. \tag{1.6}$$

In particular, HLCP (1.5) is monotone if

$$Qu + Rv = 0 \text{ implies } u^T v \geq 0, \text{for any } u, v \in \mathbb{R}^n. \tag{1.7}$$

Since SLCP (1.3) is obtained by taking $Q = M$ and $R = -I$ in (1.5), it follows that SLCP (1.3) is monotone if and only if M is positive semidefinite.

In this paper we introduce a homogenization technique that can be applied to any monotone MLCP and yields a monotone MNCP of the form (1.2) with a homogeneous $F(\cdot)$. In the context of interior-point methods, homogenization generally refers to an artificial transformation of a given (complementarity or constrained optimization) problem to a homogeneous problem. The homogeneity causes the transformed problem to have nicer properties than those of the original problem, such as a trivial feasible starting point and solvability in any circumstances. The solution sets of the two problems are related however: once the homogeneous problem is solved, a solution to the original problem or a certificate that a solution does not exist is readily available.

Certificates of infeasibility for linear programming are produced with the simplex method by detecting the unboundedness of either the primal or dual problem. However, the simplex method can be applied only to linear programming; its extension for linear complementarity problems (*e.g.*, Lemke's method) does not offer such certificates. Standard interior-point algorithms do not offer certificates of infeasibility either. Moreover, numerical issues arise with the interior-point-based implementations when solving an infeasible problem, since some of the problem's variables diverge. Over the past fifteen years, interior-point methods that provide infeasibility certificates always have been used in conjunction with a homogenization mechanism.

Having a proof of infeasibility is important for two reasons. First, the model that gives rise to the infeasible problem may be defective. For example, obtaining infeasible problems in modeling physical phenomena would indicate such a situation. Second, the problem may be infeasible because of invalid data or human error in the input process.

The homogenization of a constrained optimization problem is a concept that has been used with interior-point methods since their appearance, not necessarily as a technique that detects infeasibility. Karmakar's algorithm [12], generally considered to be the first efficient interior-point algorithm, transforms the original linear problem into a homogeneous linear problem (called "canonical form") in order to obtain a feasible starting point. Anstreicher [4] used homogenization to devise a polynomial-time interior algorithm that solves a linear program with no assumptions of a non-empty interior of the primal and/or dual problem. Because a Phase I-Phase II technique is employed, and the solution of a linear system twice larger than with conventional methods needs to be found at each iteration, Anstreicher's algorithm is expensive in practice.

The homogeneous interior-point algorithm for linear programming introduced by Ye, Todd, and Mizuno [21] uses a *self-dual embedding* technique to incorporate the original linear problem together with its dual problem in a larger homogeneous linear problem that turns out to be self-dual. This algorithm was the first homogenization technique capable of providing certificates of infeasibility of the original problem and has become a standard for homogeneous interior-point methods because of its properties:

- It solves the problem without any regularity assumptions concerning the existence of optimal, feasible or interior feasible points.
- It can start at any positive point, feasible or infeasible.
- Each iteration requires the solution of a linear system whose dimension is almost the same as for standard (primal-dual) interior-point algorithms.
- If the problem has a solution, the algorithm will find it; if the problem is infeasible or unbounded, then the algorithm will detect this situation by providing a "certificate" of infeasibility for at least one of the primal or dual problems.
- It is a one-phase algorithm and has $O(\sqrt{n}L)$-iteration complexity.

In addition, the method improves the behavior and computational cost of Anstreicher's algorithm.

Ye [20] showed that this technique is also suitable for monotone SLCP. The homogenization yields a self-dual homogeneous monotone linear complementarity problem that possesses the same properties as the model for linear programming. The homogeneous linear complementarity problem is self-dual in the sense that if the original linear complementarity problem arises from a linear program, then the homogeneous linear complementarity problem represents the self-dual embedding of the linear program.

A homogeneous model was proposed in [2] for nonlinear complementarity problems, more specifically for NCP (1.1), in the form

$$\begin{array}{rcl} xs & = & 0 \\ \tau\kappa & = & 0 \\ s & = & \tau f(x/\tau) \\ \kappa & = & -x^T f(x/\tau) \\ x, s, \tau, \kappa & \geq & 0. \end{array} \tag{1.8}$$

This model is also called *augmented* since it contains two additional one-dimensional complementarity variables. The homogenization preserves the monotonicity of the problem, but the homogeneous model is nonlinear even if the original complementarity problem is linear. For example, if applied to SLCP (1.3) it yields the following homogeneous NCP:

$$\begin{array}{rcl} xs & = & 0 \\ \tau\kappa & = & 0 \\ s & = & Mx + \tau b \\ \kappa & = & -x^T Mx/\tau - x^T b \\ x, s, \tau, \kappa & \geq & 0. \end{array} \tag{1.9}$$

Although considerable research has been devoted to homogenization of SLCP, to the best of our knowledge, no homogenization technique exists that can be applied directly to MLCP (1.3), in spite of the fact that many problems from science and engineering are naturally formulated in this form. On the other hand, MLCP (1.4) can be transformed into an SLCP of the form (1.3) (see, for example, [3, 9, 10]. While the matrices defining MLCP (1.4) are sparse, the matrix M from (1.3) becomes dense, thus considerably increasing the computational cost of solving the SLCP, in terms of both execution time and memory requirements. The primary objective of this paper is to provide a monotone homogeneous augmented model for MLCP with preservation of the sparsity structure, which can be efficiently solved by means of path-following interior-point methods.

The paper is organized as follows. Section 2 discusses the connection between the MLCP (1.4) and HLCP (1.5). Section 3 introduces the homogeneous model for MLCP (1.4). Its properties are studied in Section 4. Several well known results on the existence and properties of central path for nonlinear monotone complementarity are reviewed in Section 5 and used in Section 6 to show that our homogeneous model possesses the desirable properties characteristic of homogenization techniques. The numerical algorithm and the simulation results are presented in Sections 7 and 8, respectively. Section 9 summarizes the findings of our analysis and numerical experiments and discusses future research directions.

Throughout this paper we use the MATLAB-like notation $[\,u\,;\,v\,;\,w\,]$ to denote the column vector $[u^T v^T w^T]^T$. Also, we denote by $[A\,B\,C]$, or $[A,\,B,\,C]$, the matrix formed by columns of the matrices A, B, and C in that order. Given a matrix P, we denote by $\operatorname{Ran} P$ its range (or column space) and by $\operatorname{Ker} P$ its kernel (or null space).

2. Equivalence between MLCP and HLCP. In this section we show that MLCP (1.4) is equivalent to an HLCP of the form

$$\begin{array}{rcl} xs & = & 0 \\ E^T Ax + E^T Bs & = & E^T b \\ x, s & \geq & 0, \end{array} \tag{2.1}$$

for any matrix E whose columns form a basis of $\operatorname{Ker}(C^T)$. As mentioned in the introduction, the equivalence between different formulations of the linear complementarity problems is well studied. However, we give a simple proof in order to clarify notation and to highlight some properties to be used in later sections. The feasible sets of the linear complementarity problems (1.4) and (2.1) are denoted respectively by

$$\mathcal{F} = \{(x, s, y) \in \mathbb{R}^{2n+m}\,;\; Ax + Bs + Cy = b, x \geq 0, s \geq 0\}, \tag{2.2}$$
$$\mathcal{F}_E = \{(x, s) \in \mathbb{R}^{2n}\,;\; E^T Ax + E^T Bs = E^T b, x \geq 0, s \geq 0\}, \tag{2.3}$$

and the solutions sets by

$$\mathcal{F}^* = \{(x^*, s^*, y^*) \in \mathcal{F}\,;\; x^{*T} s^* = 0\},\; \mathcal{F}_E^* = \{(x^*, s^*) \in \mathcal{F}_E\,;\; x^{*T} s^* = 0\}. \tag{2.4}$$

LEMMA 2.1.
(i) $(x, s) \in \mathcal{F}_E$ *if and only if there is* $y \in \mathbb{R}^m$ *such that* $(x, s, y) \in \mathcal{F}$.
(ii) $(x^*, s^*) \in \mathcal{F}_E^*$ *if and only if there is* $y^* \in \mathbb{R}^m$ *such that* $(x^*, s^*, y^*) \in \mathcal{F}^*$.
(iii) MLCP (1.4) *is monotone if and only if* HLCP (2.1) *is monotone.*
(iv) $[\,A\ B\ C\,]$ *is full row rank if and only if* $[\,E^T A\ E^T B\,]$ *is full row rank.*

Proof. (i) and (ii) follow from the observation that $z \in \operatorname{Ran} C \Leftrightarrow E^T z = 0$.
(iii) The monotonicity of (1.4) can be equivalently expressed as $x^T s \geq 0$ whenever $Ax + Bs \in \operatorname{Ran} C$. Since the last relation holds if and only if $E^T Ax + E^T Bs = 0$, it follows that the monotonicity of (1.4) implies the monotonicity of (2.1) and vice-versa.
(iv) Suppose that $[\,A\ B\ C\,]$ is full row rank and that $u^T [\,E^T A\ E^T B\,] = 0$ for some vector $u \in \mathbb{R}^k$, where $k = \dim \operatorname{Ker} C^T$. If we denote $v = Eu$, then $v^T [\,A\ B\ C\,] = 0$;

and since $[\,A\ B\ C\,]$ is full row rank, we must have $v = 0$. Because the columns of E are linearly independent, this implies $u = 0$, showing that $[\,E^T A\ E^T B\,]$ is full row rank. Conversely, let us assume that $[\,E^T A\ E^T B\,]$ is full rank and $\bar{v}^T[\,A\ B\ C\,] = 0$ for some vector $\bar{v} \in \mathbb{R}^{m+n}$. It follows that $\bar{v}^T C = 0$, so that there is $\bar{u} \in \mathbb{R}^k$ such that $\bar{v} = E\bar{u}$. But then $\bar{u}^T[\,E^T A\ E^T B\,] = 0$. Since $[\,E^T A\ E^T B\,]$ is full row rank, this implies $\bar{u} = 0$. Therefore $\bar{v}^T[\,A\ B\ C\,] = 0$ implies $\bar{v} = 0$, so $[\,A\ B\ C\,]$ is full row rank. □

Lemma 2.1 shows that unless $k = n$, the MLCP (1.4) is equivalent to an overdetermined HLCP. Therefore we assume $k = n$, so that $\dim \operatorname{Ran} C = (m+n) - \dim \operatorname{Ker} C^T = m$.

ASSUMPTION 2.2. *The matrix C has full column rank.*
MLCPs of the form (1.4) satisfying Assumption 2.2 are also present in the study of Monteiro and Pang [16] on the behavior of path-following interior-point algorithms. Overdetermined horizontal forms were studied by Güler in [10] in the context of maximal monotone operators; the author concludes that in the monotone case interior-point methods can be used to solve such forms if and only if Assumption 2.2 is satisfied. Under Assumption 2.2, the matrices $E^T A$ and $E^T B$ from (2.1) have $n + m$ rows and n columns, so HLCP (2.1) has the same form as HLCP (1.5).

LEMMA 2.3 (cf. Theorem 11 of [19]). *HLCP (1.5) is monotone if and only if $Q + R$ is nonsingular and $-QR^T$ is positive semidefinite.*

LEMMA 2.4 (cf. Corollary 18 of [19]). *If HLCP (1.5) is monotone and feasible, then it is solvable.*

COROLLARY 2.5. *If Assumption 2.2 is satisfied, then* MLCP (1.4) *is monotone if and only if $-E^T AB^T E$ is positive semidefinite for any matrix E whose columns form a basis of* $\operatorname{Ker} C^T$.

LEMMA 2.6. *If Assumption 2.2 is satisfied and* MLCP (1.4) *is monotone, then the matrix $[\,A\ B\ C\,]$ is full row rank.*

Proof. Since HLCP (2.1) has the same form as HLCP (1.5), Corollary 2.5 and Lemma 2.3 imply hat HLCP (2.1) is also monotone. Then according to a result of [5], it follows that $[\,E^T A\ \ E^T B\,]$ is full row rank. But then according to Lemma 2.1 so is $[\,A\ B\ C\,]$. □

In view of these results we will assume for the remainder of this paper that MLCP (1.4) is monotone and that Assumption 2.2 is satisfied.

3. Augmented homogeneous model. We propose the following augmented mixed homogeneous complementarity problem (HMCP) related to MLCP (1.4):

$$\begin{aligned} xs &= 0 \\ \tau\kappa &= 0 \\ Ax + Bs + Cy - \tau b &= 0 \\ \bar{x}^T\bar{s}/\tau + \kappa &= 0 \\ x, \tau, s, \kappa &\geq 0, \end{aligned} \tag{3.1}$$

where

$$[\bar{x}\,;\bar{s}\,;\bar{y}] = P_{\operatorname{Ker}[ABC]}\ [x\,;s\,;y] + \tau\bar{b}, \tag{3.2}$$

$$\bar{b} = [\,u^*\,;\,v^*\,;\,w^*\,] = [A\,B\,C]^T\,(AA^T + BB^T + CC^T)^{-1}b. \tag{3.3}$$

We note that $\bar{b}$ is the least-squares solution of $Ax + Bs + Cy = b$.

HMCP (3.1) contains two additional complementary variables τ and κ and one additional equation (constraint). The latter is not linear and therefore the augmented model can be written as an MNCP of the form (1.2) with $F : \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^{n+m+1}$,

$$F\left([x;\tau],[s;\kappa],y\right) = F\left(x,\tau,s,\kappa,y\right) = \begin{bmatrix} Ax + Bs + Cy - \tau b \\ \bar{x}^T \bar{s}/\tau + \kappa \end{bmatrix}. \tag{3.4}$$

Notation and terminology. The concepts of feasibility and solvability as defined for the mixed linear complementarity problem are not applicable for our augmented complementarity problem because the homogenization process causes the domain of the problem not to be a closed set anymore. Indeed, the second component of F from (3.4) is not defined for $\tau = 0$. Therefore it is useful to recall the concepts of *asymptotic* feasibility and solvability for the generic MNCP (1.2), in case the map F is not necessarily defined on the boundary of $\mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^m$, so that $\mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m \subseteq \mathrm{dom}\,(F) \subseteq \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^m$. These notions have been considered also for more general problems in [22].

Definition 3.1. MNCP (1.2) *is called asymptotically feasible if there is a bounded sequence* $\{(x^k, s^k, y^k)\} \subseteq \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m$ *such that*

$$\lim_{k\to\infty} F(x^k, s^k, y^k) \quad = \quad 0.$$

Moreover, any limit point (x^*, s^*, y^*) *of the sequence* $\{(x^k, s^k, y^k)\}$ *is called an asymptotically feasible point.*

Definition 3.2. MNCP (1.2) *is called asymptotically solvable if there is an asymptotically feasible point* (x^*, s^*, y^*) *such that* $x^* s^* = 0$.

Observe that both asymptotic feasibility and asymptotic solvability would be equivalent to the corresponding concepts defined for MLCP (1.4) if the domain of the problem were closed.

The study of nonlinear mixed complementarity in the context of interior-point methods employs several concepts not present in the linear case. The definitions given below were initially introduced by Monteiro and Pang in the context of implicitly defined mixed nonlinear complementarity problems over the nonnegative orthant [16] and the cone of positive semidefinite matrices [17]. Yoshise [22] has adapted the concepts to work in an asymptotic approach needed for the study of a homogenization technique for explicit nonlinear monotone complementarity problems over symmetric cones.

Definition 3.3. *The map* $F(x, s, y)$ *is called* (x, s)*-equilevel-monotone on its domain if for any* (x, s, y) *and* (x', s', y') *that lie in the domain of* F *and satisfy* $F(x, s, y) = F(x', s', y')$*, it holds that* $(x - x')^T(s - s') \geq 0$.

Definition 3.4. *The map* $F(x, s, y)$ *is called* (x, s)*-everywhere-monotone on the domain of* F *if there exist continuous functions* ϕ *from the domain of* F *to the set* $\mathbb{R}^{n+m}$ *and* $c : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}$ *such that* $c(r, r) = 0$ *and*

$$(x - x')^T(s - s') \geq (r - r')^T(\phi(x, s, y) - \phi(x', s', y')) + c(r, r')$$

holds for any (x, s, y) *and* (x', s', y') *in the domain of* F *satisfying* $F(x, s, y) = r$ *and* $F(x', s', y') = r'$.

By taking $r = r'$, it follows that (x, s)-everywhere-monotonicity implies (x, s)-equilevel-monotonicity.

Definition 3.5. *The map $F(x,s,y)$ is called y-bounded on its domain if for any sequence $\{(x^k,s^k,y^k)\}$ in its domain such that both $\{(x^k,s^k)\}$ and $\{F(x^k,s^k,y^k)\}$ are bounded sequences, the sequence $\{y^k\}$ is also bounded.*

Definition 3.6. *The map $F(x,s,y)$ is called y-injective on its domain if for any (x,s,y) and (x,s,y') lying in the domain of F and satisfying $F(x,s,y)=F(x,s,y')$, we have $y=y'$.*

We remark that the above two definitions are satisfied by MLCP (1.4) only under Assumption 2.2, which is assumed to hold together with the monotonicity of (1.4).

4. Properties of the augmented HMCP. This section presents the properties of HMCP (3.1). First, we prove that the augmented problem is an (everywhere-) monotone nonlinear homogeneous complementarity problem possessing the y-bondedness and y-injectiveness properties. Second, we show that the HMCP (3.1) is solvable under the assumption of monotonicity of MLCP (1.4) and its solution can be used as a certificate of the solvability or infeasibility of the original problem.

The orthogonal projection of $[\,x\,;\,s\,;\,y\,]$ onto $\operatorname{Ker}[ABC]$ is essential for ensuring that the transformation from MLCP (1.4) to HMCP (3.1) preserves monotonicity, as shown in the following lemma.

Lemma 4.1. *The mapping F defined by (3.2)-(3.4) is as follows*

(i) continuous and homogeneous (of degree 1) on its domain;

(ii) (x,s)-equilevel-monotone on its domain;

(iii) y-bounded on its domain;

(iv) y-injective on its domain;

(v) (x,s)-everywhere-monotone on its domain.

Proof. (i) We first show that the mappings $[\,x\,;\,\tau\,;\,s\,;\,\kappa\,;\,y\,]\mapsto\bar{x}$, $[\,x\,;\,\tau\,;\,s\,;\,\kappa\,;\,y\,]\mapsto\bar{s}$, and $[\,x\,;\,\tau\,;\,s\,;\,\kappa\,;\,y\,]\mapsto\bar{y}$ are linear in $[\,x\,;\,\tau\,;\,s\,;\,\kappa\,;\,y\,]$. This is obvious once we write

$$[\,\bar{x}\,;\,\bar{s}\,;\,\bar{y}\,]=P\,[\,x\,;\,s\,;\,y\,]+\tau\bar{b}=\begin{bmatrix} P & \bar{b} & 0\end{bmatrix}[\,x\,;\,s\,;\,y\,;\,\tau\,;\,\kappa\,].$$

The continuity of F readily follows from the above observation and from the fact that $\tau>0$ on the domain of F.

Now since $\bar{x}$ and $\bar{s}$ are linear functions of $[\,x\,;\,\tau\,;\,s\,;\,\kappa\,;\,y\,]$, we have

$$F(tx,t\tau,ts,t\kappa,ty)=\begin{bmatrix} Atx+Bts+Cty-t\tau b\\ \overline{tx}^T\overline{ts}/(t\tau)+t\kappa\end{bmatrix}=tF(x,\tau,s,\kappa,y),\quad\forall t\in\mathbb{R}$$

and hence F is homogeneous.

(ii) is a consequence of (v).

(iii) Consider the sequence $\{(x^k,\tau^k,s^k,\kappa^k,y^k)\}$ in the domain of F such that $\{(x^k,\tau^k,s^k,\kappa^k)\}$ and $\{F(x^k,\tau^k,s^k,\kappa^k,y^k)\}$ are bounded.

Since C has full column rank, we can write

$$\begin{aligned}\|y^k\|&=\|(C^TC)^{-1}C^TCy^k\|\le\|(C^TC)^{-1}C^T\|\|Cy^k\|\\&=\|(C^TC)^{-1}C^T\|\|(Ax^k+Bs^k+Cy^k-\tau^kb)-(Ax^k+Bs^k-\tau^kb)\|\\&\le\|(C^TC)^{-1}C^T\|\left(\|Ax^k+Bs^k+Cy^k-\tau^kb\|+\|Ax^k+Bs^k-\tau^kb\|\right)\\&\le\|(C^TC)^{-1}C^T\|\left(M_1+\|\,[A\,B\,-b]\,\|M_2\right),\end{aligned}$$

where M_1 and M_2 are the bounds for $\{\|F(x^k,\tau^k,s^k,\kappa^k,y^k)\|\}$ and $\{\|(x^k,\tau^k,s^k,\kappa^k)\|\}$, respectively. Therefore $\{y^k\}$ is bounded, implying that F is y-bounded according to Definition 3.5.

(iv) If $F(x,\tau,s,\kappa,y) = F(x,\tau,s,\kappa,y')$, then $Cy = Cy'$, implying that $y = y'$ since C is assumed to have full column rank.

(v) Consider (x,τ,s,κ,y) and (x',τ',s',κ',y') in the domain of F and let

$$\begin{aligned}
[r\,;\,\gamma] &= F(x,\tau,s,\kappa,y) = \left[Ax + Bs + Cy - \tau b\,;\, \bar{x}^T\bar{s}/\tau + \kappa\right],\\
[r'\,;\,\gamma'] &= F(x',\tau',s',\kappa',y') = \left[Ax' + Bs' + Cy' - \tau' b\,;\, \bar{x'}^T\bar{s'}/\tau' + \kappa'\right].
\end{aligned}$$

Since

$$P_{\mathrm{Ker}[ABC]} = I - [A\,B\,C]^T\, G\, [A\,B\,C]\,, \quad G := (AA^T + BB^T + CC^T)^{-1}, \tag{4.1}$$

equation (3.2), which defines $(\bar{x},\bar{s},\bar{y})$, is equivalent to

$$\begin{aligned}
[\,\bar{x}\,;\,\bar{s}\,;\,\bar{y}\,] &= [\,x\,;\,s\,;\,y\,] - [A\,B\,C]^T\, G(Ax + Bs + Cy) + \tau\bar{b}\\
&= [\,x\,;\,s\,;\,y\,] - [A\,B\,C]^T\, G(Ax + Bs + Cy - \tau b) = [\,x\,;\,s\,;\,y\,] - [A\,B\,C]^T\, Gr,
\end{aligned}$$

where the expression (3.3) of $\bar{b}$ was used to obtain the second equality. Therefore we can write

$$[\,x - x'\,;\,s - s'\,;\,y - y'\,] = [\,\bar{x} - \bar{x'}\,;\,\bar{s} - \bar{s'}\,;\,\bar{y} - \bar{y'}\,] + [A\,B\,C]^T\, G(r - r'),$$

which gives

$$\begin{aligned}
&(x - x')^T(s - s') = \left(\bar{x} - \bar{x'} + A^T G(r - r')\right)^T \left(\bar{s} - \bar{s'} + B^T G(r - r')\right) =\\
&\left(\bar{x} - \bar{x'}\right)^T \left(\bar{s} - \bar{s'}\right) + (r - r')^T\, G\left(A(s - s') + B(x - x') - AB^T G\,(r - r')\right).
\end{aligned} \tag{4.2}$$

By multiplying (3.2) with $[A\,B\,C]$ and using (3.3), we obtain

$$A\bar{x} + B\bar{s} + C\bar{y} = \tau b, \quad A\bar{x'} + B\bar{s'} + C\bar{y'} = \tau' b.$$

We can then write

$$A\bar{x}/\tau + B\bar{s}/\tau + C\bar{y}/\tau = b = A\bar{x'}/\tau' + B\bar{s'}/\tau' + C\bar{y'}/\tau',$$

and the monotonicity of MLCP (1.4) implies $\left(\bar{x}/\tau - \bar{x'}/\tau'\right)^T\left(\bar{s}/\tau - \bar{s'}/\tau'\right) \geq 0$. Multiplying the previous inequality with $\tau\tau'$ and manipulating the terms, we get

$$\frac{\tau'}{\tau}\bar{x}^T\bar{s} + \frac{\tau}{\tau'}\bar{x'}^T\bar{s'} \geq \bar{x'}^T\bar{s} + \bar{x'}^T\bar{s}. \tag{4.3}$$

Using (4.3) and the expressions of γ and γ', we deduce successively that

$$\begin{aligned}
(\tau - \tau')(\kappa - \kappa') &= (\tau - \tau')(\gamma - \gamma') - (\tau - \tau')(\bar{x}^T\bar{s}/\tau - \bar{x'}^T\bar{s'}/\tau')\\
&= (\tau - \tau')(\gamma - \gamma') - (\bar{x}^T\bar{s} + \bar{x'}^T\bar{s'}) + (\frac{\tau'}{\tau}\bar{x}^T\bar{s} + \frac{\tau}{\tau'}\bar{x'}^T\bar{s'})\\
&\geq (\tau - \tau')(\gamma - \gamma') - (\bar{x}^T\bar{s} + \bar{x'}^T\bar{s'}) + \bar{x'}^T\bar{s} + \bar{x'}^T\bar{s}\\
&= (\tau - \tau')(\gamma - \gamma') - (\bar{x} - \bar{x'})^T(\bar{s} - \bar{s'}).
\end{aligned} \tag{4.4}$$

Adding (4.2) and (4.4) and using the expressions of r and r', we obtain

$$\begin{aligned}
&[x - x'\,;\,\tau - \tau']^T\,[s - s'\,;\,\kappa - \kappa']\\
&\geq [r - r'\,;\,\gamma - \gamma']^T\,(\phi(x,\tau,s,\kappa,y) - \phi(x',\tau',s',\kappa',y'))\,,
\end{aligned}$$

where $\phi : \left(\mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}_+\right) \times \mathbb{R}^m \to \mathbb{R}^{n+m+1}$ is given by

$$\phi(x, \tau, s, \kappa, y) = \left[G(As + Bx) - GAB^T G(Ax + Bs + Cy - \tau b)\,;\, \tau\right]. \tag{4.5}$$

The function ϕ is clearly continuous; and by taking $c := 0$ it follows that F is (x, s)-everywhere-monotone on $\left(\mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}_+\right) \times \mathbb{R}^m$, according to Definition 3.4. ∎

THEOREM 4.2. HMCP (3.1) *is asymptotically feasible and every asymptotically feasible point is an asymptotically complementarity solution.*

Proof. One can easily verify that HMCP (3.1) is asymptotically feasible by considering the sequence $(x^l, \tau^l, s^l, \kappa^l, y^l) := ((1/2)^l e, (1/2)^l, (1/2)^l e, (1/2)^l, 0)$ and letting $l \to \infty$.

Let (x, τ, s, κ, y) be any asymptotically feasible point of HMCP (3.1). Hence, there is a sequence $\{(x^k, \tau^k, s^k, \kappa^k, y^k)\} \subseteq \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m$ such that

$$\lim_{k\to\infty} (x^k, \tau^k, s^k, \kappa^k, y^k) = (x, \tau, s, \kappa, y), \quad \lim_{k\to\infty} F(x^k, \tau^k, s^k, \kappa^k, y^k) = 0.$$

It follows that

$$\begin{array}{rcl} Ax + Bs + Cy - \tau b & = & 0 \\ x, \tau, s, \kappa & \geq & 0 \\ \lim\limits_{k\to\infty} \bar{x^k}^T \bar{s^k}/\tau^k + \kappa & = & 0 \end{array} \tag{4.6}$$

From (3.2) it follows that the limit $[\,\bar{x}\,;\,\bar{s}\,;\,\bar{y}\,] = \lim\limits_{k\to\infty} [\,\bar{x^k}\,;\,\bar{s^k}\,;\,\bar{y^k}\,]$ exists. Using the notation from (4.1), we have

$$\begin{array}{rcl} [\,\bar{x}\,;\,\bar{s}\,;\,\bar{y}\,] - [\,x\,;\,s\,;\,y\,] & = & \tau\bar{b} - (I - P)[\,x\,;\,s\,;\,y\,] = \tau\bar{b} - P_{\mathrm{Ran}[ABC]^T}[\,x\,;\,s\,;\,y\,] \\ & = & [A\ B\ C]^T G(\tau b - Ax + Bs + Cy). \end{array}$$

The first feasibility equation from (4.6) implies $x = \bar{x}$, $s = \bar{s}$ and $y = \bar{y}$, while the third equation from (4.6) yields the complementarity condition $x^T s + \tau\kappa = 0$. ∎

The next theorem shows that the maximal asymptotic complementarity solutions to HMCP (3.1) represent certificates of solvability or infeasibility of the MLCP (1.4). Moreover, when MLCP (1.4) is solvable, a solution is obtained from the solution of the HMCP (3.1) at no cost. We recall that a solution is maximal if the number of positive components of the vectors subject to nonnegativity and complemenatarity constraints in the solution is as large as possible. Note that the indexes of those positive components are invariant among all maximal solutions for HMCP (3.1) (see [10])

THEOREM 4.3. *Let $(x^*, \tau^*, s^*, \kappa^*, y^*)$ be a maximal asymptotic complementarity solution of* HMCP (3.1) *corresponding to* MLCP (1.4). *Then the following statements hold:*

(i) MLCP (1.4) *has a solution if and only if $\tau^* > 0$. In this case, a solution of* MLCP (1.4) *is given by $(x^*/\tau^*, s^*/\tau^*, y^*/\tau^*)$.*

(ii) MLCP (1.4) *is infeasible if and only if $\kappa^* > 0$.*

Proof. (i) If $(x^*, \tau^*, s^*, \kappa^*, y^*)$ is a maximal asymptotic complementarity solution of the HMCP (3.1) with $\tau^* > 0$, then

$$Ax^*/\tau^* + Bs^*/\tau^* + Cy^*/\tau^* = \tau^* b/\tau^* = b, \quad (x^*/\tau^*)^T (s^*/\tau^*) = \frac{{x^*}^T s^*}{{\tau^*}^2} = 0,$$

so that $(x^*/\tau^*, s^*/\tau^*, y^*/\tau^*)$ is solution for MLCP (1.4). Now let $(\hat{x}, \hat{s}, \hat{y})$ be a solution for MLCP (1.4), and define $\hat{\tau} = 1$, $\hat{\kappa} = 0$. We show that $(\hat{x}, \hat{\tau}, \hat{s}, \hat{\kappa}, \hat{y})$ is a solution HMCP (3.1). The two complementarity conditions of HMCP (3.1) are obviously satisfied as well as the first feasibility condition. As in the proof of Theorem 4.2, we can show that the first feasibility condition for $(\hat{x}, \hat{\tau}, \hat{s}, \hat{\kappa}, \hat{y})$ implies $\bar{\hat{x}} = \hat{x}$, $\bar{\hat{s}} = \hat{s}$ and $\bar{\hat{y}} = \hat{y}$. Therefore we can write

$$\frac{\bar{\hat{x}}^T \bar{\hat{s}}}{\hat{\tau}} + \hat{\kappa} = \frac{\hat{x}^T \hat{s}}{1} + 0 = \hat{x}^T \hat{s} = 0,$$

which proves that the second feasibility equation of the HMCP (3.1) holds. Since $(\hat{x}, \hat{\tau}, \hat{s}, \hat{\kappa}, \hat{y})$ is a solution of HMCP (3.1) and $\hat{\tau} = 1$, it follows that any maximal solution $(x^*, \tau^*, s^*, \kappa^*, y^*)$ of HMCP (3.1) must have $\tau^* > 0$.

(ii). First we show that if $(x^*, 0, s^*, \kappa^*, y^*)$, with $\kappa^* > 0$, is an asymptotic solution for HMCP (3.1), then MLCP (1.4) is infeasible. Assume the opposite, namely, that there exist $x \geq 0$, $s \geq 0$ and $y \in \mathbb{R}^m$ such that $Ax + Bs + Cy = b$. Since $(x^*, 0, s^*, \kappa^*, y^*)$ asymptotically solves HMCP (3.1), we can consider the sequences $x^k > 0$ with $x^k \to x^*$, $\tau^k > 0$ with $\tau^k \to 0$, $s^k > 0$ with $s^k \to s^*$, $y^k \to y^*$ and $\kappa^k > 0$ with $\kappa^k \to \kappa^* > 0$ satisfying

$$Ax^k + Bs^k + Cy^k \to \tau^k b, \quad \bar{x^k}^T \bar{s^k}/\tau^k \to -\kappa^*. \tag{4.7}$$

As in the proof of Theorem 4.2, (4.7) implies

$$\bar{x^k} \to x^* \text{ and } \bar{s^k} \to s^*. \tag{4.8}$$

According to (3.2), the left multiplication of $[\,\bar{x^k}\,;\,\bar{s^k}\,;\,\bar{y^k}\,]$ with $[A\,B\,C]$ causes the orthogonal projection term to vanish and we have

$$[ABC][\,\bar{x^k}\,;\,\bar{s^k}\,;\,\bar{y^k}\,] = \tau^k[ABC]\bar{b} = \tau^k b.$$

Since $\tau^k > 0$, we deduce that

$$A\bar{x^k}/\tau^k + B\bar{s^k}/\tau^k + C\bar{y^k}/\tau^k = b.$$

Then we can write $A(\bar{x^k}/\tau^k - x) + B(\bar{s^k}/\tau^k - s) + C(\bar{y^k}/\tau^k - y) = 0$, which implies $(\bar{x^k}/\tau^k - x)^T(\bar{s^k}/\tau^k - s) \geq 0$ by the monotonicity of MLCP (1.4), and therefore $\tau^k x^T s - (x^T\bar{s^k} + s^T\bar{x^k}) \geq -\bar{x^k}^T\bar{s^k}/\tau^k$. By considering the limit when $\tau^k \to 0$ and taking into account (4.7) and (4.8), we obtain $\kappa^* \leq -(x^T s^* + s^T x^*) \leq 0$, which contradicts the fact that κ^* is positive.

Conversely, assume that MLCP (1.4) is infeasible. We want to prove that there is a complementarity solution $(x^*, \tau^*, s^*, \kappa^*, y^*)$ of the HMCP (3.1) that has $\kappa^* > 0$. Consider the set

$$\mathcal{P} = \{Ax + Bs + Cy - b \;:\; x, s \in \mathbb{R}^n_+,\; y \in \mathbb{R}^m\}.$$

One can easily verify that $\mathcal{P}$ is closed and convex. The infeasibility of MLCP is equivalent to $0 \notin \mathcal{P}$. Then there is a separating hyperplane between 0 and $\mathcal{P}$, that is, there is a vector $a \in \mathbb{R}^{m+n}$, $a \neq 0$ and $\xi > 0$ so that

$$a^T(Ax + Bs + Cy - b) \geq \xi > 0,\; \forall x, s \geq 0, \forall y \in \mathbb{R}^m. \tag{4.9}$$

Let us take $s = 0$ and $y = 0$ in (4.9). Then for any $x \geq 0$ we must have $x^T A^T a = a^T Ax \geq \xi + a^T b$. If the jth component of $A^T a$ is negative, then $x^T A^T a$ can be made smaller than $\xi + a^T b$ by taking $x_i = 0$ for $i \neq j$ and x_j sufficiently large. Hence, $A^T a \geq 0$. Similarly, $B^T a \geq 0$. Taking $x = s = 0$ in (4.9) leads to $y^T C^T a \geq \xi + a^T b$ for any $y \in \mathbb{R}^m$. This implies that $C^T a = 0$, so that $a \in \operatorname{Ker} C^T$. Therefore we can write $a = Eu$ with $u \in \mathbb{R}^n$, where $E \in \mathbb{R}^{(m+n)\times n}$ is the matrix introduced in Lemma 2.1 whose columns form basis of $\operatorname{Ker} C^T$. We have $(A^T a)^T (B^T a) = u^T E^T A B^T E u \leq 0$, since $-E^T A B^T E$ is positive semidefinite (see Corollary 2.5). On the other hand $(A^T a)^T (B^T a) \geq 0$. Thus

$$(A^T a)^T (B^T a) = 0. \tag{4.10}$$

According to Corollary 2.5, equation (4.10) indicates that the nonnegative function $q \mapsto -q^T E^T A B^T E q$ vanishes at $q = u$. This implies by the first-order optimality conditions that $E^T A B^T E u + (E^T A B^T E)^T u = 0$, or, equivalently, $E^T (A B^T a + B A^T a) = 0$. Hence $A B^T a + B A^T a \in \operatorname{Ran} C$. Therefore we have $A B^T a + B A^T a + C y' = 0$, with $y' \in \mathbb{R}^m$. Denoting $x' = B^T a$ and $s' = A^T a$, we can write that

$$Ax' + Bs' + Cy' = 0, \text{ with } \ x', s' \geq 0, \ y' \in \mathbb{R}^m \text{ and } (x')^T s' = 0. \tag{4.11}$$

Now consider

$$x(t) = x' + t^2 e, \ \tau(t) = t, \ s(t) = s' + t^2 e, \ y(t) = y', \quad t > 0.$$

Using (4.11), we obtain

$$\lim_{t\to 0} (Ax(t) + Bs(t) + Cy(t) - \tau(t) b) = \lim_{t\to 0} \big(t^2 (A + B) e - tb\big) = 0 \tag{4.12}$$

and

$$\begin{bmatrix} \overline{x(t)} \\ \overline{s(t)} \\ \overline{y(t)} \end{bmatrix} = P \begin{bmatrix} x' + t^2 e \\ s' + t^2 e \\ y' \end{bmatrix} + t \begin{bmatrix} u^* \\ v^* \\ w^* \end{bmatrix} = \begin{bmatrix} x' \\ s' \\ y' \end{bmatrix} + t \begin{bmatrix} u^* \\ v^* \\ w^* \end{bmatrix} + t^2 P \begin{bmatrix} e \\ e \\ 0 \end{bmatrix},$$

with the first equality being the definition (3.2) and the last equality holding because $[\, x' \,;\, s' \,;\, y' \,] \in \operatorname{Ker} [ABC]$ (see (4.11)). From the last equation in (4.11) it follows that

$$\lim_{t\to 0} \frac{\overline{x(t)}^T \overline{s(t)}}{t} = (s')^T u^* + (x')^T v^* = a^T (Au^* + Bv^*) = a^T (Au^* + Bv^* + Cw^*) = a^T b.$$

By taking x,s, and y to be zero in (4.9) we deduce that $\kappa^* = -a^T b \geq \xi > 0$. Thus, $(x^*, \tau^*, s^*, \kappa^*, y^*) = (B^T a, 0, A^T a, -a^T b, y')$ is an asymptotic solution of HMCP (3.1) with $\kappa^* > 0$. □

If $(x^*, \tau^*, s^*, \kappa^*, y^*)$ is a maximal complementarity solution to HMCP (3.1), then we must have either $\tau^* > 0$ and $\kappa^* = 0$, or $\tau^* = 0$ and $\kappa^* > 0$. Indeed, if $\tau^* = 0$ and $\kappa^* = 0$, then according to Theorem 4.3 the MLCP (1.4) is feasible but not solvable; this cannot happen in the monotone case however, because, according to Lemma 2.4, feasibility implies solvability. The fourth alternative, both $\tau^* > 0$ and $\kappa^* > 0$, cannot hold because of the complementarity condition.

Thus any maximal complementarity solution $(x^*, \tau^*, s^*, \kappa^*, y^*)$ of HMCP (3.1) provides either a solution or a certificate of infeasibility for MLCP (1.4) . Table 4.1 displays all possible combinations of τ^* and κ^* and the corresponding solvability status of the MLCP (1.4). “NA” stands for “not available” (or not possible).

$\tau^* \setminus \kappa^*$	$= 0$	> 0
$= 0$	NA	infeasibility certificate for MLCP
> 0	$(x^*, s^*, y^*)/\tau^*$ solution of MLCP	NA

Table 4.1
Solvability and infeasibility certificates given by τ^* and κ^*

5. General theory of the existence of central paths for nonlinear complementarity problems. In this section we briefly present several results concerning the properties of an interior-point mapping, which we later use to characterize the central path of HMCP (3.1). The results are part of a framework introduced by Monteiro and Pang [16] to study nonlinear monotone implicitly defined complementarity problems over the non-negative orthant and over the cone of symmetric positive semidefinite matrices [17]. Yoshise [22, 13] proved the same type of results for nonlinear monotone implicitly defined complementarity problems over symmetric cones. While the analysis from [16, 17] requires the complementarity problem to be defined on the entire cone, the results from [22] can be used for complementarity problems not defined on the boundary of the cone. Yoshise's results hold for symmetric cones. Here we specialize them to the nonnegative orthant, which is a symmetric cone. Consider a nonlinear complementarity problem in implicit form,

$$xs = 0,\ F(x, s, y) = 0,\ x, s \geq 0, \tag{5.1}$$

where $F : \mathrm{dom}\,(F) \to \mathbb{R}^{m+n}$ is a continuous map satisfying $\mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m \subseteq \mathrm{dom}\,(F) \subseteq \mathbb{R}^n_{+} \times \mathbb{R}^n_{+} \times \mathbb{R}^m$. The trajectory of the interior point map $H : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m \to \mathbb{R}^{2n+m}$ given by

$$H(x, s, y) = [xs\,;\, F(x, s, y)] \tag{5.2}$$

is characterized by means of homeomorphic continuous maps in the following theorem.

Theorem 5.1 (cf. Theorem 3.10 of [22]). *Suppose that the continuous map F is (x, s)-equilevel-monotone, y-bounded, and y-injective on its domain. Then the map H defined by* (5.2) *satisfies the following properties:*

(i) H is proper with respect to $\mathbb{R}^n_{++} \times F\left(\mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m\right)$.

(ii) H maps $\mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m$ homeomorphically onto $\mathbb{R}^n_{++} \times F\left(\mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m\right)$.

Under the (x, s)-everywhere monotonicity assumption on F, the set $F\left(\mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m\right)$ is convex and open, as shown by the following theorem. The convexity of this set turns out to be a key property in Section 6 in proving crucial properties of the central path associated with our homogeneous model.

Theorem 5.2 (cf. Theorem 3.12 of [22]). *Suppose that the continuous map F is (x, s)-everywhere-monotone, y-bounded, and y-injective on its domain. Then the set $F\left(\mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}^m\right)$ is an open convex set.*

6. Existence and properties of the central path for HMCP. In this section we prove the existence, uniqueness and convergence of the central path for HMCP (3.1). We also show that the sequences of interior points generated by path-following interior-point algorithms have limit points that are solutions to HMCP (3.1). Consider the map

$$H(x, \tau, s, \kappa, y) = [xs\,; \tau\kappa\,; F(x, \tau, s, \kappa, y)]\,, \tag{6.1}$$

and choose a strictly feasible initial point $(x^0, \tau^0, s^0, \kappa^0, y^0)$. For simplicity we set $(x^0, \tau^0, s^0, \kappa^0, y^0) := (e, 1, e, 1, 0)$. Define

$$[\hat{p}^0\,;\hat{r}^0] := H(x^0, \tau^0, s^0, \kappa^0, y^0) = [e\,;1\,;F(e, 1, e, 1, 0)]\,.$$

THEOREM 6.1. *If* MLCP (1.4) *is monotone, then the following statements hold for* HMCP (3.1)*:*

(i) For any $t \in (0, 1]$ there exist $x(t) > 0$, $\tau(t) > 0$, $s(t) > 0$, $\kappa(t) > 0$, and $y(t) \in \mathbb{R}^m$ such that

$$H(x(t), \tau(t), s(t), \kappa(t), y(t)) = t[\hat{p}^0\,;\hat{r}^0]. \tag{6.2}$$

(ii) The set $\mathcal{C}$ containing all the points $(x(t), \tau(t), s(t), \kappa(t), y(t))$ given by (i) forms a bounded path in $\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times \mathbb{R}^m$. Moreover, any accumulation point $(x(0), \tau(0), s(0), \kappa(0), y(0))$ of $\mathcal{C}$ is an asympotic solution of HMCP (3.1).

Proof. (i) According to Lemma 4.1, F satisfies the conditions of Theorem 5.2. Thus, the set $F\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times\ \mathbb{R}^m\right)$ is open and convex. Since

$$H\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times \mathbb{R}^m\right) = \mathbb{R}^{n+1}_{++} \times F\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times \mathbb{R}^m\right),$$

we obtain that $H\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times\ \mathbb{R}^m\right)$ is also open and convex. HMCP (3.1) is asymptotically feasible by Theorem 4.2, *i.e.*, $0 \in \mathrm{cl}\left(F\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times\ \mathbb{R}^m\right)\right)$. Since

$$\mathrm{cl}\left(H\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times \mathbb{R}^m\right)\right) = \mathbb{R}^{n+1}_{+} \times \mathrm{cl}\left(F\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times \mathbb{R}^m\right)\right),$$

we have $0 \in \mathrm{cl}\left(H\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times \mathbb{R}^m\right)\right)$. $H\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times\ \mathbb{R}^m\right)$ being open and convex implies $t[\hat{p}^0\,;\hat{r}^0] \in H\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times \mathbb{R}^m\right)$ for all $t \in (0, 1]$. Then the conclusion from (i) follows from the fact that the map H is a homeomorphism from $\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times \mathbb{R}^m$ onto $H\left(\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++}\ \times \mathbb{R}^m\right)$ (according to Theorem 5.1).

(ii) The homeomorphism of H also implies that $\mathcal{C}$ is a path in $\mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times \mathbb{R}^m$. We now prove the boundedness of $\mathcal{C}$. Assume $(x(t), \tau(t), s(t), \kappa(t), y(t)) \in \mathcal{C}$. Then $F(x(t), \tau(t), s(t), \kappa(t), y(t)) = t\hat{r}^0 = tF(x^0, \tau^0, s^0, \kappa^0, y^0)$; and by the homogeneity of F we obtain $F(x(t), \tau(t), s(t), \kappa(t), y(t)) = F(tx^0, t\tau^0, ts^0, t\kappa^0, ty^0)$. According to Lemma 4.1, F is equilevel-monotone, so that

$$(x(t) - tx^0)^T(s(t) - ts^0) + (\tau(t) - t\tau^0)(\kappa(t) - t\kappa^0) \geq 0,$$

or equivalently

$$x(t)^T s^0 + s(t)^T x^0 + \tau(t)\kappa^0 + \kappa(t)\tau^0 \leq \frac{x(t)^T s(t)}{t} + \frac{\tau(t)\kappa(t)}{t} + t(x^0)^T s^0 + t\tau^0\kappa^0.$$

Moreover, any point on the path must have $x(t)s(t) = te$ and $\tau(t)\kappa(t) = t$. Observe that the first equality gives $x(t)^T s(t) = tn$. Also, we have $(x^0)^T s^0 = n$ and $\tau^0\kappa^0$. Since $x(t) > 0$, $\tau(t) > 0$, $s(t) > 0$, $\kappa(t) > 0$, the above inequality implies

$$\|\,[x(t); \tau(t); s(t); \kappa(t)]\,\|_1 = [x(t); \tau(t); s(t); \kappa(t)]^T\,[e; 1; e; 1] \leq (n+1)(t+1) \leq 2(n+1),$$

which proves the boundedness of $[\,x(t);\ \tau(t);\ s(t);\ \kappa(t)\,]$. Because $0 < t \leq 1$, we have

$$\|F(x(t), \tau(t), s(t), \kappa(t), y(t))\| = t\|\hat{r}^0\| \leq \|\hat{r}^0\|.$$

Using y-boundedness of F we deduce that $y(t)$ is bounded as well (see Lemma 4.1). Hence the set $\mathcal{C}$ is bounded. Therefore, at least one accumulation point $(x(0), \tau(0), s(0), \kappa(0), y(0))$ must exist. According to (6.2) and Definition 3.2, we conclude that any accumulation point is an asympotic solution of HMCP (3.1). □

The following theorem proves that any solution to HMCP (3.1) found by means of a path-following interior-point algorithm possesses the maximal complementarity property.

THEOREM 6.2. *If* $z^* := (x^*, \tau^*, s^*, \kappa^*, y^*)$ *is an asymptotic solution of* HMCP (3.1) *and* $z(0) := (x(0), \tau(0), s(0), \kappa(0)), y(0))$ *is an accumulation point of the path* $\mathcal{C}$*, then* $z_i^* > 0$ *implies* $[z(0)]_i > 0$*, for any* $i \in \{1, 2, \ldots, 2n+2\}$.

Proof. Consider $t \in (0, 1]$, and the corresponding point $z(t) := (x(t), \tau(t), s(t), \kappa(t)), y(t)) \in \mathcal{C}$ for which we have

$$\begin{array}{lcl} [r(t)\,;\gamma(t)] := F(x(t), \tau(t), s(t), \kappa(t), y(t)) & = & t[r^0\,;\gamma^0] \\ x(t)s(t) & = & te \\ \tau(t)\kappa(t) & = & t, \end{array} \tag{6.3}$$

where $[r^0\,;\gamma^0] = \hat{r}^0$. Since (z^*, y^*) is an asympotic solution of HMCP (3.1), there is a sequence $\{z^k\} := \{(x^k, \tau^k, s^k, \kappa^k, y^k)\} \subset \mathbb{R}^{n+1}_{++} \times \mathbb{R}^{n+1}_{++} \times \mathbb{R}^m$ such that

$$\begin{array}{lcl} (x^k, \tau^k, s^k, \kappa^k, y^k) & \to & (x^*, \tau^*, s^*, \kappa^*, y^*) \\ \left[r^k\,;\,\gamma^k\right] := F(x^k, \tau^k, s^k, \kappa^k, y^k) & \to & 0 \\ x^k s^k & \to & x^* s^* = 0 \\ \tau^k \kappa^k & \to & \tau^* \kappa^* = 0. \end{array} \tag{6.4}$$

The sequence $\{z^k\}$ is bounded since it is convergent. Moreover, by Theorem 6.1 the set $\mathcal{C}$ is also bounded; therefore, there must be $\epsilon > 0$ such that

$$\begin{array}{lcll} \|z^k\| & \le & 1/\epsilon, & \forall k \text{ and} \\ \|z(t)\| & \le & 1/\epsilon, & \forall t \in (0, 1]\,. \end{array} \tag{6.5}$$

For a fixed $t \in (0, 1]$, (6.4) also implies that there exists a $k(t)$ positive integer such that

$$\left.\begin{array}{l} x_i^k s_i^k < t\epsilon/(n+1),\ \ i \in \{1, 2, \ldots, 2n+m+2\}, \\ \tau^k \kappa^k < t\epsilon/(n+1), \\ \|r^k\| < t\epsilon, \end{array}\right. \qquad \forall k \ge k(t),$$

which implies

$$(x^k)^T s^k + \tau^k \kappa^k < t\epsilon \ \text{ and } \ \|r^k\| < t\epsilon, \ \ \forall k \le k(t). \tag{6.6}$$

Since F is everywhere-monotone, we can write

$$\begin{aligned} & \left[x^k - x(t)\,;\,\tau^k - \tau(t)\right]^T \left[s^k - s(t)\,;\,\kappa^k - \kappa(t)\right] \\ \ge & \left[r^k - r(t)\,;\,\gamma^k - \gamma(t)\right]^T \left(\phi(z^k) - \phi(z(t))\right), \end{aligned}$$

where ϕ is the continuous *linear* function given by (4.5). By manipulating the terms in the above inequality, we obtain

$$\begin{aligned} s(t)^T x^k + x(t)^T s^k + \kappa(t)\tau^k + \tau(t)\kappa^k \le {} & (x^k)^T s^k + x(t)^T s(t) + \tau^k \kappa^k + \tau(t)\kappa(t) \\ & + \left[r^k - r(t)\,;\,\gamma^k - \gamma(t)\right]^T \left(\phi(z(t)) - \phi(z^k)\right). \end{aligned}$$

By using $[x(t)]_i[s(t)]_i = t$, $i \in \{1, 2, \ldots, 2n+2\}$, and $\tau(t)\kappa(t) = t$ given by (6.3), we can transform the previous inequality to

$$\begin{aligned}
t(z^k)^T z(t)^{-1} &\le (x^k)^T s^k + tn + \tau^k\kappa^k + t + \left[r^k - tr^0\,;\,\gamma^k - t\gamma^0\right]^T \left(\phi(z(t)) - \phi(z^k)\right)\\
&\le (x^k)^T s^k + \tau^k\kappa^k + t(n+1) + \|\left[r^k - tr^0\,;\,\gamma^k - t\gamma^0\right]\|\|\phi(z^k) - \phi(z(t))\|\\
&\le (x^k)^T s^k + \tau^k\kappa^k + t(n+1) + \\
&\quad + \left(\|\left[r^k\,;\,\gamma^k\right]\| + t\|\left[r^0\,;\,\gamma^0\right]\|\right)\|\phi\|\left(\|z^k\| + \|z(t)\|\right) \text{ (by linearity of } \phi)\\
&\le t\epsilon + t(n+1) + t\left(\epsilon + \|\left[r^0\,;\,\gamma^0\right]\|\right)\|\phi\|\left(1/\epsilon + 1/\epsilon\right),
\end{aligned}$$

where the last inequality follows by applying (6.5) and (6.6).

To conclude, we have proved that

$$\forall t \in (0,1], \exists k(t) \text{ such that } (z^k)^T z(t)^{-1} \le \bar{\mu}, \ \ \forall k \ge k(t),$$

where $\bar{\mu} := n + 1 + \epsilon + 2\left(\epsilon + \|\left[r^0\,;\,\gamma^0\right]\|\right)\|\phi\|/\epsilon$ does not depend on either t or $k(t)$. By the convergence of $\{z^k\}$ to z^*, we obtain that

$$(z^*)^T z(t)^{-1} \le \bar{\mu}, \ \ \forall t \in (0,1]. \tag{6.7}$$

Consider $z_i^* > 0$. If the accumulation point $z(0)$ of $\mathcal{P}$ satisfies $[z(0)]_i = 0$, then a sequence $\{t^l\}$ of positive numbers converging to 0 and satisfying $\lim_{l\to\infty}\left[z(t^l)\right]_i = 0$ exists. It follows that $\{z_i^*/\left[z(t^l)\right]_i\}$ is unbounded. But this is a contradiction, since $z_i^*/\left[z(t^l)\right]_i \le (z^*)^T z(t^l)^{-1} \le \bar{\mu}$ according to (6.7). Hence $[z(0)]_i > 0$. ◻

Corollary 6.3. *If $(x^*, \tau^*, s^*, \kappa^*, y^*)$ is an asymptotic solution of* HMCP (3.1) *with $\tau^* > 0$ ($\kappa^* > 0$), then any accumulation point $(x(0), \tau(0), s(0), \kappa(0), y(0))$ of the path $\mathcal{C}$ satisfies $\tau(0) > 0$ ($\kappa(0) > 0$, respectively).*

As we have shown in Section 4, the HMCP (3.1) always has a solution for which the pair (τ^*, κ^*) possesses strict complementarity. Corollary 6.3 shows that the solutions found by path-following algorithms are valid certificates (in the sense of Theorem 4.3) of solvability or infeasibility of the original MLCP (1.4). In addition, in the case when the original MLCP has a solution, Theorem 6.2 indicates that path-following algorithms for the HMCP retrieve maximal complementarity solutions to the MLCP.

7. Numerical method for solving HMCP. The interior-point method used in this work is similar to Mehrotra's predictor-corrector algorithm for linear programing problems [15]. Mehrotra's algorithm emerged in the last decades as the practical interior-point method for solving linear programming problems, being implemented in optimization solvers such as OB1 [14], HOPDM [8], PcX [6], LIPSOL [25], OOQP [7]. It has been also successfully generalized for convex quadratic programming [7], as well as for standard monotone linear complementarity problems [24].

Our adaptation of Mehrotra's algorithm is listed in Algorithm 7 and is aimed at solving the HMCP (3.1) as a monotone nonlinear complementarity. An alternative class of algorithms is the *homogeneous interior point algorithms* (*e.g.*, [2, 11, 23]), which are specialized for the solution of the homogeneous models. The design and analysis of such algorithm for HMCP (3.1) are outside the scope of this work and will be considered in future work.

The predictor direction (7.1) from Algorithm 7 is a pure Newton direction for $H(x, \tau, s, \kappa, y) = 0$. The corrector step (7.2) aims to improve centrality and to compensate for the errors made by the predictor step because of the nonlinearity of the

Algorithm 1 A path-following predictor-corrector algorithm for solving HMCP (3.1)

Set $(x_0, \tau_0, s_0, \kappa_0, y_0) = (e, 1, e, 1, 0)$ and let $k = 0$.
repeat

Let $(x, \tau, s, \kappa, y) \leftarrow (x_k, \tau_k, s_k, \kappa_k, y_k)$.
Let $\mu = [\, x\, ;\, \tau\,]^T [\, s\, ;\, \kappa\,]/(n+1)$;

(stopping criteria)
Return (x, τ, s, κ, y) if $\|H(x, \tau, s, \kappa, y)\| \le 10^{-8}$ and $\mu \le 10^{-8}$.

(predictor step)
Compute $(u_p, \alpha_p, v_p, \beta_p, w_p)$ from

$$\begin{bmatrix} S & 0 & X & 0 & 0 \\ 0 & \kappa & 0 & \tau & 0 \\ A & -b & B & 0 & C \\ d_x^T & d_\tau^T & d_s^T & 1 & d_y^T \end{bmatrix} \begin{bmatrix} u \\ \alpha \\ v \\ \beta \\ w \end{bmatrix} = \begin{bmatrix} -xs \\ -\tau\kappa \\ \tau b - Ax - Bs - Cy \\ -\frac{1}{\tau}\bar{x}^T\bar{s} - \kappa \end{bmatrix}. \tag{7.1}$$

Compute $\theta_p = \arg\max\{\theta \in (0, 1] : (x, \tau, s, \kappa) + \theta(u_p, \alpha_p, v_p, \beta_p) \ge 0\}$.
Let $\mu_p = ([\, x\, ;\, \tau\,] + \theta_p[\, u_p\, ;\, \alpha_p\,])^T([\, s\, ;\, \kappa\,] + \theta_p[\, v_p\, ;\, \beta_p\,])/(n+1)$.
Let centering parameter $\sigma = (\mu_p/\mu)^3$.

(corrector step)
Compute (u, α, v, β, w) from

$$\begin{bmatrix} S & 0 & X & 0 & 0 \\ 0 & \kappa & 0 & \tau & 0 \\ A & -b & B & 0 & C \\ d_x^T & d_\tau^T & d_s^T & 1 & d_y^T \end{bmatrix} \begin{bmatrix} u \\ \alpha \\ v \\ \beta \\ w \end{bmatrix} = \begin{bmatrix} \sigma\mu e - xs - u_p v_p \\ \sigma\mu - \tau\kappa - \alpha_p\beta_p \\ \tau b - Ax - Bs - Cy \\ -\frac{1}{\tau}\bar{x}^T\bar{s} - \kappa \end{bmatrix}. \tag{7.2}$$

Compute $\theta_{max} = \max\{\theta : [\, x;\, \tau;\, s;\, \kappa\,] + \theta[\, u;\, \alpha;\, v;\, \beta\,] \ge 0\}$.
Compute steplength $\theta_c \in (0, \theta_{max})$ according to Algorithm 7.
Let $(x_{k+1}, \tau_{k+1}, s_{k+1}, \kappa_{k+1}, y_{k+1}) \leftarrow (x_k, \tau_k, s_k, \kappa_k, y_k) + \theta_c(u, \alpha, v, \beta, w)$.
Let $k \leftarrow k + 1$.

continue

complementarity equations in $H(x, \tau, s, \kappa, y) = 0$. The former aim is achieved by incorporating $\sigma\mu e - xs$ and $\sigma\mu - \tau\kappa$ terms in the right-hand side of (7.2), and the latter is achieved by the second-order correction terms $-u_p v_p$ and $-\alpha_p\beta_p$. The point $[\, \sigma\mu e\, ;\, \sigma\mu\,]$ on the central path targeted by the corrector is controlled by the centering parameter σ, which is a trade-off between optimality (σ close to zero) and centrality (σ close to one), depending on the progress toward optimality made along the predictor direction.

The linear systems (7.1) and (7.2) are obtained by linearizing the interior point map H given by (5.2). The vectors d_x, d_s, and d_y are given by

$$[d_x\, ;\, d_s\, ; d_y] = \frac{1}{\tau} P_{\mathrm{Ker}[ABC]}[\, \bar{s}\, ;\, \bar{x}\, ;\, \bar{y}\,]$$

and

$$d_\tau = \frac{1}{\tau}[\, \bar{s}\, ;\, \bar{x}\, ;\, 0]\bar{b} - \frac{1}{\tau^2}\bar{x}^T\bar{s}.$$

The projection of a given vector z onto $\mathrm{Ker}\,[ABC]$ is computed based on

$$P_{\mathrm{Ker}[ABC]}z = (I - P_{\mathrm{Ran}[ABC]^T})z = z - [A\,B\,C]^T\,(AA^T + BB^T + CC^T)^{-1}\,[A\,B\,C]^T\,z.$$

For this we perform a Cholesky factorization of $AA^T + BB^T + CC^T$ once and solve with the Cholesky factors each time $(AA^T + BB^T + CC^T)^{-1}$ needs to be applied to the vector $[A\,B\,C]^T\,z$.

We compute the steplength along the corrector direction by enforcing decrease in the merit function

$$\phi(x,\tau,s,\kappa,y) = \zeta(x^T s + \tau\kappa) + \|F(x,\tau,s,\kappa,y)\|. \tag{7.3}$$

Here ζ is a positive parameter used to balance between complementarity $x^T s$ and feasibility $\|F(x,\tau,s,\kappa,y)\|$. Clearly, if the point (x,τ,s,κ,y) satisfies $\phi(x,\tau,s,\kappa,y) = 0$, then (x,τ,s,κ,y) is a complementarity solution. One can easily prove that the corrector direction is a descent direction for $\phi(x,\tau,s,\kappa,y)$.

Algorithm 2 Procedure for computing the step size θ_c.

Set $\theta_c \leftarrow \theta_{max}$;
Set $k \leftarrow 0$;
Repeat
    Set $[\,x^+\,;\,\tau^+\,;\,s^+\,;\,\kappa^+\,;\,y^+\,] = [\,x\,;\,\tau\,;\,s\,;\,\kappa\,;\,y\,] + \theta_c[\,u\,;\,\alpha\,;\,v\,;\,\beta\,;\,w\,]$;
    If (7.4) and (7.5) are satisfied then
        Accept and return θ_p;
    Else
        $\theta_c = c_3^{2^k}\theta_c$;
    If $\theta_c \le c_4$ then
        Return error;
    Set $k \leftarrow k+1$;
continue

The use of the $\|F(x,\tau,s,\kappa,y)\|$ term in the merit function is a departure from the original Mehrotra algorithm and is required by the nonlinearity of $F(x,\tau,s,\kappa,y)$. When F is linear, as is the case for linear complementarity, including linear programming and convex quadratic programming, the feasibility improves along the entire corrector direction (under monotonicity assumption). For nonlinear problems the corrector direction is only a locally descent direction for $F(x,\tau,s,\kappa,y)$, assuming monotonicity. For this reason we use the Armijo rule for enforcing a “sufficient decrease” in the merit function; namely, we require the corrector step length θ_c to satisfy

$$\phi(x^+,\tau^+,s^+,\kappa^+,y^+) \le c_1\theta_c\nabla\phi(x,\tau,s,\kappa,y)^T[\,u\,;\,\alpha\,;\,v\,;\,\beta\,;\,w\,], \tag{7.4}$$

where $[\,x^+\,;\,\tau^+\,;\,s^+\,;\,\kappa^+\,;\,y^+\,] = [\,x\,;\,\tau\,;\,s\,;\,\kappa\,;\,y\,]+\theta_c[\,u\,;\,\alpha\,;\,v\,;\,\beta\,;\,w\,]$ is the candidate for update, and c_1 is a constant of the algorithm. Additionally we enforce a “away from boundary” condition on the corrector step length that has the role of preventing the iterates from converging prematurely toward the boundary of the nonnegative orthant, which can cause serious numerical difficulties. This condition takes the form

$$x_i^+ s_i^+ \ge c_2\mu^+, \text{ for } i = 1,2,\ldots,n, \text{ and } \tau^+\kappa^+ \ge c_2\mu^+, \tag{7.5}$$

where $\mu^+ = ((x^+)^T s^+ + \tau^+ \kappa^+)/(n+1)$, and c_2 is a constant of the algorithm. The complete procedure for computing the step length is presented in Algorithm 2. A similar line-search technique has been used for the numerical solution of the homogeneous model for standard monotone linear complementarity problems [1].

8. Numerical experiments. In this section we report on the performance of the homogenization technique on a class of randomly generated monotone MLCPs. The predictor-corrector algorithm presented in the previous section was implemented in MATLAB 7.9. The runs have been performed on a machine equipped with a Intel dual-core 2.0 GHz CPU and 4 gigabytes of memory. The parameters of the Algorithm 2 are $c_1 = 10^{-4}$, $c_2 = 10^{-6}$, $c_3 = 0.85$, $c_4 = 10^{-4}$, and $\zeta = 1/\sqrt{n+1}$.

8.1. Test problems. We apply the homogenization technique to a class of randomly generated sparse MLCPs. Our generation technique initially generates a sparse SLCP of the form (1.3) by generating a random positive semidefinite M. M does not necessarily have to be symmetric. The vector b can chosen such that the problem has or does not have a solution. An SLCP with a unique solution is obtained by generating a positive definite M. Then, the SLCP is transformed to an HLCP of the form (1.5) by multiplying equation (1.3) with a sparse nonsingular matrix and rearranging the variables. We obtain a monotone MLCP of the form (1.5) with $A = G_n Q$, $B = G_n R$, $C = G_m K$, and the right-hand $G_n b$ by generating a sparse matrix $G \in \mathbb{R}^{(m+n)\times(m+n)}$ whose columns represent an orthonormal basis of $\mathbb{R}^{(m+n)}$ and a sparse $m \times m$ nonsingular matrix K. Matrices G_n and G_m denote the first n and last m columns of G, respectively. The solution of the resulting MLCP has the same properties as the solution of the initial SLCP, which allows straightforward generation of MLCPs with no solution, unique solution, or multiple solutions. We refer the reader to [18] for a detailed presentation and validation proofs of this generation technique.

n	m	τ	κ	μ	$\|\|r\|\|$
500	125	1.51e+000	4.11e-010	1.95e-009	2.00e-014
1000	250	1.17e+000	3.69e-009	9.90e-009	2.90e-013
1500	375	1.11e+000	8.27e-010	2.26e-009	3.00e-014
2000	500	1.10e+000	3.01e-009	7.90e-009	3.90e-013
2500	625	1.13e+000	1.49e-009	3.96e-009	4.00e-014
3000	750	1.19e+000	1.67e-009	4.63e-009	6.40e-013
3500	875	1.22e+000	8.95e-010	2.79e-009	7.00e-014
4000	1000	1.26e+000	5.82e-010	1.93e-009	1.20e-013
4500	1125	1.32e+000	6.93e-010	2.10e-009	1.20e-013
5000	1250	1.27e+000	1.59e-009	4.92e-009	5.20e-013

TABLE 8.1

Certificates of solvability ($\tau > 0$, $\kappa = 0$) are properly retrieved by solving HMCP *for feasible* MLCP*s. The number of the complementarity variables and free variables of the* MLCP *is denoted by n and m, respectively. The left two columns show the complementarity measure μ and the norm of the feasibility residual for the retrieved solution of the* MLCP.

8.2. HMCP certificates and solution to MLCP. Our first set of experiments solves HMCPs corresponding to randomly generated MLCPs. Our homogeneous technique was applied to MLCPs having multiple solutions and to infeasible MLCPs of various sizes. A solution to the MLCP was always retrieved in the solvable case, and a certificate of infeasibility, that is, $\kappa > 0$, was found for all infeasible instances, as shown in Table 8.1 and Table 8.2. We also found that the solution to MLCP found by rescaling the HMCP solution with $1/\tau$ is accurate. This is shown by the last two columns of Table 8.1, where we list the complementarity measure μ and the norm of the feasibility residual $\|r\| = \|Ax + Bs + Cy - b\|$ of the MLCP's solution.

n	m	τ	κ
500	125	5.33e-008	5.93e+000
1000	250	1.11e-007	1.27e+001
1500	375	1.05e-007	1.90e+001
2000	500	1.43e-008	2.23e+001
2500	625	1.85e-007	2.01e+001
3000	750	5.11e-007	1.52e+001
3500	875	5.66e-007	1.75e+001
4000	1000	7.60e-007	1.57e+001
4500	1125	9.57e-007	1.48e+001
5000	1250	9.85e-007	2.11e+001

Table 8.2

Certificates of infeasibilty ($\tau = 0$, $\kappa > 0$) are properly retrieved by solving HMCP*s for infeasible* MLCP*s. The number of the complementarity variables and free variables of the* MLCP *are denoted by n and m, respectively.*

8.3. Performance comparison with other homogeneous models. To the best of our knowledge, the homogeneous model proposed in this work is the first homogenization technique that can be applied directly to monotone horizontal linear complementarity problems. Alternatively, one can transform a given monotone MLCP or HMCP to a monotone SLCP and can apply the homogenization technique of Andersen and Ye [2]. In this section we compare these two approaches in terms of computational cost, both execution time and memory usage.

A monotone MLCP can be transformed to an equivalent monotone SLCP by first removing the free variables using the technique from Lemma 2.1, yielding a monotone HLCP. The orthonormal basis of C^T required by Lemma 2.1 is found by performing a sparse singular value decomposition of C^T. The monotone HLCP then is transformed to a monotone SLCP by using the reduction method from [9]. The reduction requires finding a maximal set of linear independent columns of the SLCP's matrix Q. Let $Q_{i_1}, Q_{i_2}, \ldots, Q_{i_L}$ denote this set, where subscripts denote columns of matrices. Also define the matrices S and T by

$$S_j = \begin{cases} Q_j, & \text{if } j \in \{i_1, i_2, \ldots, i_L\} \\ R_j, & \text{otherwise} \end{cases} \quad \text{and} \quad T_j = \begin{cases} R_j, & \text{if } j \in \{i_1, i_2, \ldots, i_L\} \\ Q_j, & \text{otherwise} \end{cases}.$$

One can easily show that the matrix C is invertible and SLCP$(S^{-1}T, S^{-1}b)$ is monotone [9]. Additionally, the solutions to the SLCP coincide with the solutions to the HLCP modulo a rearrangement of the variables. Our implementation uses an LU factorization to find the maximal set of linear independent columns of Q. The matrix $S^{-1}T$ is computed by solving for each column of T and is close to being dense, irrespective of the sparsity of the MLCP's data.

We compare our HMCP homogenization approach of MLCP with the homogenization technique of Andersen and Ye [2] for SLCP. For this we have implemented in MATLAB the homogeneous interior-point proposed by the authors for the solution of their homogeneous model for SLCPs. This implementation is denoted "Homog−SLCP," while our implementation is denoted "HMCP−MLCP." For the Homog−SLCP approach we report execution times obtained with sparse and dense linear algebra, in order to rule out possible cases where the sparse linear solvers can still take advantage of the (precarious) sparsity of SLCP matrices and be faster than dense linear solvers. Additionally, we report the execution time spent in the transformation of the MLCP to SLCP, which is denoted by "Conv." We chose not to include in the Homog−SLCP time, even though the transformation contributes to total execution in the Homog−SLCP approach. Tables 8.3 and 8.4 show these execution times as well as the number of iterations of each algorithm on the test problems used in Section 8.2.

		HMCP − MLCP		Conv	Homog−SLCP		
n	m	iter	t	t	iter	t_{sparse}	t_{dense}
500	125	12	1.29	1.94	13	44.96	4.31
1000	250	12	7.76	12.30	15	608.90	28.27
1500	375	13	26.42	39.93	13	3187.58	72.79
2000	500	12	59.11	98.95	14	8284.77	164.53
2500	625	12	114.70	201.80	11	14550.27	243.53
3000	750	12	247.08	363.57	14	36358.05	515.56
3500	875	12	361.29	577.16	12	OOM	846.33
4000	1000	12	552.47	869.26		OOM	OOM
4500	1125	12	861.67	1205.81		OOM	OOM
5000	1250	11	1089.22	1699.90		OOM	OOM

TABLE 8.3

HMCP is applied to sparse MLCPs (HMCP − MLCP), while the Andersen&Ye homogenization algorithm solves an equivalent SLCP (Homog−SLCP). Solvable MLCPs are used in this study. We report the execution times (in seconds) and the number of iterations, denoted "iter," needed by the two algorithms. Execution times obtained with both sparse (t_{sparse}) and dense (t_{dense}) linear algebra kernels are listed for Homog−SLCP. We also list the execution times of the transformation from MLCP to SLCP.

		HMCP − MLCP		Conv	Homog−SLCP		
n	m	iter	t	t	iter	t_{sparse}	t_{dense}
500	125	13	1.34	1.80	12	41.48	3.99
1000	250	12	7.48	12.52	13	363.46	24.80
1500	375	12	29.20	40.22	13	2257.53	73.06
2000	500	12	60.08	98.33	13	8689.07	155.90
2500	625	12	123.84	201.74	11	16256.95	246.69
3000	750	12	214.75	369.63	11	28970.54	409.69
3500	875	12	448.54	577.56	11	OOM	724.17
4000	1000	12	674.99	876.33		OOM	OOM
4500	1125	12	959.86	1194.37		OOM	OOM
5000	1250	12	1311.67	1612.35		OOM	OOM

TABLE 8.4

Same experiment as in Table 8.3 is performed for infeasible sparse MLCPs.

A first observation is that the transformation from MLCP to SLCP is more expensive than solving the MLCP with HMCP. The overhead is caused by the expensive linear algebra requirements of the transformation and the loss of sparsity that gradually occurs during the transformation process.

The Homog−SLCP implementation is significantly slower than HMCP − MLCP, despite a similar number of iteration of the two algorithms. This is the consequence of the sparsity loss occurring during the transformation; HMCP − MLCP solves sparse linear systems, while Homog−SLCP solves dense linear systems. Also, the storage requirement of Homog−SLCP grows rapidly with the problem sizes, and the algorithm runs out of memory (OOM) even for moderately sized problems, which is not the case for HMCP − MLCP.

9. Concluding remarks. In this paper we have introduced a new homogenization technique for monotone horizontal mixed linear complementarity problems. We have proved that the transformed problem offers an infeasibility certificate or provides a solution of the original problem. We have also shown that interior-point path-following methods can be used effectively to obtain the numerical solutions of the homogeneous problem and safely retrieve the certificates of infeasibility or solvability. Numerical experiments performed on randomly generated problems show that the proposed numerical method is considerably faster and can solve larger problems than with previously proposed homogenization methods. The proposed solution procedure, an algorithm based on the well-known Mehrotra predictor-corrector, performs well in practice. Future work will be devoted to the design and analysis of a path-following numerical homogeneous algorithm possessing provable convergence and $O(\sqrt{n}L)$ iteration-complexity that is achieved by homogeneous algorithms for standard monotone complementarity problems.

Acknowledgments. The work of Cosmin Petra was supported in part by the U.S. Department of Energy under Contract DE-AC02-06CH11357. The work of Florian Potra was supported by the National Science Foundation under Grant No. DMS-1311923. This research used resources of the Laboratory Computing Resource Center and the Argonne Leadership Computing Facility at Argonne National Laboratory, which is supported by the Office of Science of the U.S. Department of Energy under contract DE-AC02-06CH11357.

REFERENCES

[1] E.D. Andersen and Y. Ye, *A computational study of the homogeneous algorithm for large-scale convex optimization*, Comput. Optim. Appl., 10 (1998), pp. 243–280.
[2] ———, *On a homogeneous algorithm for the monotone complementarity problem*, Mathematical Programming, 84 (1999), pp. 375–399.
[3] M. Anitescu, G. Lesaja, and F. A. Potra, *Equivalence between different formulations of the linear complementarity problem*, Optimization Methods & Software, 7 (1997), pp. 265–290.
[4] K. Anstreicher, *On interior algorithms for linear programming with no regularity assumptions*, Operations Research Letters, 11(7) (1992), pp. 209–212.
[5] J. F. Bonnans and C. C. Gonzaga, *Convergence of interior point algorithms for the monotone linear complementarity problem*, Mathematics of Operations Research, 21 (1996), pp. 1–25.
[6] J. Czyzyk, S. Mehrotra, and S. J. Wright, *PCx user guide*, Tech. Report OTC 96/01, Optimization Technology Center, Argonne National Laboratory and Northwestern University, October 1996.
[7] E. M. Gertz and S. J. Wright, *Object-oriented software for quadratic programming*, ACM Transactions on Mathematical Software, 29 (2003), pp. 58–81.
[8] J. Gondzio, *HOPDM (version 2.12) - a fast LP solver based on a primal-dual interior point method*, European Journal of Operational Research, 85 (1995), pp. 221–225.
[9] M. S. Gowda, *Reducing a monotone horizontal LCP to an LCP*, Applied Mathematics Letters, 8 (1995), pp. 97–100.
[10] O. Güler, *Generalized linear complementarity problems*, Mathematics of Operations Research, 20 (1995), pp. 441–448.
[11] K. Huang and S. Mehrotra, *Solution of monotone complementarity and general convex programming problems using modified potential reduction interior point method*, tech. report, Dept. of Industrial Engineering and Management Sciences, Northwestern University, 2013.

[12] N. K. Karmarkar, *A new polynomial-time algorithm for linear programming*, Combinatorica, 4 (1984), pp. 373–395.
[13] Y. Lin and A. Yoshise, *A homogeneous model for mixed complementarity problems over symmetric cones*, Vietnam Journal of Mathematics, 35 (2007), pp. 541–562.
[14] I.J. Lustig, R. E. Marsten, and D. F. Shanno, *On implementing Mehrotra's predictor–corrector interior-point method for linear programming*, SIAM Journal on Optimization, 2 (1992), pp. 435–449.
[15] S. Mehrotra, *On the implementation of a primal-dual interior point method*, SIAM Journal on Optimization, 2 (1992), pp. 575–601.
[16] R. D. C. Monteiro and Jong-Shi Pang, *Properties of an interior-point mapping for mixed complementarity problems*, Mathematics of Operations Research, 21 (1996), pp. 629–654.
[17] ———, *On two interior-point mappings for nonlinear semidefinite complementarity problems*, Mathematics of Operations Research, 23 (1998), pp. 39–60.
[18] C. G. Petra, *Homogenization of Mixed Horizontal LCPs and Applications*, PhD thesis, University of Maryland, Baltimore County, July 2009.
[19] R. Sznajder and M. S. Gowda, *Generalizations of P0- and P-properties; extended vertical and horizontal linear complementarity problems*, Linear Algebra and its Applications, 223–224 (1995), pp. 695–715.
[20] Y. Ye, *On homogeneous and self-dual algorithms for LCP*, Mathematical Programming, 76 (1997), pp. 211–221.
[21] Y. Ye, M. J. Todd, and S. Mizuno, *An $O(\sqrt{n}L)$-iteration homogeneous and self-dual linear programming algorithm*, Mathematics of Operations Research, 19 (1994), pp. 53–67.
[22] A. Yoshise, *Interior point trajectories and a homogeneous model for nonlinear complementarity problems over symmetric cones.*, SIAM Journal on Optimization, 17 (2006), pp. 1129 – 1153.
[23] A Yoshise, *A homogeneous algorithm for monotone complementarity problems over symmetric cones*, Frontiers Science Series, 49 (2007), p. 83.
[24] D. Zhang and Y. Zhang, *A Mehrotra-type predictor-corrector algorithm with polynomiality and Q-subquadratic convergence*, Ann. Oper. Res., 62 (1996), pp. 131–150.
[25] Y. Zhang, *Solving large-scale linear programs by interior-point methods under the matlab environment*, Tech. Report TR96-01, University of Maryland Baltimore County, February 1996.