Performance of leader-follower multiagent systems in directed networks

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Abstract

We consider the leader-follower multiagent systems in which the leader executes the desired trajectory and the followers implement the consensus algorithm subject to stochastic disturbances. The performance of the leader-follower systems is quantified by using the steady-state variance of the deviation of followers from the leader. We study the asymptotic scaling of the variance in directed lattices in 1, 2, and 3 dimensions. We show that in 1D and 2D the variance of the followers' deviation increases to infinity as one moves away from the leader, while in 3D it remains bounded.

I. Introduction

A leader-follower multiagent system consists of a leader, who provides the desired trajectory of the multiagent system, and a set of followers, who update their states using local relative feedback. This control strategy has a variety of applications including formation of unmanned air vehicles, control of rigid robotic bodies, and distributed estimation in sensor networks [1]–[13].

A fundamental question concerning the performance of leader-follower strategy is how well the followers are able to keep track the trajectory of the leader when they are subject to stochastic disturbances. In large networks, the asymptotic scaling of the variance of followers' deviation from the desired trajectory is determined by the network architecture. In this paper, we focus on directed lattices in 1, 2, and 3 dimensions. We show that as one moves away from the leader, the variance of followers increases unboundedly in 1D and 2D. In 3D, the variance of followers is bounded above by a constant that is independent of the number of followers. These results have

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a strong resemblance to performance limitation of distributed consensus in undirected tori [10]. For directed networks, our results on the asymptotic scaling of performance appear to be among the first work in the literature.

Our contributions are twofold. First, we obtain analytical expressions for the steady-state variance of the deviation of the followers from the leader. These expressions allow us to study the distribution of energy in leader-follower multiagent systems with directed lattices as the controller architecture. Second, we characterize the asymptotic scaling trends of the variance of followers in 1D, 2D, and 3D directed lattices. We show that in 1D and 2D the variance of followers scales asymptotically as a square-root function and a logarithmic function, respectively, and in 3D the variance remains bounded regardless of the network size.

This paper is organized as follows. In Section II, we present our main results for the performance of leader-follower multiagent systems on directed lattices. In particular, we obtain analytical expressions for the variance of followers and derive the asymptotic scaling trends in large networks. In Section III, we consider the extensions to double-integator models and local errors between followers as an alternative performance measure. We also discuss connections of our proof techniques with random walks on undirected lattices. In Section IV, we conclude the paper and in Appendix, we provide the detailed proofs.

II. Leader-follower multiagent systems on directed lattices

We consider the performance of leader-follower multiagent systems on directed lattices. By exploiting the lower triangular Toeplitz structure of the resulting Laplacian matrices, we obtain analytical expressions for the variance of followers and establish its asymptotic scaling trends in large networks.

A. 1D lattice

Consider a set of N agents on a line whose dynamics are modeled by the single-integrators

$$\dot{x}_n(t) = u_n(t) + d_n(t), \quad n = 1, \ldots, N,$$

where $x_n(t)$ denotes the deviation of the nth vehicle from its desired trajectory, $u_n(t)$ is the control input, and $d_n(t)$ is a zero-mean, unit-variance stochastic disturbance. A virtual leader, indexed by 0, is assumed to execute the desired trajectory at all time. Thus, $x_0(t) \equiv 0$, and

$\dot{x}_0(t) = 0$. The followers implement the consensus algorithm. Namely, each follower updates its state information using the relative differences between itself and the agent ahead

$$\dot{x}_n(t) \,=\, -\,(x_n(t) \,-\, x_{n-1}(t)) \,+\, d_n(t), \quad n \,=\, 1, \ldots, N.$$

Note that we have

$$\dot{x}_1(t) \,=\, -x_1(t) \,+\, d_1(t)$$

for the first follower after the leader, because $x_0(t) \equiv 0$.

By stacking the states of all followers into a vector, $x(t) \,=\, [\, x_1(t) \,\cdots\, x_N(t)\,]^T \,\in\, \mathbb{R}^N$, the state-space representation of the leader-follower system is given by

$$\dot{x}(t) \,=\, -Lx(t) \,+\, d(t), \tag{1}$$

where $L \in \mathbb{R}^{N\times N}$ is the Laplacian matrix of the 1D lattice. In particular, L is lower triangular Toeplitz with 1 on the main diagonal, -1 on the first subdiagonal, and zero everywhere else,

$$L \,\sim\, \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \tag{2}$$

When the disturbance, $d(t) \,=\, [\, d_1(t) \,\cdots\, d_N(t)]^T \,\in\, \mathbb{R}^N$, is absent, the deviation of the followers asymptotically converges to zero. In other words, the followers converge to the desired trajectory, that is, the trajectory of the leader. In the presence of the disturbance, however, the followers converge to the desired state in the mean value. The steady-state variance of the followers can be used to quantify the deviation from the desired state

$$P_n \,:=\, \lim_{t\to\infty} E\{x_n^2(t)\}, \quad n \,=\, 1, \ldots, N,$$

where $E\{\cdot\}$ denotes the expectation operator.

We are interested in the scaling trend of the variance distribution as one moves away from the leader. Intuitively, the followers who are farther away from the leader have larger steady-state variance. It turns out that the variance of the followers increases as a square-root function of the number of followers. This result is detailed in Lemma 1.

Lemma 1. *The steady-state variance of the nth follower in the 1D lattice* (1) *is given by*

$$P_n \;=\; \sum_{i=1}^{n} \frac{(2i-2)!}{2\cdot 2^{2i-2}((i-1)!)^2} \;=\; \frac{n\,(2n)!}{2^{2n}\,n!\,n!}, \quad n=1,\ldots,N. \tag{3}$$

The total variance normalized by the number of followers is

$$\Pi_N \;:=\; \frac{1}{N}\sum_{n=1}^{N} P_n \;=\; \frac{(2N+1)!}{3\cdot 2^{2N} N!N!}.$$

Furthermore,

$$\lim_{n\to\infty} \frac{P_n}{\sqrt{n}} \;=\; \sqrt{\frac{1}{\pi}}, \quad \lim_{N\to\infty} \frac{\Pi_N}{\sqrt{N}} \;=\; \frac{2}{3\sqrt{\pi}}.$$

This result has appeared in [11], [12]. We provide the proof in Appendix A for completeness.

B. 2D lattice

We next consider the leader-follower system that consists of a virtual leader and $N \times N$ followers in the formation of a 2D lattice. A follower at the nth row and the mth column of the 2D lattice, indexed by (n,m), updates its state using the relative differences between itself and its two neighbors

$$\dot{x}_{n,m} \;=\; -\,(x_{n,m} - x_{n,m-1}) \;-\; (x_{n,m} - x_{n-1,m}) \;+\; d_{n,m},$$

for $n,m = 1,\ldots,N$. Here, we drop the dependence on time to ease the notation. Similar to the 1D case, we assume that the followers on the boundary of the formation have direct access to the state of the leader. In particular, the followers on the first column and the first row implement

$$\begin{aligned}\dot{x}_{n,1} \;&=\; -\,(x_{n,1} - x_{n-1,1}) \;-\; (x_{n,1} - x_{n,0}) \;+\; d_{n,1},\\ \dot{x}_{1,m} \;&=\; -\,(x_{1,m} - x_{0,m}) \;-\; (x_{1,m} - x_{1,m-1}) \;+\; d_{1,m},\end{aligned}$$

where $x_{n,0} = x_{0,m} = x_0 \equiv 0$.

Let $x = [\mathbf{x}_1^T \cdots \mathbf{x}_N^T]^T \in \mathbb{R}^{N^2}$ be the state of followers where $\mathbf{x}_n \;=\; [\,x_{n,1} \;\cdots\; x_{n,N}\,]^T \in \mathbb{R}^N$ denotes the state of followers on the nth row of the lattice. Then the state-space representation of the leader-follower system is given by

$$\dot{x} \;=\; -\,L_2 x \;+\; d, \tag{4}$$

where the Laplacian matrix $L_2 \in \mathbb{R}^{N^2 \times N^2}$ is lower triangular block Toeplitz

$$L_2 \sim \begin{bmatrix} K_2 & 0 & 0 \\ -I & K_2 & 0 \\ 0 & -I & K_2 \end{bmatrix},$$

where $I \in \mathbb{R}^{N \times N}$ is the identity matrix and $K_2 \in \mathbb{R}^{N \times N}$ is lower triangular Toeplitz with 2 on its main diagonal, -1 on the first subdiagonal, and zero everywhere else,

$$K_2 \sim \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}.$$

In what follows, we derive the analytical expression for the variance of each follower. The steady-state covariance matrix of leader-follower system can be expressed as

$$P = \int_0^\infty \mathrm{e}^{-L_2 t}\, \mathrm{e}^{-L_2^T t}\, \mathrm{d}t \in \mathbb{R}^{N^2 \times N^2}.$$

Let $P_n \in \mathbb{R}^{N \times N}$ be the nth diagonal block of P and let $(P_n)_m$ be the mth diagonal element of P_n. We have the following result.

Lemma 2. *For the leader-follower system in 2D lattice* (4)*, the steady-state variance of the follower at the nth row and mth column is given by*

$$(P_n)_m = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{(2i+2j-4)!}{4 \cdot 4^{2i+2j-4}((i-1)!(j-1)!)^2} \tag{5}$$

for $n, m = 1, \ldots, N$.

The proof of Lemma 2 is similar to the proof of Lemma 1; see Appendix B for details.

Since we are summing up positive quantity in (5), we conclude that $(P_n)_m$ is monotonically increasing as both n and m increase. In other words, the variance of the follower grows as one moves away from the leader. We next show that the variance of the followers on the diagonal of the lattice scales asymptotically as a *logarithmic* function.

Proposition 1. *Consider the leader-follower system in 2D lattice* (4)*. Let V_n be the steady-state variance of the follower at the nth row and the nth column of the lattice for $n = 1, \ldots, N$. Then*

V_n scales asymptotically as a logarithmic function of n, denoted as

$$V_n \sim O(\log(n)).$$

Proof. We begin by writing V_n as

$$V_n = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(i,j)$$

where

$$f(i,j) = \frac{(2i+2j)!}{4\cdot 4^{2(i+j)}\, i!\, i!\, j!\, j!}.$$

In other words, V_n is the summation of a positive function f over the square $\mathcal{S}_n := \{(i,j) \mid 0 \leq i,j \leq n-1\}$. Let Δ_n be the summation of f over the triangle

$$\mathcal{T}_n := \{(i,j) \mid 0 \leq i \leq n-1, 0 \leq j \leq n-i\}, \tag{6}$$

with vertices $(0,0)$, $(0,n-1)$, and $(n-1,0)$

$$\Delta_n := \sum_{i=0}^{n-1}\sum_{j=0}^{n-i} f(i,j).$$

Then $\Delta_n < V_n < \Delta_{2n}$, because the triangle $\mathcal{T}_n$ is a subset of the square $\mathcal{S}_n$ which itself is a subset of the triangle $\mathcal{T}_{2n}$.

To show $V_n \sim O(\log(n))$ for large n, it suffices to show $\Delta_n \sim O(\log(n))$. We compute the summation of f along the line segment $i+j=k$

$$\begin{aligned} S_k &:= \sum_{i=0}^{k} f(i,k-i) = \sum_{i=0}^{k} \frac{(2k)!}{4\cdot 4^{2k}\, i!\, i!\, (k-i)!\, (k-i)!} \\ &= \frac{1}{4\cdot 4^{2k}} \frac{(2k)!}{k!k!} \sum_{i=0}^{k} \frac{k!k!}{i!\, i!\, (k-i)!\, (k-i)!} \\ &= \frac{1}{4\cdot 4^{2k}} \left(\frac{(2k)!}{k!k!}\right)^2, \quad k = 0,1,\ldots,n-1, \end{aligned} \tag{7}$$

where we have used the fact that $\sum_{i=0}^{k}\binom{k}{i}^2 = \binom{2k}{k} = \frac{(2k)!}{k!k!}$. From the expression (3) and the approximation (17), we conclude that S_k behaves similar to the harmonic series for large k,

$$S_k \approx \frac{1}{4\pi k}, \quad k \gg 1.$$

It follows that

$$\Delta_n \;=\; \sum_{k=0}^{n-1} S_k \;\sim\; O(\log(n)).$$

This completes the proof. □

From Proposition 1, it follows that the total variance of the followers on the main diagonal normalized by N scales logarithmically for large N,

$$\frac{1}{N}\sum_{n=1}^{N} V_n \;\sim\; O(\log(N)).$$

C. 3D lattice

While the variance of the followers increases unboundedly with the size of lattices in 1D and 2D, it turns out that in 3D, the variance of the followers is bounded by a constant that is independent of the lattice size. For undirected networks, similar results have been shown for distributed consensus [10] and distributed estimation [2], [3].

Consider the leader-follower system that consists of a virtual leader and $N \times N \times N$ followers on the 3D lattice. The coordinates of the follower at the nth row, mth column of the lth cross-section is denoted by (n, m, l) for $n, m, l = 1, \ldots, N$. The follower updates its state using local feedback subject to disturbance

$$\begin{aligned}\dot{x}_{n,m,l} \;=\; &-\left(x_{n,m,l} - x_{n-1,m,l}\right) \;-\; \left(x_{n,m,l} - x_{n,m-1,l}\right)\\ &-\left(x_{n,m,l} - x_{n,m,l-1}\right) \;+\; d_{n,m,l}.\end{aligned}$$

Similar to the 1D and 2D cases, the followers on the boundary, indexed by $(1, m, l)$, $(n, 1, l)$, and $(n, m, 1)$, have access to the state of the leader, $x_0 \equiv 0$.

The state-space representation of the leader-follower system in 3D lattice is given by

$$\dot{x} \;=\; -\,L_3 x \;+\; d, \tag{8}$$

where the Laplacian matrix $L \in \mathbb{R}^{N^3 \times N^3}$ is lower triangular block Toeplitz

$$L_3 \;\sim\; \begin{bmatrix} K & 0 & 0 \\ -I & K & 0 \\ 0 & -I & K \end{bmatrix}$$

where $K \in \mathbb{R}^{N^2 \times N^2}$ is also lower triangular block Toeplitz

$$K \sim \begin{bmatrix} K_3 & 0 & 0 \\ -I & K_3 & 0 \\ 0 & -I & K_3 \end{bmatrix}$$

where $K_3 \in \mathbb{R}^{N \times N}$ is lower triangular Toeplitz with 3 on the main diagonal, -1 on the first subdiagonal, and 0 everywhere else

$$K_3 \sim \begin{bmatrix} 3 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & -1 & 3 \end{bmatrix}.$$

Similar to the 1D and 2D cases, we obtain expression for the steady-state variance of followers.

Lemma 3. *Consider the leader-follower system on 3D lattice* (8)*. The steady-state variance of the follower at coordinates* (n, m, l) *of the 3D lattice can be expressed as*

$$((P_n)_m)_l = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} \frac{6^{6-(2i+2j+2k)}(2i+2j+2k-6)!}{6((i-1)!(j-1)!(k-1)!)^2}. \tag{9}$$

The proof is similar to the proof of Lemma 2; see Appendix C.

From (9), we see that $((P_n)_m)_l$ is monotonically increasing as n, m, and l increase. In other words, the variance of the follower grows as one moves away from the leader. We next show that the variance of the followers on the diagonal of the 3D lattice is bounded above by a constant independent of lattice size.

Proposition 2. *Consider the leader-follower system on 3D lattice* (8)*. Let* V_n *be the steady-state variance of the follower at the coordinates* (n, n, n) *of the 3D lattice for* $n = 1, \ldots, N$*. Then* V_n *is bounded above by a constant that is independent of network size, denoted as* $V_n \sim O(1)$*.*

The proof technique is analogous to the proof of Proposition 1; see details in Appendix D.

III. Discussions

A. Connections with random walks

The connections between random walks and distributed estimation and control problems have been studied by several authors; see [1]–[3], [10], [12], [13], [15], [16]. All existing work focuses

on *undirected* networks. We next show that the asymptotic scaling for the variance of followers in directed lattices can be expressed as

$$V_n \sim \frac{1}{2D}\sum_{k=0}^{n-1} u_{2k}$$

where $D = 1$, 2, or 3 is the dimension and u_{2k} is the probability of a random walk of length $2k$ returning to the starting point on the undirected lattices.

Recall that for 1D lattice, u_{2k} is given by [17, Section 7.2]

$$u_{2k} = \frac{1}{2^{2k}}\binom{2k}{k} = \frac{(2k)!}{2^{2k}\,k!\,k!}.$$

From expression (3), it follows that

$$P_n = \frac{1}{2}\sum_{k=0}^{n-1} u_{2k}. \tag{10}$$

In other words, the steady-state variance of the nth follower can be expressed as the sum of the probability of a random walk returning to the starting point of length $2k$ for $k = 0, 1, \ldots, n-1$. In 2D lattice, u_{2k} is given by [17, Section 7.3]

$$u_{2k} = \left(\frac{1}{2^{2k}}\binom{2k}{k}\right)^2 = \frac{1}{4^{2k}}\left(\frac{(2k)!}{k!k!}\right)^2.$$

From expression (7), it follows that $S_k = (1/4)u_{2k}$. Then the summation of the positive function f over the triangle $\mathcal{T}_n$ in 2D lattice (6) can be expressed as

$$\Delta_n = \frac{1}{4}\sum_{k=0}^{n-1} u_{2k}. \tag{11}$$

In 3D lattice, u_{2k} is given by [17, Section 7.3]

$$u_{2k} = \sum_{j=0}^{p}\sum_{k=0}^{p-j}\frac{1}{2^{2p}}\left(\frac{(2p)!}{p!p!}\right)\left(\frac{p!}{3^p j!k!(p-j-k)!}\right)^2.$$

From expression (18), it follows that $G_k = (1/6)u_{2k}$. Then the summation of the positive function over the triangular pyramid is given by (see Appendix C)

$$T_n = \frac{1}{6}\sum_{k=0}^{n-1} u_{2k}. \tag{12}$$

From (10), (11), and (12), we observe that the asymptotic scaling for the variance in directed

lattices can be expressed as

$$V_n \sim \frac{1}{2D}\sum_{k=0}^{n-1} u_{2k}, \quad D = 1, 2, 3.$$

B. Local errors

For distributed consensus on undirected networks, the steady-state variance of the local error between two neighboring agents is upper bounded by constants that are independent of both the size and the dimension of the lattice [10]. Similar results can be obtained for leader-follower systems in directed lattices. We focus on the 1D case and omit the calculations for 2D and 3D cases because they are more involved.

Consider the local error between two neighboring agents $y_n(t) := x_n(t) - x_{n-1}(t)$ for $n = 1, \ldots, N$. The steady-state variance is given by

$$\lim_{t\to\infty} E\{y_n^2(t)\} = P_n + P_{n-1} - 2P_{n(n-1)}.$$

A similar calculation as in the proof of Lemma 1 shows that

$$P_{n(n-1)} = \frac{n(2n)!}{2^{2n}(n!)(n!)} - \frac{1}{2} = P_n - \frac{1}{2}$$

for $n \geq 2$. It follows that

$$\lim_{n\to\infty}\lim_{t\to\infty} E\{y_n^2(t)\} = \lim_{n\to\infty}\{P_{n-1} - P_n + 1\} = 1.$$

In other words, the steady-state variance of the local error is upper bounded by a constant in 1D lattice.

C. Double-integrator model in 1D lattice

We consider the double-integrator model for followers using relative position and absolute velocity feedback. In 1D formations, we show that the steady-state variance of followers has the same scaling trend as the single-integrator model for followers using relative position feedback.

The state-space representation of the leader-follower system in 1D with double-integrator model is given by

$$\dot{x}(t) = Ax(t) + Bd(t)$$

where $A = \begin{bmatrix} 0 & I \\ -I & -L \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ I \end{bmatrix}$. The steady-state covariance matrix is determined by the solution of the Lyapunov equation

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0.$$

Expanding the 2-by-2 block matrix $\mathcal{P} = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}$, we have three matrix equations

$$\begin{aligned} P_2 + P_2^T &= 0 \\ P_3 - P_1 - P_2 L^T &= 0 \\ -LP_3 - P_3 L^T - P_2 - P_2^T + I &= 0. \end{aligned}$$

It is readily verified that the unique, positive-definite solution satisfies $\{P_1 = P_3, P_2 = 0\}$ where P_3 is the solution of the Lyapunov equation $LP_3 + P_3 L^T = I$. Note that P_3 is equal to the covariance matrix P in (13). Therefore, the steady-state variance of the position and the velocity of followers with the double-integrator model is the same as the variance of the position of followers with the single-integrator model.

For double-integrator model using *relative velocity* feedback, numerical computations suggest that the variance of followers grows as an *exponential* function of the lattice size. For asymmetric bidirectional control of vehicular platoons, the exponential growth of $\mathcal{H}_\infty$-norm is shown in [18]. The connection between the scaling trend of variance in the double-integrator model and the exponential growth of $\mathcal{H}_\infty$-norm of vehicular platoons is a topic of future work.

IV. Conclusions

We have obtained explicit formulas for the steady-state variance distribution of leader-follower multiagent systems on directed lattices in 1, 2, and 3 dimensions. We show that the variance of followers scales as a square-root function of the distance from the leader in 1D lattice, it scales as a logarithmic function along the diagonal of the 2D lattice, and it is bounded by a network-size independent constant in 3D lattice.

APPENDIX

A. Proof of Lemma 1

We begin with the steady-state covariance matrix

$$P := \lim_{t\to\infty} E\{x(t)x^T(t)\} = \int_0^\infty \mathrm{e}^{-Lt}\mathrm{e}^{-L^Tt}\mathrm{d}t. \tag{13}$$

We compute the matrix exponential by using the inverse Laplace transform

$$\mathrm{e}^{-Lt} = \mathcal{L}^{-1}\{(sI + L)^{-1}\}.$$

Since L is a lower triangular Toeplitz matrix (see (2)), it follows that $(sI + L)^{-1}$ is also lower triangular Toeplitz

$$(sI + L)^{-1} \sim \begin{bmatrix} (s+1)^{-1} & 0 & 0 \\ (s+1)^{-2} & (s+1)^{-1} & 0 \\ (s+1)^{-3} & (s+1)^{-2} & (s+1)^{-1} \end{bmatrix}.$$

In particular, $(s+1)^{-i}$ is the ith entry of the first column. By using the formula for inverse Laplace transform

$$\mathcal{L}^{-1}\{(s+1)^{-i}\} = \frac{t^{i-1}}{(i-1)!}\mathrm{e}^{-t}, \quad i = 1, \ldots, n,$$

we obtain the nth diagonal element of the matrix $\mathrm{e}^{-Lt}\mathrm{e}^{-L^Tt}$

$$\left(\mathrm{e}^{-Lt}\mathrm{e}^{-L^Tt}\right)_n = \sum_{i=1}^n \left(\frac{t^{i-1}}{(i-1)!}\mathrm{e}^{-t}\right)^2.$$

Performing the integration from 0 to ∞ yields

$$\begin{aligned} P_n &= \sum_{i=1}^n \frac{1}{((i-1)!)^2}\int_0^\infty \frac{\tau^{2(i-1)}\mathrm{e}^{-\tau}}{2^{2i-1}}\mathrm{d}\tau \\ &= \sum_{i=1}^n \frac{1}{((i-1)!)^2}\cdot\frac{\Gamma(2i-1)}{2^{2i-1}}, \end{aligned}$$

where we have used the change of variable $\tau = 2t$ and the formula for the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-\tau}\mathrm{d}\tau. \tag{15}$$

Since $\Gamma(z) = (z-1)!$, we have the desired formula (3)

To show the asymptotic scaling of P_n, we use Stirling's formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n. \tag{16}$$

With some algebra, we get

$$P_n \approx \sqrt{\frac{n}{\pi}}. \tag{17}$$

Summing P_n with respect to n yields the expression for the average variance $\Pi_N = \frac{(2N+1)!}{3\cdot 2^{2N} N! N!}$. By applying Stirling's formula, we obtain $\Pi_N \approx \frac{2}{3}\sqrt{N/\pi}$.

B. Proof of Lemma 2

Since L_2 is lower triangular block Toeplitz, it follows that $(sI + L_2)^{-1}$ is also lower triangular block Toeplitz. In particular, $(sI + K_2)^{-i}$ is the ith block entry of the first column. By using the inverse Laplace transform

$$\mathcal{L}^{-1}\{(sI + K_2)^{-i}\} = \frac{t^{i-1}\,\mathrm{e}^{-K_2 t}}{(i-1)!}, \quad i = 1, \ldots, N,$$

we obtain the nth diagonal block of $\mathrm{e}^{-Lt}\mathrm{e}^{-L^T t}$, that is,

$$\left(\mathrm{e}^{-Lt}\,\mathrm{e}^{-L^T t}\right)_n = \sum_{i=1}^{n} \frac{t^{2i-2}}{((i-1)!)^2}\,\mathrm{e}^{-K_2 t}\,\mathrm{e}^{-K_2^T t}.$$

An analogous calculation shows that the mth diagonal element of $\mathrm{e}^{-K_2 t}\,\mathrm{e}^{-K_2^T t}$ is given by

$$\left(\mathrm{e}^{-K_2 t}\,\mathrm{e}^{-K_2^T t}\right)_m = \sum_{j=1}^{m} \frac{t^{2j-2}}{((j-1)!)^2}\,\mathrm{e}^{-2t}\,\mathrm{e}^{-2t}.$$

Putting it together, we have

$$(P_n)_m = \int_0^\infty \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{t^{2i-2}}{((i-1)!)^2}\frac{t^{2j-2}}{((j-1)!)^2}\,\mathrm{e}^{-4t}\,\mathrm{d}t.$$

Performing the integration yields the desired formula (5).

C. Proof of Lemma 3

The steady-state covariance matrix is given by

$$P = \int_0^\infty \mathrm{e}^{-L_3 t}\,\mathrm{e}^{-L_3^T t}\,\mathrm{d}t.$$

The matrix exponential $\mathrm{e}^{-L_3 t} = \mathcal{L}^{-1}\{sI + L_3\} \in \mathbb{R}^{N^3 \times N^3}$ is lower triangular block Toeplitz with the ith block of the first column being

$$\mathcal{L}^{-1}\{(sI + K)^{-i}\} = \frac{t^{i-1}}{(i-1)!}\mathrm{e}^{-Kt} \in \mathbb{R}^{N^2 \times N^2}.$$

Since the jth block of the first column of e^{-Kt} is

$$\mathcal{L}^{-1}\{(sI + K_3)^{-j}\} = \frac{t^{j-1}}{(j-1)!}\mathrm{e}^{-K_3 t} \in \mathbb{R}^{N \times N},$$

and since the kth element of the first column of $\mathrm{e}^{-K_3 t}$ is

$$\mathcal{L}^{-1}\{(s+3)^{-k}\} = \frac{t^{k-1}}{(k-1)!}\mathrm{e}^{-3t},$$

putting the above calculations together, we have

$$((P_n)_m)_l = \int_0^\infty \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} \frac{t^{2i+2j+2k-6} \cdot \mathrm{e}^{-6t}}{((i-1)!(j-1)!(k-1)!)^2}\mathrm{d}t.$$

Performing the integration yields the desired formula (9).

D. Proof of Proposition 2

By setting $n = m = l$ in (9) yields the variance of the follower (n, n, n)

$$V_n = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} f(i,j,k)$$

where

$$g(i,j,k) = \frac{(2i+2j+2k)!}{6 \cdot 6^{2(i+j+k)}(i!j!k!)^2}.$$

In other words, V_n is the summation of the positive function g over the cube $\mathcal{C}_n := \{0 \le i, j, k \le n-1\}$. Let T_n be the summation of g over the triangular pyramid $\mathcal{P}_n := \{0 \le p \le n-1, 0 \le j \le p, 0 \le k \le p-j\}$, whose vertices are given by $\{(0,0,0), (n-1,0,0), (0,n-1,0), (0,0,n-1)\}$. It follows that $T_n < V_n < T_{2n}$. This is because $\mathcal{P}_n$ is a subset of $\mathcal{C}_n$ which itself is a subset of $\mathcal{P}_{2n}$. Thus, it suffices to show that $T_n \sim O(1)$.

We compute the sum of g across the triangle segment of the pyramid $T_n = \sum_{p=0}^{n-1} G_p$, where

G_p is the sum of g over the triangle $i+j+k=p$,

$$\begin{aligned} G_p &= \sum_{j=0}^{p}\sum_{k=0}^{p-j} f(p-j-k,j,k) \\ &= \sum_{j=0}^{p}\sum_{k=0}^{p-j} \frac{1}{6\cdot 2^{2p}}\left(\frac{(2p)!}{p!p!}\right)\left(\frac{p!}{3^p j!k!(p-j-k)!}\right)^2. \end{aligned} \tag{18}$$

To evaluate the summation, we employ a probability argument [17]. Consider dropping p balls into three boxes A, B, and C. The probability of dropping j balls into A, k balls into B, and $p-j-k$ balls into C is $\frac{p!}{3^p j!k!(p-j-k)!}$. Since the largest probability occurs when the same number of balls drop in three boxes, it follows that

$$G_p \le \frac{1}{6\cdot 2^{2p}}\cdot\frac{(2p)!}{p!p!}\cdot\frac{p!}{3^p(\lfloor\frac{p}{3}\rfloor!)^3}\sum_{j=0}^{p}\sum_{k=0}^{p-j}\left(\frac{p!}{3^p j!k!(p-j-k)!}\right)^2$$

where $\lfloor\frac{p}{3}\rfloor$ denotes the largest integer that is no greater than $p/3$. Note that

$$\sum_{j=0}^{p}\sum_{k=0}^{p-j}\left(\frac{p!}{3^p j!k!(p-j-k)!}\right)^2 = 1,$$

because it is the sum of probability of all outcomes of dropping three balls in three boxes. Therefore, for large n,

$$G_p \le \frac{1}{6\cdot 2^{2p}}\cdot\frac{(2p)!}{p!p!}\cdot\frac{p!}{3^p(\lfloor\frac{p}{3}\rfloor!)^3} \approx c\,p^{-3/2},$$

where c is a constant and we have used Stirling's formula (16). It follows that

$$T_n = \sum_{p=1}^{n} G_p \approx \sum_{p=1}^{n} c\,p^{-3/2} \sim O(1).$$

This completes the proof.

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