Stability of Multiobjective Predictive Control: An Utopia-Tracking Approach*

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Abstract

We propose a multiobjective strategy for model predictive control (MPC) that we term *utopia-tracking* MPC. The controller minimizes, in some norm, the distance of its cost vector to that of the unreachable steady-state utopia point. Stability is ensured by using a terminal constraint to a selected point along the steady-state Pareto front. One of the key advantages of this approach is that multiple objectives can be handled systematically without having to compute the entire Pareto front or selecting weights. In addition, general cost functions (i.e., economic, regularization) can be used.

Keywords: stability, predictive control, multiobjective, Lyapunov, utopia, Pareto, economic.

1 Introduction

Conflicting objectives arise naturally in model predictive control (MPC) applications. Traditional trade-offs include tracking performance and robustness (e.g., H_2/H_∞ control) or economic performance and sustainability. Specific domains where reconciling objectives is critical include chemical plants [5, 17] and energy systems [20, 21].

A key technical challenge in dealing with multiple objectives is that the Pareto front is computationally expensive to build in real-time environments. In addition, even when such a front is built, expert knowledge is still needed to obtain a preferred solution. Traditional approaches such as weighting and expert systems are limited since the system conditions and priorities change under different operating modes or economic environments. It is thus desired to allow the MPC controller to handle trade-offs automatically as conditions change. Stability is another technical issue arising in MPC with multiple objectives. In particular, with the advent of MPC formulations able to handle general cost functions [4, 16, 8], it is natural to wonder whether multiobjective extensions are possible.

Stability of multiobjective MPC formulations has been studied by numerous researchers. In [1], the MPC control action is chosen among the set of Pareto optimal solutions based on a time-varying, state-dependent decision criterion. In [18], the control action minimizes the maximum of a finite

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number of objectives. In [12], the MPC controller switches objectives depending on the value of the state vector under stabilizing constraints. This type of expert knowledge is also used in [10], where a lexicographic formulation and logic are used to prioritize the objectives. In these works, the multiple cost functions are assumed to be positive definite as in traditional MPC formulations.

In this work, we propose a new strategy, called *utopia-tracking* MPC, to handle multiple objective functions. We establish conditions for nominal asymptotic stability and propose numerical implementation schemes.

The key idea is to minimize the distance of the cost function to that of the steady-state utopia point (unreachable point given by the intersection of the minima of the independent objectives). A key property of the controller is that it can exploit the system dynamics to leave the steady-state Pareto front and get closer to the utopia point compared with any solution along the steady-state Pareto front. Stability is ensured by using a terminal constraint to a reachable point along the Pareto front.

Our proposed approach is novel because it can handle general cost functions (e.g., economic, regularization, tracking) that are required to satisfy only a Lipschitz continuity property. In addition, the strategy does not require the construction of the Pareto front, nor does it require the selection of weighting factors.

The paper is structured as follows. We start with basic definitions in Section 2. Definitions of steady-state multiobjective optimization are presented in Section 3. In Section 4 we analyze the stability of the utopia-tracking controller. In Section 5 we discuss computational issues. We present a numerical study in Section 6 and close in Section 7 with conclusions and directions for future work.

2 Preliminaries

We consider a discrete-time dynamic system of the form

$$x_{k+1} = f(x_k, u_k), \tag{2.1}$$

where $x_k \in \Re^{n_x}$ are the states and $u_k \in \Re^{n_u}$ are the controls. The mapping $f : \Re^{n_x \times n_u} \to \Re^{n_x}$ is assumed to be Lipschitz in both arguments with constant $L_f \geq 0$ and is assumed to satisfy $f(x^s, u^s) = x^s$ at an equilibrium point (x^s, u^s). We will define the vector $\mathbf{u}^T := [u_0^T, ..., u_{N-1}^T]^T \in \Re^{N \times n_u}$.

The state and controls are required to satisfy the constraints $\forall\, k$:

$$x_k \in \mathcal{X}, \qquad u_k \in \mathcal{U}. \tag{2.2}$$

The sets $\mathcal{X} \subseteq \Re^{n_x}$ and $\mathcal{U} \subseteq \Re^{n_u}$ are assumed to be compact and to contain the equilibrium point.

Definition 1 (**Admissible Set**.) *Given $N+1$ time steps $k = 0, ..., N$, the admissible set is given by*

$$\mathcal{W}_N := \{(x, \mathbf{u}) \,|\, x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ x_N = x^s\}.$$

The set of admissible states $\mathcal{Z}_N$ is then given by

$$\mathcal{Z}_N =: \{x \,|\, \exists\, \mathbf{u}_N \text{ such that } (x, \mathbf{u}_N) \in \mathcal{W}_N\}. \tag{2.3}$$

Definition 2 (**$\mathcal{K}$-Function [9].**) *A continuous function $\alpha : \Re \to \Re$ is called a $\mathcal{K}$ function if $\alpha(s) = 0$ for $s = 0$ and $\alpha(s) > 0$ for $s > 0$ and it is strictly increasing.*

Definition 3 *(**p-Norm**.) The p-norm $\|\cdot\|_p$ with $p \geq 1$ is a $\mathcal{K}$-function of the form*

$$\|s\|_p = \left(\sum_{i=1}^{n_s} |s_i|^p \right)^{\frac{1}{p}},$$

for a vector $s \in \Re^{n_s}$ with elements $s_i,\ i = 1, ..., n_s$.

We have that with $\|s\|_p = 0$ if $s = 0$ and $\|s\|_p > 0$ otherwise for all $p \geq 1$. In addition, we know that the p-norm is Lipschitz continuous with constant equal to 1. Well-known norms are the $\mathcal{L}_1, \mathcal{L}_2$ and the $\mathcal{L}_\infty$ norms given by

$$\|s\|_1 = \sum_{i=1}^{n_s} |s_i|, \quad \|s\|_2 = \sqrt{\sum_{i=1}^{n_s} (s_i)^2}, \quad \|s\|_\infty = \max\{|s_1|, |s_2|, ..., |s_{n_s}|\}.$$

Using these definitions, we can establish the following definition of a Lyapunov function.

Definition 4 **(Lyapunov Function.)** *A continuous function $V(\cdot) : \Re^{n_x} \to \Re$ is called a Lyapunov function if there exist an invariant set $\mathcal{X}$ and $\mathcal{K}$ functions $\alpha_L(\cdot), \alpha_U(\cdot)$, and $\Delta\alpha(\cdot)$ such that, $\forall x \in \mathcal{X}$,*

$$\alpha_L(\|x\|_p) \leq V(x) \leq \alpha_U(\|x\|_p) \tag{2.4a}$$

$$\Delta V(x) \leq -\Delta\alpha(\|x\|_p). \tag{2.4b}$$

We note that the general p-norm can be used to bound the Lyapunov function since this is a $\mathcal{K}$-function and the composition of $\mathcal{K}$-functions is a function of the same form.

3 Steady-State Multiobjective Optimization

Consider the following multiobjective steady-state problem:

$$\min_{x,u} \ [\Phi_1(x, u), \Phi_2(x, u), ..., \Phi_M(x, u)] \tag{3.5a}$$

$$\text{s.t. } x = f(x, u) \tag{3.5b}$$

$$x \in \mathcal{X}, u \in \mathcal{U}, \tag{3.5c}$$

where the cost functions $\Phi_i : \Re^{n_x \times n_u} \to \Re,\ i \in \mathcal{M} := \{1, ..., M\}$ are assumed to be Lipschitz continuous in both arguments. We define the cost vector as,

$$\Phi(\cdot, \cdot)^T := [\Phi_1(\cdot, \cdot), \Phi_2(\cdot, \cdot), ..., \Phi_M(\cdot, \cdot)]^T, \tag{3.6}$$

with Lipschitz constant L_Φ. No further assumptions are needed about the properties of these functions, as in [18, 1].

The cost functions are assumed to be conflicting so that one cannot be minimized without increasing the other [1]. This situation gives rise to the concept of a Pareto solution.

Definition 5 *(**Steady-State Pareto Solution [3].**) A feasible point (x_p, u_p) for the multiobjective problem* (3.5) *is said to be Pareto optimal if and only if there exists no other feasible point (x, u) such that $\Phi_i(x, u) \leq \Phi_i(x_p, u_p)$, $\forall i \in \mathcal{M}$ and $\Phi_i(x, u) < \Phi_i(x_p, u_p)$ for at least one index $i \in \mathcal{M}$.*

[1]If any pair of functions is not conflicting, then the sum of objectives can be defined as a single objective.

The family of Pareto solutions forms the so-called Pareto front, which represents a limiting curve of performance in the cost space. In this work, we will not follow the traditional approach of constructing the Pareto front and then chosing a suitable point along it [10]. The first reason is that this seems impractical in real-time environments. The second reason is that expert knowledge is needed to select the point and the selection criterion might need change as the conditions of the system change (e.g., prices). While automatic criterion selection procedures can be used with lexicographic programming, these also are computationally expensive [11]. In this work, we try to overcome some of these limitations by following an *utopia*-tracking approach [7, 5].

Definition 6 *(***Steady-State Utopia Point***.) The steady-state utopia point is a point given by the solution $(x_i^{L,s}, u_i^{L,s})$ with coordinates $\Phi_i(x_i^{L,s}, u_i^{L,s})$ in the cost space. The coordinates are given by the solution of,*

$$\min_{x,u} \ \Phi_i(x,u) \tag{3.7a}$$

$$s.t.\ x = f(x,u) \tag{3.7b}$$

$$x \in \mathcal{X}, u \in \mathcal{U}. \tag{3.7c}$$

for $i \in \mathcal{M}$,

The utopia cost vector will be denoted as $\Phi^{L,s}$. The utopia point is unattainable since the costs are conflicting; however, it can still be used as a reference point. For instance, it is possible to compute the closest point along the Pareto front to the utopia point (also known as the *compromise solution*.)

Definition 7 *(***Steady-State Compromise Solution***.) The steady-state compromise solution is a point (x^s, u^s) with cost $\Phi(x^s, u^s)$ given by the solution of the minimum distance problem,*

$$\min_{x,u} \ \|\Phi(x,u) - \Phi^{L,s}\|_p \tag{3.8a}$$

$$s.t.\ x = f(x,u) \tag{3.8b}$$

$$x \in \mathcal{X}, u \in \mathcal{U}, \tag{3.8c}$$

in some norm $\|\cdot\|_p$, where

$$\|\Phi(x,u) - \Phi^{L,s}\|_p = \left(\sum_{i \in \mathcal{M}} |\Phi_i(x,u) - \Phi_i^{L,s}|^p \right)^{\frac{1}{p}}. \tag{3.9}$$

The individual costs of the compromise solution are given by $\Phi_i(x^s, u^s),\ i \in \mathcal{M}$. We will denote the above problem as the *steady-state utopia-tracking problem*. A schematic representation of the utopia-tracking approach is presented in Figure 1. We highlight that, for the single objective case, the compromise solution and the utopia point coincide so that $\Phi_1(x^s, u^s) = \Phi_1^{L,s}$.

Remark: We emphasize that the choice of the compromise solution as equilibrium point is not strictly necessary. Other possibilities include the Kalai-Smorodinsky solution, the egalitarian solution, and the Nash solution [6].

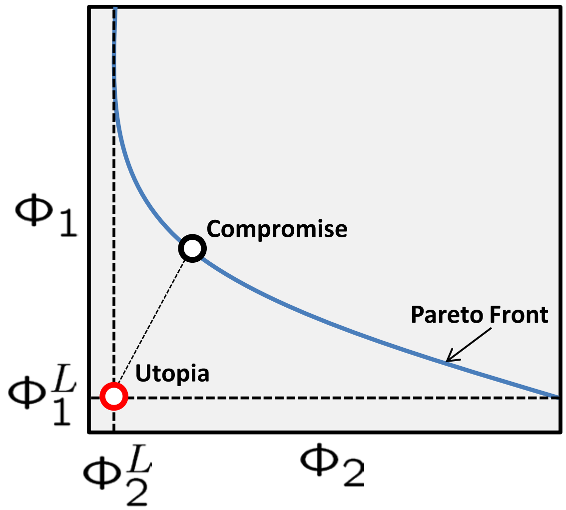

Figure 1: Schematic representation of Pareto front, compromise solution, and utopia point.

4 Multiobjective Predictive Control

We start by making the following assumption about controllability [4, 8].

Definition 8 (**Weak Controllability**.) *There exists a $\mathcal{K}$-function $\gamma(\cdot)$ such that, for every $x \in \mathcal{X}$, there exists $(x, \mathbf{u}_N) \in \mathcal{Z}_N$ such that*

$$\sum_{k=0}^{N-1} \|u_k - u^s\|_p \leq \gamma(\|x - x^s\|_p).$$

This assumption is essential in establishing boundedness of general cost functions for MPC controllers. Under this assumption, the following result can be established.

Lemma 1 *Consider the general MPC problem,*

$$\min_{x_k, u_k} \sum_{k=0}^{T-1} \|\varphi(x_k, u_k, x^s, u^s)\|_p \tag{4.10a}$$

$$s.t.\ x_{k+1} = f(x_k, u_k),\ k = 0, ..., T-1 \tag{4.10b}$$

$$x_0 = x_\ell \tag{4.10c}$$

$$x_T = x^s \tag{4.10d}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U},\ k = 0, ..., T, \tag{4.10e}$$

with Lipschitz continuous cost $\varphi : \Re^{n_x \times n_u} \to \Re$. Assume there exists $L \geq 0$ such that

$$\|\varphi(x, u, x^s, u^s)\|_p \leq L(\|x - x^s\|_p + \|u - u^s\|_p).$$

If weak controllability and Lipschitz continuity of the system $f(\cdot, \cdot)$ holds, then there exists a $\mathcal{K}$-function $\alpha_U(\cdot)$ such that,

$$\sum_{k=0}^{T-1} \|\varphi(x_k, u_k, x^s, u^s)\|_p \leq \alpha_U(\|x - x^s\|_p), \tag{4.11}$$

for all $(x, \mathbf{u}) \in \mathcal{Z}_N$.

Proof: The proof is a straightforward extension of the upper bounding strategy used in [4, 8]. Applying the Lipschitz property assumed, the system can be propagated forward in time and substituted in the objective. Under Lipschitz continuity of the system the result follows. □

In the following subsections, we propose three strategies to deal with multiple objectives. In the first strategy (state-tracking MPC), the controller tracks directly the state of the compromise solution. In the second strategy (cost-tracking MPC), the controllers track the compromise solution in the cost space. The third strategy is the utopia-tracking MPC controller which tracks the steady-state utopia point in the cost space using the compromise solution as terminal condition. We will see that this last strategy can exploit the dynamic transition to leave the Pareto front and get closer to the steady-state utopia point thus maximizing performance.

4.1 State-Tracking MPC

We consider the state-tracking (ST) problem,

$$\min_{x_k,u_k} V_{ST}(x_\ell, \mathbf{u}) := \sum_{k=0}^{T-1} \|x_k - x^s\|_p + \|u_k - u^s\|_p \tag{4.12a}$$

$$\text{s.t. } x_{k+1} = f(x_k, u_k),\ k = 0, ..., T-1 \tag{4.12b}$$

$$x_0 = x_\ell \tag{4.12c}$$

$$x_T = x^s \tag{4.12d}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U},\ k = 0, ..., T. \tag{4.12e}$$

Here, ℓ is the current time instant, and T is the horizon length. The solution of this problem is given by the vector $\mathbf{u}_\ell$ from where the control u_ℓ is injected. The associated control law is given by $u = h_{ST}(x)$. The objective of the state-tracking controller is to steer the system states from the initial state x to the compromise steady-state solution x^s in minimum time.

Note that the above formulation is a generalization of traditional tracking controllers where the $\mathcal{L}_2$ norm is used [15]. This norm is often chosen because of computational reasons (see Section 5.) Any p-norm, however, can be used in principle to ensure stability as we see in the following result.

Theorem 1 *(***Stability of Tracking MPC***.) The minimum-distance steady-state point x^s under the control law $h_{ST}(x)$ given by the tracking MPC formulation* (4.12) *is an asymptotically stable equilibrium with region of attraction $\mathcal{Z}_N$.*

Proof: Under weak controllability, the cost function satisfies conditions of Lemma 1. Consequently, it is bounded from above by a $\mathcal{K}$-function. The existence of a $\mathcal{K}$-function as lower bound is trivial for this choice of cost function. To show that the cost function is nonincreasing we establish the following:

$$\begin{aligned}
&V_{ST}(x_{\ell+1}, \mathbf{u}_{\ell+1}) - V_{ST}(x_\ell, \mathbf{u}_\ell)\\
&= \sum_{k=\ell+1}^{\ell+T} (\|x_k - x^s\|_p + \|u_k - u^s\|_p) - \sum_{k=\ell}^{\ell+T-1} (\|x_k - x^s\|_p + \|u_k - u^s\|_p)\\
&\leq -(\|x_\ell - x^s\|_p + \|h_{ST}(x_\ell) - u^s\|_p)\\
&\leq -\alpha(\|x_\ell - x^s\|_p).
\end{aligned}$$

The first inequality follows from the terminal constraint. The last inequality follows since $\|x_\ell - x^s\|_p + \|h_{ST}(x_\ell) - u^s\|_p \geq \|x_\ell - x^s\|_p$. The proof is complete. □

We note that the objective of state-tracking MPC is to reach the steady-state point in minimum time and not in an *economically* optimal manner. Here, economic performance is interpreted as the distance to the utopia point since this is the limiting point. In addition, the controller performance is affected by the trade-off between the state and control regularization terms, and it often requires weighting factors. The proposed multiobjective formulations of the following subsections can be used to avoid these limitations.

4.2 Cost-Tracking MPC

To deal with the limitations of tracking MPC in dealing with multiple objectives, we first propose the following *cost-tracking* (CT) MPC controller:

$$\min_{x_k, u_k} V_{CT}(x_\ell, \mathbf{u}) := \sum_{k=0}^{T-1} \|\Phi(x_k, u_k) - \Phi(x^s, u^s)\|_p \tag{4.13a}$$

$$\text{s.t. } x_{k+1} = f(x_k, u_k),\ k = 0, ..., T-1 \tag{4.13b}$$

$$x_0 = x_\ell \tag{4.13c}$$

$$x_T = x^s \tag{4.13d}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U},\ k = 0, ..., T. \tag{4.13e}$$

The control law of this controller is given by $u_\ell = h_{CT}(x_\ell)$. The objective of the controller is to minimize the cost distance to the compromise steady-state solution. We will see that the cost function is a natural Lyapunov function.

Assumption 1 *There exists a $\mathcal{K}$-function $\alpha_L(\cdot)$ such that*

$$\|\Phi(x, u) - \Phi(x^s, u^s)\|_p \geq \alpha_L(\|x - x^s\|_p). \tag{4.14}$$

Theorem 2 *Under weak controllability and Assumption 1, the steady-state x^s under the control law $h_{CT}(x)$ given by the multiobjective MPC formulation* (4.13) *is an asymptotically stable equilibrium point with region of attraction $\mathcal{Z}_N$.*

Proof: From Assumption 1, the cost function is bounded from below by a $\mathcal{K}$-function. Under weak controllability, Lemma 1 holds immediately with $L = L_\Phi$. Consequently, the cost is bounded from above by a $\mathcal{K}$-function. To show that the cost is nonincreasing, we establish the following:

$$\begin{aligned} V_{CT}(x_{\ell+1}, \mathbf{u}_{\ell+1}) - V_{CT}(x_\ell, \mathbf{u}_\ell) &= \sum_{k=\ell+1}^{\ell+T} \|\Phi(x_k, u_k) - \Phi(x^s, u^s))\|_p - \sum_{k=\ell}^{\ell+T-1} \|\Phi(x_k, u_k) - \Phi(x^s, u^s))\|_p \\ &\leq -\|\Phi(x_\ell, u_\ell) - \Phi(x^s, u^s)\|_p \\ &\leq -\alpha_L(\|x_\ell - x^s\|_p). \end{aligned}$$

The last inequality also follows from Assumption 1. The proof is complete. □

A key property of the cost-tracking approach is that the nature of the cost functions does not affect the upper bound assumption of the minimum distance cost. Assumption 1 is the most restrictive assumption that we have found and it is often difficult to verify in practice. In [8], the authors propose to use a regularization term for the case in which the condition does not hold. In particular, the lower bound condition can be guaranteed to hold under the satisfaction of the so-called strong second order condition [22]. This condition requires that the optimization problem is locally stable so that the optimal solution is well-defined. In other words, the cost is zero only at $x = x^s$ and strictly positive otherwise.

4.3 Utopia-Tracking MPC

We now consider an alternative *utopia-tracking* (UT) formulation that minimizes directly the distance to the utopia point:

$$\min_{x_k, u_k} V_{UT}(x_\ell, \mathbf{u}) := \sum_{k=0}^{T-1} \left\|\Phi(x_k, u_k) - \Phi^{L,s}\right\|_p \tag{4.15a}$$

$$\text{s.t. } x_{k+1} = f(x_k, u_k),\ k = 0, ..., T-1 \tag{4.15b}$$

$$x_0 = x_\ell \tag{4.15c}$$

$$x_T = x^s \tag{4.15d}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U},\ k = 0, ..., T. \tag{4.15e}$$

The control law is given by $u_\ell = h_{UT}(x_\ell)$. Since this controller minimizes the distance to the utopia directly, it can exploit the system dynamics to leave the steady-state Pareto front and get closer to the utopia point.

The main technical difficulty in establishing stability of the UT controller is that the cost function $V_{UT}(x, u)$ is nonzero at $x = x^s, u = u^s$ since the utopia point $\Phi^{L,s}$ is *unreachable*. Consequently, the cost function does not qualify as a Lyapunov function. To establish stability for this formulation, we follow the approach proposed in [4]. An alternative strategy for linear systems was presented in [16].

We define the partial Lagrange function of the steady-state utopia-tracking problem (3.8):

$$\mathcal{L}(x, u, \lambda) \quad := \quad \|\Phi(x, u) - \Phi^{L,s}\|_p + (x - f(x, u))\lambda,$$

where $\lambda \in \Re^{n_x}$ is a Lagrange multiplier. At x^s, u^s, λ^s we have that the partial Lagrange function reaches a minimum given by $\mathcal{L}(x^s, u^s, \lambda^s) = \|\Phi(x^s, u^s) - \Phi^{L,s}\|$ since $0 = x^s - f(x^s, u^s)$. With this, an artificial origin has been introduced if $(x, u) = (x^s, u^s)$. We need the following assumptions.

Assumption 2 (**Strong Duality**.) *There exists a multiplier λ^s such that the pair u^s, x^s uniquely solves,*

$$\min_{x,u}\ \mathcal{L}(x, u, \lambda^s),\ \text{s.t. } (x, u) \in \mathcal{X} \times \mathcal{U}.$$

From strong duality we have that $\mathcal{L}(x, u, \lambda^s) - \mathcal{L}(x^s, u^s, \lambda^s) \geq 0,\ \forall (x, u) \in \mathcal{X} \times \mathcal{U}$. We require the following assumption:

Assumption 3 *There exists a $\mathcal{K}$-function $\alpha_L(\cdot)$ such that*

$$\mathcal{L}(x, u, \lambda^s) - \mathcal{L}(x^s, u^s, \lambda^s) \geq \alpha_L(\|x - x^s\|_p). \tag{4.16}$$

We can now define the utopia-tracking MPC problem (4.15) in terms of the partial Lagrange function:

$$\min_{u_k} V_{UT}(x, \mathbf{u}) := \sum_{k=0}^{T-1} \left(\mathcal{L}(x_k, u_k, \lambda^s) - \mathcal{L}(x^s, u^s, \lambda^s)\right) \tag{4.17a}$$

$$\text{s.t. } x_{k+1} = f(x_k, u_k),\ k = 0, ..., T-1 \tag{4.17b}$$

$$x_0 = x \tag{4.17c}$$

$$x_T = x^s \tag{4.17d}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U},\ k = 0, ..., T. \tag{4.17e}$$

The control law is given by $u_\ell = h_{UT}(x_\ell)$. As shown in [4] (see Lemma 2), formulations (4.17) and (4.15) are equivalent.

Theorem 3 *Under weak controllability, strong duality, and Assumption 3, the steady-state x^s under the control law $h_{UT}(x)$ given by utopia-tracking MPC formulation* (4.17) *is an asymptotically stable equilibrium point with region of attraction $\mathcal{Z}_N$.*

Proof: From Assumption 3, the cost is bounded from below by a $\mathcal{K}$-function. To prove that it is bounded above, we first establish the following:

$$\left\| \sum_{k=0}^{T-1} \left(\mathcal{L}(x_k, u_k, \lambda^s) - \mathcal{L}(x^s, u^s, \lambda^s)\right) \right\| \leq \sum_{k=0}^{T-1} \|\mathcal{L}(x, u, \lambda^s) - \mathcal{L}(x^s, u^s, \lambda^s)\|_p.$$

We also have that

$$\begin{aligned}
&\|\mathcal{L}(x, u, \lambda^s) - \mathcal{L}(x^s, u^s, \lambda^s)\|_p \\
&\leq \left\| \|\Phi(x, u) - \Phi^{L,s}\|_p + (x - f(x, u))\lambda - \left(\|\Phi(x^s, u^s) - \Phi^{L,s}\|_p + (x^s - f(x^s, u^s))\lambda^s\right) \right\| \\
&\leq \left\| \|\Phi(x, u) - \Phi^{L,s}\|_p - \|\Phi(x^s, u^s) - \Phi^{L,s}\|_p \right\|_p + \|(x - f(x, u))\lambda^s - (x^s - f(x^s, u^s))\lambda^s\|_p \\
&\leq \|\Phi(x, u) - \Phi(x^s, u^s)\|_p + L_f \left(\|x - x^s\|_p + \|u - u^s\|_p\right) \|\lambda^s\|_p \\
&\leq \left(L_\Phi + L_f \|\lambda^s\|_p\right) \left(\|x - x^s\|_p + \|u - u^s\|_p\right).
\end{aligned}$$

Consequently, Lemma 1 holds with $L = L_\Phi + \|\lambda^s\|_p$, and the cost is bounded above. To show that it is nonincreasing we establish the following:

$$\begin{aligned}
&V_{UT}(x_{\ell+1}, \mathbf{u}_{\ell+1}) - V_{UT}(x_\ell, \mathbf{u}_\ell) \\
&\quad = \sum_{k=\ell+1}^{\ell+T} \left(\mathcal{L}(x_k, u_k, \lambda^s) - \mathcal{L}(x^s, u^s, \lambda^s)\right) - \sum_{k=\ell}^{\ell+T-1} \left(\mathcal{L}(x_k, u_k, \lambda^s) - \mathcal{L}(x^s, u^s, \lambda^s)\right) \\
&\quad \leq - \left(\mathcal{L}(x_\ell, u_\ell, \lambda^s) - \mathcal{L}(x^s, u^s, \lambda^s)\right) \\
&\quad \leq -\alpha_L(\|x - x^s\|_p).
\end{aligned}$$

The last inequality follows from Assumption 3. The proof is complete. □

As with the cost-tracking controller, the most restrictive assumption that we have found is Assumption 3. Under strong duality and second order conditions the condition holds.

Remark: In the case of a single objective, the compromise solution and the utopia solution coincide. Consequently, the cost-tracking and the utopia-tracking controllers are equivalent.

5 Computational Considerations

The choice of the norm in the controller cost has important implications for computational performance. In particular, the $\mathcal{L}_2$ norm is smooth, whereas $\mathcal{L}_1$ and $\mathcal{L}_\infty$ are not. Another issue is that the cost functions can have drastically different values. Here we propose reformulations to overcome these limitations. In addition, since the utopia-tracking controller works in the cost space, we propose to use terminal constraints on the costs directly.

5.1 Formulations

The solution of the individual problems (3.7) also yields upper bounds $\Phi_i^{U,s},\ i \in \mathcal{M}$, given by the maximum of the remaining costs not minimized. Consequently, we can use these together with the utopia costs $\Phi_i^{L,s},\ i \in \mathcal{M}$, to scale the controller cost without affecting its properties.

The scaled $\mathcal{L}_2$ problem has the following form:

$$\min_{x_k,u_k} \sum_{k=0}^{T-1} \left\| \frac{\Phi(x_k,u_k) - \Phi(x^s,u^s)}{\Phi^{U,s} - \Phi^{L,s}} \right\|_2 := \sum_{k=0}^{T-1} \sqrt{\sum_{i\in\mathcal{M}} \left(\frac{\Phi_i(x_k,u_k) - \Phi_i^{L,s}}{\Phi_i^{U,s} - \Phi_i^{L,s}} \right)^2} \tag{5.18a}$$

$$\text{s.t. } x_{k+1} = f(x_k,u_k),\ k = 0,...,T-1 \tag{5.18b}$$

$$x_0 = x_\ell \tag{5.18c}$$

$$x_T = x^s \tag{5.18d}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U},\ k = 0,...,T. \tag{5.18e}$$

The square root in the objective function can introduce numerical ill-conditioning since the first derivative diverges as the argument approaches zero. A typical alternative is to minimize the square of the norm. The solution can differ from that of the $\mathcal{L}_2$ norm, but the stability theory developed in the preceding section still applies in this case.

To deal rigorously with the $\mathcal{L}_2$ variant of (5.21), we consider the following formulation:

$$\min_{x_k,u_k} \sum_{k=0}^{T-1} z_k \tag{5.19a}$$

$$\text{s.t. } x_{k+1} = f(x_k,u_k),\ k = 0,...,T-1 \tag{5.19b}$$

$$x_0 = x_\ell \tag{5.19c}$$

$$x_T = x^s \tag{5.19d}$$

$$z_k^2 = \sum_{i\in\mathcal{M}} \left(\frac{\Phi_i(x_k,u_k) - \Phi_i^{L,s}}{\Phi_i^{U,s} - \Phi_i^{L,s}} \right)^2,\ k = 0,...,T-1 \tag{5.19e}$$

$$z_k \geq 0,\ k = 0,...,T-1 \tag{5.19f}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U},\ k = 0,...,T, \tag{5.19g}$$

which is better-conditioned. Here,

$$\sum_{k=0}^{T-1} \left\| \frac{\Phi(x_k,u_k) - \Phi(x^s,u^s)}{\Phi^{U,s} - \Phi^{L,s}} \right\|_2 = \sum_{k=0}^{T-1} z_k. \tag{5.20}$$

It is possible to reformulate the $\mathcal{L}_1$ variant as follows. Introducing variables $y^+_{k,i}, y^-_{k,i} \geq 0,\ i \in \mathcal{M}$, we can define the absolute value $y^+_{k,i} - y^-_{k,i} = \Phi_i(x_k, u_k) - \Phi_i^{L,s}$ [19]. After scaling we have

$$\min_{x_k,u_k} \sum_{k=0}^{T-1} \sum_{i\in\mathcal{M}} (y^+_{k,i} + y^-_{k,i}) \tag{5.21a}$$

$$\text{s.t. } x_{k+1} = f(x_k, u_k),\ k = 0, ..., T-1 \tag{5.21b}$$

$$x_0 = x_\ell \tag{5.21c}$$

$$x_T = x^s \tag{5.21d}$$

$$y^+_{k,i} - y^-_{k,i} = (\Phi_i(x_k, u_k) - \Phi_i^{L,s})/(\Phi_i^{U,s} - \Phi_i^{L,s}),\ k = 0, ..., T-1,\ i \in \mathcal{M} \tag{5.21e}$$

$$y^+_{k,i}, y^-_{k,i} \geq 0\ k = 0, ..., T-1,\ i \in \mathcal{M} \tag{5.21f}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U},\ k = 0, ..., T, \tag{5.21g}$$

where

$$\sum_{k=0}^{T-1} \left\| \frac{\Phi(x_k, u_k) - \Phi(x^s, u^s)}{\Phi^{U,s} - \Phi^{L,s}} \right\|_1 = \sum_{k=0}^{T-1} \sum_{i\in\mathcal{M}} (y^+_{k,i} + y^-_{k,i}). \tag{5.21h}$$

We can reformulate the $\mathcal{L}_\infty$ variant as [2] follows:

$$\min_{x_k,u_k} \sum_{k=0}^{T-1} \eta_k + \sum_{k=0}^{T-1} \sum_{i\in\mathcal{M}} (y^+_{k,i} + y^-_{k,i}) \tag{5.22a}$$

$$\text{s.t. } x_{k+1} = f(x_k, u_k),\ k = 0, ..., T-1 \tag{5.22b}$$

$$x_0 = x_\ell \tag{5.22c}$$

$$x_T = x^s \tag{5.22d}$$

$$y^+_{k,i} - y^-_{k,i} = (\Phi_i(x_k, u_k) - \Phi_i^{L,s})/(\Phi_i^{U,s} - \Phi_i^{L,s}),\ k = 0, ..., T-1,\ i \in \mathcal{M} \tag{5.22e}$$

$$y^+_{k,i} + y^-_{k,i} \leq \eta_k,\ k = 0, ..., T-1 \tag{5.22f}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U},\ k = 0, ..., T, \tag{5.22g}$$

where

$$\sum_{k=0}^{T-1} \left\| \frac{\Phi(x_k, u_k) - \Phi(x^s, u^s)}{\Phi^{U,s} - \Phi^{L,s}} \right\|_\infty = \sum_{k=0}^{T-1} \eta_k. \tag{5.23}$$

A similar formulation for the $\mathcal{L}_\infty$ problem was used in [18].

We emphasize that the steady-state utopia point can be computed off-line or asynchronously as done in steady-state real-time optimization. Consequently, this computation does not involve additional on-line costs for the controller.

5.2 Terminal Constraint

In finite arithmetic, the terminal constraints (4.15d) are difficult to handle, particularly in large-scale problems. Experimentally, we have observed that the controller problems can be solved more reliably by using terminal constraints of the form

$$\Phi_i(x_T, u_T) = \Phi_i^{L,s},\ i \in \mathcal{M}, \tag{5.24}$$

where $u_T \in \mathcal{U}$ is an additional variable. This has the advantage that a significantly lower number of constraints needs to be handled. In addition, it seems more practical to work directly with cost values. A rigorous equivalence with traditional state terminal constraints is beyond the scope of this work. In particular, it can be a concern that different states can reach the same cost and thus the states might never stabilize. Intuitively, however, it seems that as long as the solution of the problem (3.8) is locally unique (solution satisfies strong second-order conditions [14]), the states should converge to the compromise steady-state solution.

6 Numerical Case Study

We simulated the performance of the three proposed controllers using free-radical polymerization reactor [13]. The dynamic model has the following form:

$$\frac{dC_m(t)}{dt} = -(k_p + k_{fm}) \cdot C_m(t) \cdot P_0(t) + \frac{F}{V} \cdot (C_{m,in} - C_m(t)) \tag{6.25a}$$

$$\frac{dC_i(t)}{dt} = -k_i \cdot C_i(t) + \frac{F_i(t)}{V} \cdot C_{i,in} - \frac{F}{V} \cdot C_i(t) \tag{6.25b}$$

$$\frac{dD_0(t)}{dt} = (0.5k_{tc} + k_{td}) \cdot P_0(t)^2 + k_{fm} \cdot C_m(t) \cdot P_0(t) - \frac{F}{V} \cdot D_0(t) \tag{6.25c}$$

$$\frac{dD_1(t)}{dt} = M_m \cdot (k_p + k_{fm}) \cdot C_m(t) \cdot P_0(t) - \frac{F}{V} \cdot D_1(t). \tag{6.25d}$$

Here, $C_m(t)$ is the monomer concentration, $C_i(t)$ is the initiator concentration, $D_0(t)$ is the zeroth moment and $D_1(t)$ is the first moment. These are the states. The control variable is $F_i(t)$, the initiator flowrate. The term $P_0(t)$ takes into account the total concentration of live polymer chains,

$$P_0(t) = \sqrt{\frac{2 \cdot \eta_i \cdot k_i \cdot C_i(t)}{k_{td} + k_{tc}}}. \tag{6.26}$$

The model parameter values can be found in [13]. We assume that it is desired to maximize conversion $\Phi_1(t) = X(t) = (C_{m,in} - C_m(t))/C_{m,in}$ while simultaneously maximizing the profit function,

$$\Phi_2(t) = 2500 + 3500 \cdot X(t)^{0.6} + 8.82 \times 10^{-4} M_w(t)^{0.65} - 3000 \cdot F_i(t)^{0.5}, \tag{6.27}$$

where $M_w(t) = D_1(t)/D_0(t)$ is the polymer molecular weight.

We converted the model into discrete time form using Euler discretization. All the controller implementations are available at `http://www.mcs.anl.gov/~vzavala`. We found that the $\mathcal{L}_1$ and the $\mathcal{L}_\infty$ norm reformulations are computationally more robust compared with the $\mathcal{L}_2$ norm. In all our numerical experiments, we used the $\mathcal{L}_\infty$ norm. By monitoring the second order conditions with IPOPT, we have observed that the solutions are locally stable.

We tested the ST, CT, and UT controllers using a closed-loop scenario considering two initial points at the extremes of the Pareto front. The trajectories for the CT and UT controllers are presented in Figure 2. In the transition from the initial point at the lower end of the Pareto front, both controllers leave the front since they can exploit the system dynamics to get to the compromise solution. We also note that the UT controller is able to get much closer to the steady-utopia point and then converges to the compromise solution. In the transition from the second initial point, the difference in performance is less pronounced. This can be due to the inability of the controller to visit the region surrounding the upper end of the Pareto front.

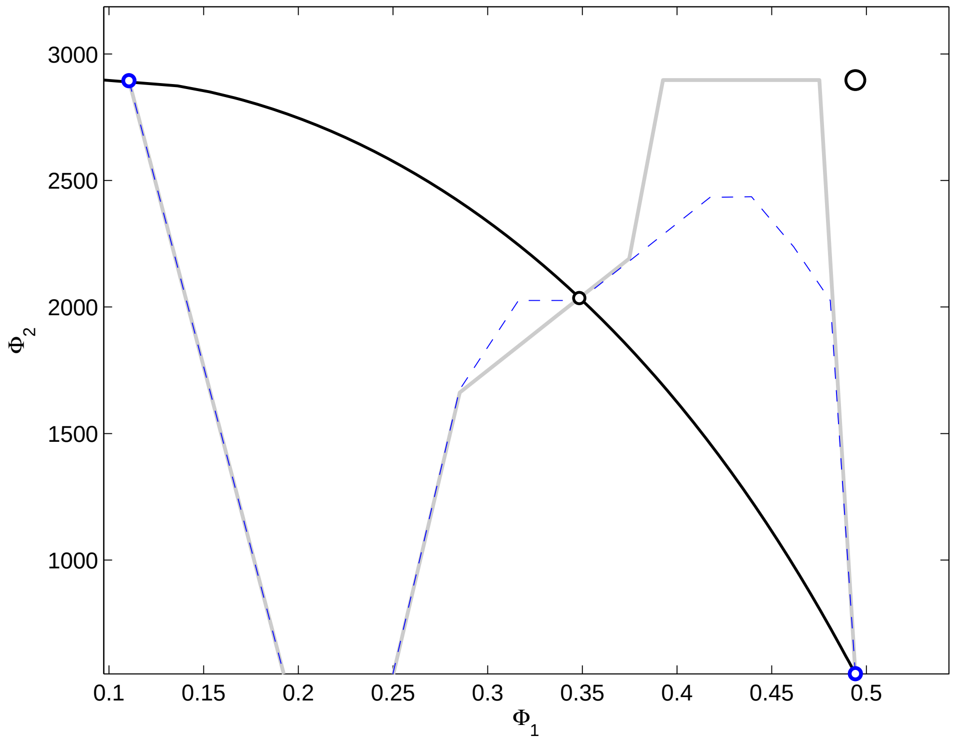

Figure 2: Phase plot of utopia-tracking (gray line) and cost-tracking controllers (dotted line). The utopia point is the large dot.

We found that the ST controller is stable but its performance is not competitive. In Figure 3 we present the evolution in time of the distance of the controllers to the utopia point $\|\Phi - \Phi^{L,s}\|_\infty$. As can be seen, the performance of the UT controller is superior, while the poorest performance is given by the ST controller.

7 Conclusions and Future Work

We have proposed a new strategy, which we term *utopia-tracking* MPC, to handle multiple objective functions in predictive control. We have established conditions for nominal asymptotic stability and propose numerical implementation schemes. The approach is able to handle general cost functions (e.g., economic, regularization, tracking) that are required to satisfy only a Lipschitz continuity property. In addition, the strategy does not require the construction of the Pareto front in real time and avoids the need of tuning weights. Interesting directions of future work are robustness and stability under different terminal conditions.

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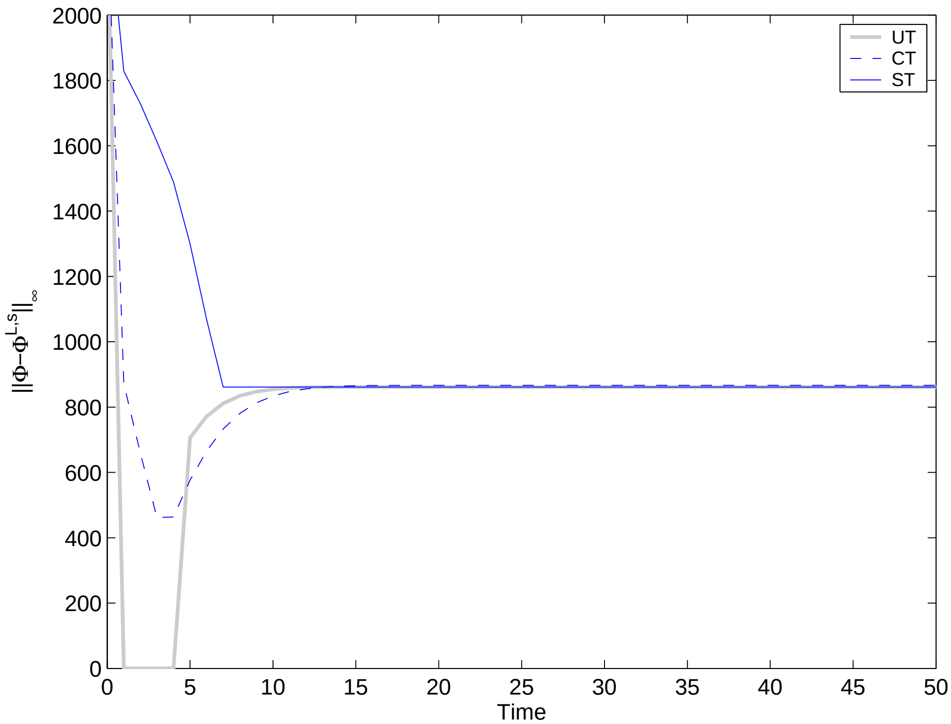

Figure 3: Time evolution of the distance of the controllers trajectories to the utopia point.

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