

## FAST AND SLOW DYNAMICS FOR THE COMPUTATIONAL SINGULAR PERTURBATION METHOD\*

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**Abstract.** The computational singular perturbation (CSP) method of Lam and Goussis is an iterative method to reduce the dimensionality of systems of ordinary differential equations with multiple time scales. In [*J. Nonlinear Sci.*, 14 (2004), pp. 59–91], the authors of this paper showed that each iteration of the CSP algorithm improves the approximation of the slow manifold by one order. In this paper, it is shown that the CSP method simultaneously approximates the tangent spaces to the fast fibers along which solutions relax to the slow manifold. Again, each iteration adds one order of accuracy. In some studies, the output of the CSP algorithm is postprocessed by linearly projecting initial data onto the slow manifold along these approximate tangent spaces. These projections, in turn, also become successively more accurate.

**Key words.** chemical kinetics, kinetic equations, dimension reduction, computational singular perturbation method, fast-slow systems, slow manifold, fast fibers, Fenichel theory, Michaelis–Menten–Henri mechanism

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**1. Introduction.** Complex chemical reaction mechanisms typically involve tens or even hundreds of reactants participating in a multitude of reactions, which occur on a broad spectrum of time scales ranging from microseconds to seconds. In these and other systems with multiple time scales, it is often the case that the fastest modes are exhausted after a brief transient period and become dependent on the slower ones. In terms of dynamics in the state space, the fast transient dynamics bring the orbits close to lower-dimensional manifolds where the dynamics are slower.

Reduction methods in chemical kinetics are designed to locate these lower-dimensional manifolds and, hence, to achieve a systematic decrease in the size and complexity of the systems. In the state space, these manifolds are parametrized by a subset of all of the variables, commonly known as *reaction progress variables*. For a fixed set of values of these variables, the values of the complementary variables are then determined by the corresponding point on the lower-dimensional manifold. As a result, one needs only to track the dynamical evolution of the progress variables on the lower-dimensional manifold; the concentrations of the complementary variables may be found from look-up tables. Moreover, the reaction progress variables may be a predetermined set of state variables, or they may be determined in the course of the computation.

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Among the reduction methods currently in use are the quasi-steady state approximation (QSSA), the intrinsic low-dimensional manifold (ILDM) method of Maas and Pope, an iterative method by Fraser and Roussel, the method of invariant manifolds (MIM) of Gorban and Karlin, the computational singular perturbation (CSP) method of Lam and Goussis, and a variety of other methods; references are given in [12], [32], and [34].

In the QSSA method [3, 28, 29], chemical intuition is employed to identify reactants whose concentrations equilibrate relatively fast. Then the right members of the ordinary differential equations for these concentrations are set to zero to yield algebraic equations, and in turn the solution of these equations is an approximation of the slow manifold.

In the ILDM method [18, 19], the Jacobian of the vector field is partitioned at each point of the phase space into a fast and a slow component, and a basis for the slow subspace is generated by means of a Schur decomposition. The ILDM, which approximates the slow manifold, is then defined as the locus of points where the vector field lies entirely in the slow subspace.

The iterative method [3, 27] is derived formally from the invariance equation—a partial differential equation that is satisfied by all invariant manifolds and thus also by the slow manifold. At each iteration, the terms of the invariance equation involving derivatives are evaluated at the approximation of the slow manifold available from the previous iteration, while all algebraic terms are left evaluated at a general point of the phase space. This reduces the invariance equation to an algebraic equation, which can be solved to yield the new approximation of the slow manifold. Asymptotic analyses of this iterative method and of the ILDM method are presented in [12].

The MIM [4, 5, 6, 7, 8] also exploits the invariance equation satisfied by a slow manifold, and it is constructed so that the reduced equations on the approximate slow manifold satisfy a thermodynamic consistency requirement. An initial approximation of the slow manifold (for example, the QSSA or the equilibrium approximation) is used to seed the method. The parametrization of this approximate manifold induces a projection operator onto its tangent space—as well as a complementary projection operator onto the tangent space of the approximate fast dynamics—and the invariance equation is expressed in terms of this complementary projection operator. Then a Newton iteration is carried out to obtain a more accurate approximation. This cycle is repeated until the desired accuracy is obtained. At any step, the projection operator, and hence the parametrization of the approximate slow manifold, may be chosen so that the reduced equations on the manifold are thermodynamically consistent.

The CSP method was first proposed by Lam and Goussis [9, 13, 14, 15, 16] and is widely used, for example, in combustion modeling [10, 17, 20, 21, 30, 31]. The method is essentially an algorithm to find successive approximations to the slow manifold and match the initial conditions to the dynamics on the slow manifold. Moreover, as is also the case with the reductions methods listed above, it is generally applicable to systems of nonlinear ordinary differential equations with simultaneous fast and slow dynamics where the long-term dynamics evolve on a low-dimensional slow manifold in the phase space.

In a previous paper [34], we focused on the slow manifold and the accuracy of the CSP approximation for fast-slow systems. In such systems, the ratio of the characteristic fast and slow times is made explicit by a small parameter  $\varepsilon$ , and the quality of the approximation can be measured in terms of  $\varepsilon$ . By comparing the CSP manifold with the slow manifold found in Fenichel's geometric singular perturbation theory [2, 11], we showed that each application of the CSP algorithm improves the asymptotic

accuracy of the CSP manifold by one order of  $\varepsilon$ .

In this paper, we complete the analysis of the CSP method by focusing on the fast dynamics. According to Fenichel's theory, the fast-slow systems we consider have, besides a slow manifold, a family of fast stable fibers along which initial conditions tend toward the slow manifold. The base points of these fibers lie on the slow manifold, and the dynamics near the slow manifold can be decomposed into a fast contracting component along the fast fibers and a slow component governed by the motion of the base points on the slow manifold. By comparing the CSP fibers with the tangent spaces of the fast fibers at their base points, we show that each application of the CSP algorithm also improves the asymptotic accuracy of the CSP fibers by one order of  $\varepsilon$ .

Summarizing the results of [34] and the present investigation, we conclude that the CSP method provides for the simultaneous approximation of the slow manifold and the tangents to the fast fibers at their base points. If one is interested in only the slow manifold, then it suffices to implement a reduced (one-step) version of the algorithm. On the other hand, if one is interested in both the slow and fast dynamics, then it is necessary to use the full (two-step) CSP algorithm. Moreover, only the full CSP algorithm allows for a linear matching of any initial data with the dynamics on the slow manifold.

The CSP method does not require an a priori separation of the variables into fast and slow nor that there is an explicit small parameter such as is used in singular perturbation theory. It requires only that there is a separation of time scales so that some (combinations of) species are fast and some are slow. For the sake of the analysis here, it is natural to take the a priori separation as given and measured by an explicit small parameter  $\varepsilon$ .

This paper is organized as follows. In section 2, we recall the relevant results from Fenichel's theory and set the framework for the CSP method. In section 3, we outline the CSP algorithm and state the main results: Theorem 3.1 concerning the approximation of the slow manifold, which is a verbatim restatement of [34, Theorem 3.1], and Theorem 3.2 concerning the approximation of the tangent spaces of the fast fibers. The proof of Theorem 3.2 is given in section 4. In section 5, we revisit the Michaelis–Menten–Henri mechanism of enzyme kinetics to illustrate the CSP method and the results of this article. Section 6 is devoted to a discussion of methods for linearly projecting initial conditions on the slow manifold.

**2. Slow manifolds and fast fibers.** Consider a general system of ordinary differential equations,

$$(2.1) \quad \frac{dx}{dt} = g(x),$$

for a vector-valued function  $x \equiv x(t) \in \mathbf{R}^{m+n}$  in a smooth vector field  $g$ . For the present analysis, we assume that  $n$  components of  $x$  evolve on a time scale characterized by the “fast” time  $t$ , while the remaining  $m$  components evolve on a time scale characterized by the “slow” time  $\tau = \varepsilon t$ , where  $\varepsilon$  is a small parameter. (The assumption that the variables can be separated once and for all into fast and slow variables, where the characteristic times are related by a fixed parameter  $\varepsilon$ , is made for the sake of the analysis, as noted in section 1. The assumption is not necessary for the applicability of the CSP method, which requires only a separation of time scales.) We collect the slow variables in a vector  $y \in \mathbf{R}^m$  and the fast variables in a vector

$z \in \mathbf{R}^n$ . Thus, the system (2.1) is equivalent to either the “fast system”

$$(2.2) \quad y' = \varepsilon g_1(y, z, \varepsilon),$$

$$(2.3) \quad z' = g_2(y, z, \varepsilon)$$

or the “slow system”

$$(2.4) \quad \dot{y} = g_1(y, z, \varepsilon),$$

$$(2.5) \quad \varepsilon \dot{z} = g_2(y, z, \varepsilon).$$

(A prime  $'$  denotes differentiation with respect to  $t$  and a dot  $\dot{\phantom{x}}$  differentiation with respect to  $\tau$ .) The fast system is more appropriate for the short-term dynamics and the slow system for the long-term dynamics of the system (2.1).

In the limit as  $\varepsilon$  tends to 0, the fast system reduces formally to a single equation for the fast variable  $z$ ,

$$(2.6) \quad z' = g_2(y, z, 0),$$

where  $y$  is a parameter, while the slow system reduces to a differential equation for the slow variable  $y$ ,

$$(2.7) \quad \dot{y} = g_1(y, z, 0),$$

with the algebraic constraint  $g_2(y, z, 0) = 0$ .

We assume that there exist a compact domain  $K$  and a smooth function  $h_0$  defined on  $K$  such that

$$(2.8) \quad g_2(y, h_0(y), 0) = 0, \quad y \in K.$$

The graph of  $h_0$  defines a critical manifold  $\mathcal{M}_0$ ,

$$(2.9) \quad \mathcal{M}_0 = \{(y, z) \in \mathbf{R}^{m+n} : z = h_0(y), y \in K\},$$

and with each point  $p = (y, h_0(y)) \in \mathcal{M}_0$  is associated a fast fiber  $\mathcal{F}_0^p$ ,

$$(2.10) \quad \mathcal{F}_0^p = \{(y, z) \in \mathbf{R}^{m+n} : z \in \mathbf{R}^n\}, \quad p \in \mathcal{M}_0.$$

The points of  $\mathcal{M}_0$  are fixed points of (2.6). If the real parts of the eigenvalues of  $D_z g_2(y, h_0(y), 0)$  are all negative, as we assume, then  $\mathcal{M}_0$  is asymptotically stable, and all solutions on  $\mathcal{F}_0^p$  contract exponentially toward  $p$ .

If  $\varepsilon$  is positive but arbitrarily small, Fenichel's theory [2, 11] guarantees that there exists a function  $h_\varepsilon$  whose graph is a slow manifold  $\mathcal{M}_\varepsilon$ ,

$$(2.11) \quad \mathcal{M}_\varepsilon = \{(y, z) \in \mathbf{R}^{m+n} : z = h_\varepsilon(y), y \in K\}.$$

This manifold is locally invariant under the system dynamics, and the dynamics on  $\mathcal{M}_\varepsilon$  are governed by the equation

$$(2.12) \quad \dot{y} = g_1(y, h_\varepsilon(y), \varepsilon),$$

as long as  $y \in K$ . Fenichel's theory also guarantees that there exists an invariant family  $\mathcal{F}_\varepsilon$ ,

$$(2.13) \quad \mathcal{F}_\varepsilon = \bigcup_{p \in \mathcal{M}_\varepsilon} \mathcal{F}_\varepsilon^p,$$

of fast stable fibers  $\mathcal{F}_\varepsilon^p$  along which solutions relax to  $\mathcal{M}_\varepsilon$ . The family is invariant in the sense that if  $\phi_t$  denotes the time- $t$  map associated with (2.1), then

$$(2.14) \quad \phi_t(\mathcal{F}_\varepsilon^p) \subset \mathcal{F}_\varepsilon^{\phi_t(p)}, \quad p \in \mathcal{M}_\varepsilon.$$

The collection of fast fibers  $\mathcal{F}_\varepsilon^p$  foliates a neighborhood of  $\mathcal{M}_\varepsilon$ . Hence, the motion of any point on  $\mathcal{F}_\varepsilon^p$  decomposes into a fast contracting component along the fiber and a slow component governed by the motion of the base point of the fiber. Also,  $\mathcal{M}_\varepsilon$  is  $\mathcal{O}(\varepsilon)$ -close to  $\mathcal{M}_0$ , with

$$(2.15) \quad h_\varepsilon(y) = h_0(y) + \varepsilon h_1(y) + \varepsilon^2 h_2(y) + \dots, \quad \varepsilon \downarrow 0,$$

and  $\mathcal{F}_\varepsilon^p$  is  $\mathcal{O}(\varepsilon)$ -close to  $\mathcal{F}_0^p$  in any compact neighborhood of  $\mathcal{M}_\varepsilon$ .

*Remark 2.1.* Typically, the manifold  $\mathcal{M}_\varepsilon$  is not unique; there is a family of slow manifolds, all having the same asymptotic expansion (2.15) to all orders in  $\varepsilon$  but differing by exponentially small amounts ( $\mathcal{O}(e^{-c/\varepsilon})$ ,  $c > 0$ ).

**3. The CSP method.** The CSP method focuses on the dynamics of the vector field  $g(x)$  rather than on the dynamics of the vector  $x$  itself.

Writing a single differential equation like (2.1) as a system of equations amounts to choosing a basis in the vector space. For example, in (2.2)–(2.3), the basis consists of the ordered set of unit vectors in  $\mathbf{R}^{m+n}$ . The coordinates of  $g$  relative to this basis are  $\varepsilon g_1$  and  $g_2$ . If we collect the basis vectors in a matrix in the usual way, then we can express the relation between  $g$  and its coordinates in the form

$$(3.1) \quad g = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \varepsilon g_1 \\ g_2 \end{pmatrix}.$$

Note that the basis chosen for this representation is the same at every point of the phase space. The CSP method is based on a generalization of this idea, where the basis is allowed to vary from point to point, so it can be tailored to the local dynamics near  $\mathcal{M}_\varepsilon$ .

Suppose that we choose, instead of a fixed basis, a (point-dependent) basis  $A$  for  $\mathbf{R}^{m+n}$ . The relation between the vector field  $g$  and the vector  $f$  of its coordinates relative to this basis is

$$(3.2) \quad g = Af.$$

Conversely,

$$(3.3) \quad f = Bg,$$

where  $B$  is the left inverse of  $A$ ,  $BA = I$  on  $\mathbf{R}^{m+n}$ . In the convention of the CSP method,  $A$  is a matrix of column vectors (vectors in  $\mathbf{R}^{m+n}$ ) and  $B$  a matrix of row vectors (functionals on  $\mathbf{R}^{m+n}$ ).

The CSP method focuses on the dynamics of the vector  $f$ . Along a trajectory of the system (2.1),  $f$  satisfies the ordinary differential equation

$$(3.4) \quad \frac{df}{dt} = \Lambda f,$$

where  $\Lambda$  is a linear operator [22, 34],

$$(3.5) \quad \Lambda = B(Dg)A + \frac{dB}{dt}A - B\frac{dA}{dt} = B[A, g].$$

Here,  $Dg$  is the Jacobian of  $g$ ,  $dB/dt = (DB)g$ ,  $dA/dt = (DA)g$ , and  $[A, g]$  is the Lie bracket of  $A$  (taken column by column) and  $g$ . The Lie bracket of any two vectors  $a$  and  $g$  is  $[a, g] = (Dg)a - (Da)g$ ; see [23].

It is clear from (3.4) that the dynamics of  $f$  are governed by  $\Lambda$ , so the CSP method focuses on the structure of  $\Lambda$ .

*Remark 3.1.* It is useful to see how  $\Lambda$  transforms under a change of basis. If  $C$  is an invertible square matrix representing a coordinate transformation in  $\mathbf{R}^{m+n}$  and  $\hat{A} = AC$  and  $\hat{B} = C^{-1}B$ , then

$$\begin{aligned} \hat{\Lambda} &= \hat{B}(Dg)\hat{A} - \hat{B}\frac{d\hat{A}}{dt} = C^{-1}B(Dg)AC - C^{-1}B\frac{d(AC)}{dt} \\ &= C^{-1}B(Dg)AC - C^{-1}B\left(\frac{dA}{dt}C + A\frac{dC}{dt}\right) \\ (3.6) \quad &= C^{-1}\Lambda C - C^{-1}\frac{dC}{dt}. \end{aligned}$$

Hence,  $\Lambda$  does not transform as a matrix, unless  $C$  is constant.

**3.1. Decompositions.** Our goal is to decompose the vector  $f$  into its fast and slow components. Suppose, therefore, that we have a decomposition of this type,  $f = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}$ , where  $f^1$  and  $f^2$  are of length  $n$  and  $m$ , respectively, but not necessarily fast and slow everywhere. The decomposition suggests corresponding decompositions of the matrices  $A$  and  $B$ , namely  $A = (A_1, A_2)$  and  $B = \begin{pmatrix} B^1 \\ B^2 \end{pmatrix}$ , where  $A_1$  is an  $(m+n) \times n$  matrix,  $A_2$  an  $(m+n) \times m$  matrix,  $B^1$  an  $n \times (m+n)$  matrix, and  $B^2$  an  $m \times (m+n)$  matrix. Then  $f^1 = B^1g$  and  $f^2 = B^2g$ .

The decompositions of  $A$  and  $B$  lead, in turn, to a decomposition of  $\Lambda$ ,

$$(3.7) \quad \Lambda = \begin{pmatrix} \Lambda^{11} & \Lambda^{12} \\ \Lambda^{21} & \Lambda^{22} \end{pmatrix} = \begin{pmatrix} B^1[A_1, g] & B^1[A_2, g] \\ B^2[A_1, g] & B^2[A_2, g] \end{pmatrix}.$$

The off-diagonal blocks  $\Lambda^{12}$  and  $\Lambda^{21}$  are, in general, not zero, so the equations governing the evolution of the coordinates  $f^1$  and  $f^2$  are coupled. Consequently,  $f^1$  and  $f^2$  cannot be identified with the fast and slow coordinates of  $g$  globally along trajectories. The objective of the CSP method is to construct local coordinate systems (that is, matrices  $A$  and  $B$ ) that lead to a block-diagonal structure of  $\Lambda$ . We will see, in the next section, that such a structure is associated with a decomposition in terms of the slow manifold and the fast fibers.

*Remark 3.2.* Note that the identity  $BA = I$  on  $\mathbf{R}^{m+n}$  implies four identities, which are summarized in the matrix identity

$$(3.8) \quad \begin{pmatrix} B^1A_1 & B^1A_2 \\ B^2A_1 & B^2A_2 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}.$$

**3.2. Block-diagonalization of  $\Lambda$ .** In this section we analyze the properties of  $\Lambda$  relative to a fast-slow decomposition of the dynamics near  $\mathcal{M}_\varepsilon$ .

Let  $\mathcal{T}_p\mathcal{F}_\varepsilon$  and  $\mathcal{T}_p\mathcal{M}_\varepsilon$  denote the tangent spaces to the fast fiber and the slow manifold, respectively, at the base point  $p$  of the fiber on  $\mathcal{M}_\varepsilon$ . (Note that  $\dim\mathcal{T}_p\mathcal{F}_\varepsilon = n$  and  $\dim\mathcal{T}_p\mathcal{M}_\varepsilon = m$ .) These two linear spaces intersect transversally, because  $\mathcal{M}_\varepsilon$  is normally hyperbolic and compact, so

$$(3.9) \quad \mathbf{R}^{m+n} = \mathcal{T}_p\mathcal{F}_\varepsilon \oplus \mathcal{T}_p\mathcal{M}_\varepsilon, \quad p \in \mathcal{M}_\varepsilon.$$

Let  $A_f$  be an  $(m+n) \times n$  matrix whose columns form a basis for  $\mathcal{T}_p\mathcal{F}_\varepsilon$  and  $A_s$  an  $(m+n) \times m$  matrix whose columns form a basis for  $\mathcal{T}_p\mathcal{M}_\varepsilon$ , and let  $A = (A_f, A_s)$ . (We omit the subscript  $p$ .) Then  $A$  is a (point-dependent) basis for  $\mathbf{R}^{m+n}$  that respects the decomposition (3.9). We recall that  $\mathcal{T}\mathcal{M}_\varepsilon \equiv \bigcup_{p \in \mathcal{M}_\varepsilon} (p, \mathcal{T}_p\mathcal{M}_\varepsilon)$  and  $\mathcal{T}\mathcal{F}_\varepsilon \equiv \bigcup_{p \in \mathcal{M}_\varepsilon} (p, \mathcal{T}_p\mathcal{F}_\varepsilon)$  are the tangent bundles of the slow manifold and the family of the fast fibers, respectively. (A general treatment of tangent bundles of manifolds is given in [1, section 1.7].)

The decomposition (3.9) induces a dual decomposition,

$$(3.10) \quad \mathbf{R}^{m+n} = \mathcal{N}_p\mathcal{M}_\varepsilon \oplus \mathcal{N}_p\mathcal{F}_\varepsilon, \quad p \in \mathcal{M}_\varepsilon,$$

where  $\mathcal{N}_p\mathcal{M}_\varepsilon$  and  $\mathcal{N}_p\mathcal{F}_\varepsilon$  are the duals of  $\mathcal{T}_p\mathcal{M}_\varepsilon$  and  $\mathcal{T}_p\mathcal{F}_\varepsilon$ , respectively, in  $\mathbf{R}^{m+n}$ . (Note that  $\dim\mathcal{N}_p\mathcal{M}_\varepsilon = n$  and  $\dim\mathcal{N}_p\mathcal{F}_\varepsilon = m$ .) The corresponding decomposition of  $B$  is  $B = \begin{pmatrix} B^{s\perp} \\ B^{f\perp} \end{pmatrix}$ , where the rows of  $B^{s\perp}$  form a basis for  $\mathcal{N}_p\mathcal{M}_\varepsilon$  and the rows of  $B^{f\perp}$  a basis for  $\mathcal{N}_p\mathcal{F}_\varepsilon$ . Furthermore,

$$(3.11) \quad \begin{pmatrix} B^{s\perp}A_f & B^{s\perp}A_s \\ B^{f\perp}A_f & B^{f\perp}A_s \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}.$$

The decompositions of  $A$  and  $B$  lead, in turn, to a decomposition of  $\Lambda$ ,

$$(3.12) \quad \Lambda = \begin{pmatrix} B^{s\perp}[A_f, g] & B^{s\perp}[A_s, g] \\ B^{f\perp}[A_f, g] & B^{f\perp}[A_s, g] \end{pmatrix}.$$

This decomposition is similar to, but different from, the decomposition (3.7). The following lemma shows that its off-diagonal blocks are zero.

LEMMA 3.1. *The off-diagonal blocks in the representation (3.12) of  $\Lambda$  are zero at each point  $p \in \mathcal{M}_\varepsilon$ .*

*Proof.* Since  $B^{s\perp}A_s = 0$  on  $\mathcal{M}_\varepsilon$  and  $\mathcal{M}_\varepsilon$  is invariant, we have

$$(3.13) \quad \frac{d}{dt}(B^{s\perp}A_s) = D(B^{s\perp}A_s)g = (DB^{s\perp})(g, A_s) + B^{s\perp}((DA_s)g) = 0.$$

( $DB^{s\perp}$  is a symmetric bilinear form; its action on a matrix must be understood as columnwise action.)

Also,  $g \in \mathcal{T}\mathcal{M}_\varepsilon$ , so  $B^{s\perp}g = 0$  on  $\mathcal{M}_\varepsilon$ . Hence, the directional derivative along  $A_s$  (taken column by column) at points on  $\mathcal{M}_\varepsilon$  also vanishes,

$$(3.14) \quad D(B^{s\perp}g)A_s = (DB^{s\perp})(A_s, g) + B^{s\perp}(Dg)A_s = 0.$$

Subtracting (3.13) from (3.14), we obtain the identity

$$(3.15) \quad B^{s\perp}[A_s, g] = B^{s\perp}((Dg)A_s - (DA_s)g) = 0.$$

The proof for the lower-left block is more involved, since the fast fibers are invariant as a family. Assume that the fiber  $\mathcal{F}_\varepsilon^p$  at  $p \in \mathcal{M}_\varepsilon$  is given implicitly by the equation  $F(q; p) = 0$ ,  $q \in \mathcal{F}_\varepsilon^p$ . Then the rows of  $(D_qF)(q; p)$  form a basis for  $\mathcal{N}_q\mathcal{F}_\varepsilon$ , so there exists an invertible matrix  $C$  such that  $B^{f\perp} = C(D_qF)$ .

Since the rows of  $(D_qF)(q; p)$  span  $\mathcal{N}_q\mathcal{F}_\varepsilon$ , we have  $(D_qF)(q; p)A_f(q) = 0$ . This identity holds, in particular, along solutions of (2.1), so

$$(3.16) \quad \begin{aligned} \frac{d}{dt}((D_qF)(q; p)A_f(q)) &= ((D_q^2F)(q; p))(g(q), A_f(q)) \\ &\quad + ((D_{pq}F)(q; p))(g(p), A_f(q)) \\ &\quad + ((D_qF)(q; p))(DA_f(q))g(q) \\ &= 0. \end{aligned}$$

The family of the fast fibers is invariant under the flow associated with (2.1), so if  $F(q; p) = 0$ , then also  $F(q(t); p(t)) = 0$  and, hence,

$$(3.17) \quad \frac{dF(q; p)}{dt} = ((D_q F)(q; p))g(q) + ((D_p F)(q; p))g(p) = 0.$$

Next, we take the directional derivative of both members of this equation along  $A_f$ , keeping in mind that  $(Dg)(p)A_f(q) = 0$  because the base point  $p$  does not vary along  $A_f$ . (Recall that the columns of  $A_f(q)$  span  $\mathcal{T}_q\mathcal{F}_\varepsilon$ .) We find that

$$(3.18) \quad \begin{aligned} & ((D_q^2 F)(q; p))(A_f(q), g(q)) + ((D_q F)(q; p))(Dg(q))A_f(q) \\ & + ((D_{pq} F)(q; p))(A_f(q), g(p)) = 0. \end{aligned}$$

But the bilinear forms  $D_q^2 F$  and  $D_{pq} F$  are symmetric, so subtracting (3.16) from (3.18) and letting  $q = p$ , we obtain the identity

$$(3.19) \quad (D_q F)(p; p)((Dg)A_f - (DA_f)g)(p) = 0.$$

Hence,  $B^{f\perp}[A_f, g](p) = C(D_q F)(p; p)[A_f, g](p) = 0$ , and the proof of the lemma is complete.  $\square$

The lemma implies that the representation (3.12) is block-diagonal,

$$(3.20) \quad \Lambda = \begin{pmatrix} B^{s\perp}[A_f, g] & 0 \\ 0 & B^{f\perp}[A_s, g] \end{pmatrix}.$$

Consequently, the decomposition (3.9) reduces  $\Lambda$ . In summary, if we can construct bases  $A_f$  and  $A_s$ , then we will have achieved a representation of  $\Lambda$  where the fast and slow components remain separated at all times and the designation of fast and slow takes on a global meaning.

**3.3. The CSP algorithm.** The CSP method is a constructive algorithm to approximate  $A_f$  and  $A_s$ . One typically initializes the algorithm with a constant matrix  $A^{(0)}$ ,

$$(3.21) \quad A^{(0)} = (A_1^{(0)}, A_2^{(0)}) = \begin{pmatrix} A_{11}^{(0)} & A_{12}^{(0)} \\ A_{21}^{(0)} & A_{22}^{(0)} \end{pmatrix}.$$

Here,  $A_{11}^{(0)}$  is an  $m \times n$  matrix,  $A_{22}^{(0)}$  is an  $n \times m$  matrix, and the off-diagonal blocks  $A_{12}^{(0)}$  and  $A_{21}^{(0)}$  are full-rank square matrices of order  $m$  and  $n$ , respectively. A common choice is  $A_{11}^{(0)} = 0$ . We follow this convention and assume, henceforth, that  $A_{11}^{(0)} = 0$ ,

$$(3.22) \quad A^{(0)} = (A_1^{(0)}, A_2^{(0)}) = \begin{pmatrix} 0 & A_{12}^{(0)} \\ A_{21}^{(0)} & A_{22}^{(0)} \end{pmatrix}.$$

(Other choices are discussed in Remark 3.6 below.) The left inverse of  $A^{(0)}$  is

$$(3.23) \quad \begin{aligned} B_{(0)} &= \begin{pmatrix} B_{(0)}^1 \\ B_{(0)}^2 \end{pmatrix} = \begin{pmatrix} B_{(0)}^{11} & B_{(0)}^{12} \\ B_{(0)}^{21} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -(A_{21}^{(0)})^{-1}A_{22}^{(0)}(A_{12}^{(0)})^{-1} & (A_{21}^{(0)})^{-1} \\ (A_{12}^{(0)})^{-1} & 0 \end{pmatrix}. \end{aligned}$$

The algorithm proceeds iteratively. For  $q = 0, 1, \dots$ , one first defines the operator  $\Lambda_{(q)}$  in accordance with (3.5),

$$(3.24) \quad \Lambda_{(q)} = B_{(q)}(Dg)A^{(q)} - B_{(q)}\frac{dA^{(q)}}{dt} = \begin{pmatrix} \Lambda_{(q)}^{11} & \Lambda_{(q)}^{12} \\ \Lambda_{(q)}^{21} & \Lambda_{(q)}^{22} \end{pmatrix},$$

and matrices  $U_{(q)}$  and  $L_{(q)}$ ,

$$(3.25) \quad U_{(q)} = \begin{pmatrix} 0 & (\Lambda_{(q)}^{11})^{-1}\Lambda_{(q)}^{12} \\ 0 & 0 \end{pmatrix}, \quad L_{(q)} = \begin{pmatrix} 0 & 0 \\ \Lambda_{(q)}^{21}(\Lambda_{(q)}^{11})^{-1} & 0 \end{pmatrix}.$$

Then one updates  $A^{(q)}$  and  $B_{(q)}$  according to the formulas

$$(3.26) \quad A^{(q+1)} = A^{(q)}(I - U_{(q)})(I + L_{(q)}),$$

$$(3.27) \quad B_{(q+1)} = (I - L_{(q)})(I + U_{(q)})B_{(q)}$$

and returns to (3.24) for the next iteration.

*Remark 3.3.* Lam [13] and Lam and Goussis [16] perform the update (3.26)–(3.27) in two steps. The first step corresponds to the postmultiplication of  $A^{(q)}$  with  $I - U_{(q)}$  and premultiplication of  $B_{(q)}$  with  $I + U_{(q)}$  and the second step to the subsequent postmultiplication of  $A^{(q)}(I - U_{(q)})$  with  $I + L_{(q)}$  and premultiplication of  $(I + U_{(q)})B_{(q)}$  with  $I - L_{(q)}$ . The nonzero entries of  $U_{(q)}$  and  $L_{(q)}$  are chosen so that  $\Lambda$  is block-diagonalized to successively higher orders in  $\varepsilon$ .

*Remark 3.4.* In the special case of a linear vector field  $g(x)$ , the block-diagonalization is commonly referred to as the power method.

**3.4. Approximation of the slow manifold.** After  $q$  iterations, the CSP condition

$$(3.28) \quad B_{(q)}^1 g = 0, \quad q = 0, 1, \dots,$$

identifies those points where the fast amplitudes vanish with respect to the then current basis. These points define a manifold that is an approximation for the slow manifold  $\mathcal{M}_\varepsilon$ .

For  $q = 0$ ,  $B_{(0)}^1$  is constant and given by (3.23). Hence, the CSP condition (3.28) reduces to the constraint  $g_2(y, z, \varepsilon) = 0$ . In general, this constraint is satisfied by a function  $z = \psi_{(0)}(y, \varepsilon)$ . The graph of this function defines  $\mathcal{K}_\varepsilon^{(0)}$ , the CSP manifold (CSPM) of order zero. Since the constraint reduces at leading order to the equation  $g_2(y, z, 0) = 0$ , which is satisfied by the function  $z = h_0(y)$ ,  $\mathcal{K}_\varepsilon^{(0)}$  may be chosen to coincide with  $\mathcal{M}_0$  to leading order; see (2.9).

For  $q = 1, 2, \dots$ , the CSP condition takes the form

$$(3.29) \quad B_{(q)}^1(y, \psi_{(q-1)}(y, \varepsilon), \varepsilon)g(y, z, \varepsilon) = 0, \quad q = 1, 2, \dots$$

The condition is satisfied by a function  $z = \psi_{(q)}(y, \varepsilon)$ , and the manifold

$$(3.30) \quad \mathcal{K}_\varepsilon^{(q)} = \{(y, z) : z = \psi_{(q)}(y, \varepsilon), y \in K\}, \quad q = 0, 1, \dots,$$

defines the CSPM of order  $q$ , which is an approximation of  $\mathcal{M}_\varepsilon$ . The following theorem regarding the quality of the approximation was proven in [34].

**THEOREM 3.1** ([34, Theorem 3.1]). *The asymptotic expansions of the CSPM  $\mathcal{K}_\varepsilon^{(q)}$  and the slow manifold  $\mathcal{M}_\varepsilon$  agree up to and including terms of  $\mathcal{O}(\varepsilon^q)$ ,*

$$(3.31) \quad \psi_{(q)}(\cdot, \varepsilon) = \sum_{j=0}^q \varepsilon^j h_j + \mathcal{O}(\varepsilon^{q+1}), \quad \varepsilon \downarrow 0, \quad q = 0, 1, \dots$$

**3.5. Approximation of the fast fibers.** We now turn our attention to the fast fibers. The columns of  $A_f(y, h_\varepsilon(y))$  span the tangent space to the fast fiber with base point  $p = (y, h_\varepsilon(y))$ , so we expect that  $A_1^{(q)}$  defines an approximation for the same space after  $q$  applications of the CSP algorithm. We denote this approximation by  $\mathcal{L}_\varepsilon^{(q)}(y)$  and refer to it as the CSP fiber (CSPF) of order  $q$  at  $p$ ,

$$(3.32) \quad \mathcal{L}_\varepsilon^{(q)}(y) = \text{span} (\text{cols} (A_1^{(q)}(y, \psi_{(q)}(y, \varepsilon), \varepsilon))).$$

We will shortly estimate the asymptotic accuracy of the approximation, but before doing so we need to make an important observation.

Each application of the CSP algorithm involves two steps; see Remark 3.3. The first step involves  $U$  and serves to push the upper-right block of  $\Lambda$  up by one order of  $\varepsilon$ , and the second step involves  $L$  and serves the same purpose for the lower-left block. The two steps are consecutive. At the first step of the  $q$ th iteration, one evaluates  $B_{(q)}^1$  on  $\mathcal{K}_\varepsilon^{(q-1)}$  to find  $\mathcal{K}_\varepsilon^{(q)}$  by solving the CSP condition (3.28) for the function  $\psi_{(q)}$ . One then uses this expression in the second step to update  $A$  and  $B$ , thus effectively evaluating  $A_1^{(q)}$  on  $\mathcal{K}_\varepsilon^{(q)}$  rather than on  $\mathcal{K}_\varepsilon^{(q-1)}$ .

The following theorem contains our main result.

**THEOREM 3.2.** *The asymptotic expansions of  $\mathcal{L}_\varepsilon^{(q)}(y)$  and  $\mathcal{T}_p\mathcal{F}_\varepsilon$ , where  $p = (y, h_\varepsilon(y)) \in \mathcal{M}_\varepsilon$ , agree up to and including terms of  $\mathcal{O}(\varepsilon^q)$  for all  $y \in K$  and for  $q = 0, 1, \dots$*

Theorem 3.2 implies that the family  $\mathcal{L}_\varepsilon^{(q)} \equiv \bigcup_{p \in \mathcal{M}_\varepsilon} (p, \mathcal{L}_\varepsilon^{(q)}(y))$  is an  $\mathcal{O}(\varepsilon^q)$ -approximation to the tangent bundle  $\mathcal{T}\mathcal{F}_\varepsilon$ .

The proof of Theorem 3.2 is given in section 4. The essential idea is to show that, at each iteration, the asymptotic order of the off-diagonal blocks of  $\Lambda_{(q)}$  increases by one and  $A_1^{(q)}$  and  $B_{(q)}^2$  become fast and fast<sup>+</sup>, respectively, to one higher order. As a consequence, in the limit as  $q \rightarrow \infty$ ,  $\Lambda_{(q)} \rightarrow \Lambda$ ,  $A^{(q)} \rightarrow A$ , and  $B_{(q)} \rightarrow B$ , where  $\Lambda$ ,  $A$ , and  $B$  are ideal in the sense described in section 3.2.

*Remark 3.5.* If, in the second step of the CSP algorithm,  $A_1^{(q)}$  were evaluated on  $\mathcal{K}_\varepsilon^{(q-1)}$  instead of on  $\mathcal{K}_\varepsilon^{(q)}$ , the approximation of  $\mathcal{T}\mathcal{F}_\varepsilon$  would still be  $\mathcal{O}(\varepsilon^q)$ -accurate. In section 5 we demonstrate this by means of an example; in Appendix B, we state the idea of the proof for the general case.

*Remark 3.6.* The statement of Theorem 3.2 as given above is tailored to suit our choice of an initial basis  $A^{(0)}$ ; see (3.22). Other choices have also been considered for systems in which there is not an explicit small parameter. Hadjinicolaou and Goussis [10] use an arbitrary basis, while Lam and Goussis [15] and Massias and Goussis [21] use the eigenvectors of the Jacobian. These other choices of  $A^{(0)}$  introduce modifications only in the degree to which the CSPM and the CSPFs of order  $q$  approximate  $\mathcal{M}_\varepsilon$  and  $\mathcal{T}\mathcal{F}_\varepsilon$ , respectively, and, for any choice of  $A^{(0)}$ , statements similar to those of Theorems 3.1 and 3.2 can be shown to be true.

**4. Proof of Theorem 3.2.** The proof of Theorem 3.2 is by induction on  $q$ . Section 4.1 contains an auxiliary lemma that shows that each successive application of

the CSP algorithm pushes  $\Lambda$  closer to block-diagonal form. The induction hypothesis is formulated in section 4.2, the hypothesis is shown to be true for  $q = 0$  in section 4.3, and the induction step is taken in section 4.4.

**4.1. Asymptotic estimates of  $\Lambda$ .** As stated in section 3, the goal of the CSP method is to reduce  $\Lambda$  to block-diagonal form. This goal is approached by the repeated application of a two-step algorithm. As shown in [34], the first step of the algorithm is engineered so that each application increases the asymptotic accuracy of the upper-right block  $\Lambda_{(q)}^{12}$  by one order of  $\varepsilon$ ; in particular,  $\Lambda_{(q)}^{12} = \mathcal{O}(\varepsilon^q)$  on  $\mathcal{K}_\varepsilon^{(q)}$  [34, eq. (5.25)]. We now complete the picture and show that each application of the second step increases the asymptotic accuracy of the lower-left block  $\Lambda_{(q)}^{21}$  by one order of  $\varepsilon$  when the information obtained in the first step of the same iteration is used. In particular,  $\Lambda_{(q)}^{21} = \mathcal{O}(\varepsilon^{q+1})$  on  $\mathcal{K}_\varepsilon^{(q+1)}$ , where  $\mathcal{K}_\varepsilon^{(q+1)}$  has been obtained in the first step of the  $(q + 1)$ th refinement.

LEMMA 4.1. For  $q = 0, 1, \dots$ ,

$$(4.1) \quad \Lambda_{(q)} = \begin{pmatrix} \Lambda_{(0,0)}^{11} + \mathcal{O}(\varepsilon) & \varepsilon^q \Lambda_{(q,q)}^{12} \\ \varepsilon^{q+1} \Lambda_{(q,q+1)}^{21} & \varepsilon \Lambda_{(1,1)}^{22} + \mathcal{O}(\varepsilon^2) \end{pmatrix},$$

when  $\Lambda_{(q)}$  is evaluated on  $\mathcal{K}_\varepsilon^{(q+1)}$ .

*Proof.* The proof is by induction. The desired estimates of  $\Lambda_{(q)}^{11}$ ,  $\Lambda_{(q)}^{12}$ , and  $\Lambda_{(q)}^{22}$  on  $\mathcal{K}_\varepsilon^{(q)}$  were established in [34, eqs. (5.24), (5.25), (5.27)]. Since the asymptotic expansions of  $\mathcal{K}_\varepsilon^{(q+1)}$  and  $\mathcal{K}_\varepsilon^{(q)}$  differ only at terms of  $\mathcal{O}(\varepsilon^{q+1})$  or higher [34, Theorem 3.1], these estimates of  $\Lambda_{(q)}^{11}$ ,  $\Lambda_{(q)}^{12}$ , and  $\Lambda_{(q)}^{22}$  are true also on  $\mathcal{K}_\varepsilon^{(q+1)}$ . It remains only to estimate  $\Lambda_{(q)}^{21}$ .

Consider the case  $q = 0$ . Let  $\Lambda_{(0,j)}^{21}$  be the coefficient of  $\varepsilon^j$  in the asymptotic expansion of  $\Lambda_{(0)}^{21}(y, \psi_{(1)}(y), \varepsilon)$ . The estimate  $\Lambda_{(0)}^{21} = \mathcal{O}(\varepsilon)$  on  $\mathcal{K}_\varepsilon^{(1)}$  follows if we can show that  $\Lambda_{(0,0)}^{21} = 0$ . It is already stated in [34, eq. (4.30)] that  $\Lambda_{(0,0)}^{21} = 0$  on  $\mathcal{K}_\varepsilon^{(0)}$ . Furthermore, [34, Theorem 3.1] implies that the asymptotic expansions of  $\psi_{(1)}$  and  $\psi_{(0)}$  agree to leading order. Thus, the asymptotic expansions of  $\Lambda_{(0)}^{21}(y, \psi_{(0)}(y), \varepsilon)$  and  $\Lambda_{(0)}^{21}(y, \psi_{(1)}(y), \varepsilon)$  also agree to leading order, and the result follows.

Now, assume that the estimate holds for  $0, 1, \dots, q$ . From (3.6) we obtain

$$(4.2) \quad \begin{aligned} \Lambda_{(q+1)}^{21} &= \Lambda_{(q)}^{21} - L_{(q)} \Lambda_{(q)}^{11} + \Lambda_{(q)}^{22} L_{(q)} - L_{(q)} \Lambda_{(q)}^{12} L_{(q)} - \Lambda_{(q)}^{21} U_{(q)} L_{(q)} \\ &\quad - L_{(q)} U_{(q)} \Lambda_{(q)}^{21} + L_{(q)} \Lambda_{(q)}^{11} U_{(q)} L_{(q)} - L_{(q)} U_{(q)} \Lambda_{(q)}^{22} L_{(q)} \\ &\quad + L_{(q)} U_{(q)} \Lambda_{(q)}^{21} U_{(q)} L_{(q)} + (DL_{(q)}) g + L_{(q)} ((DU_{(q)}) g) L_{(q)}. \end{aligned}$$

The first two terms in the right member sum to zero by virtue of the definition (3.25) of  $L_{(q)}$ . The next seven terms are all  $\mathcal{O}(\varepsilon^{q+2})$  or higher by virtue of the induction hypothesis. Finally, the last two terms are also  $\mathcal{O}(\varepsilon^{q+2})$  or higher by the induction hypothesis and [34, Lemma A.2].  $\square$

**4.2. The induction hypothesis.** The CSPF of order  $q$ ,  $\mathcal{L}_\varepsilon^{(q)}(y)$ , is defined in (3.32) to be the linear space spanned by the columns of the fast component,  $A_1^{(q)}(y, \psi_{(q)}, \varepsilon)$ , of the basis  $A^{(q)}$ . Thus, to prove Theorem 3.2, it suffices to show that the asymptotic expansions of  $A_1^{(q)}(y, \psi_{(q)}, \varepsilon)$  and the space tangent to the fast fiber,  $\mathcal{T}_p \mathcal{F}_\varepsilon$ , agree up to and including terms of  $\mathcal{O}(\varepsilon^q)$  for  $p = (y, h_\varepsilon(y))$  and for

$q = 0, 1, \dots$ . The central idea of the proof is to show that each successive application of the CSP method pushes the projection of  $A_1^{(q)}$  on  $\mathcal{TM}_\varepsilon$  along  $\mathcal{TF}_\varepsilon$  to one higher order in  $\varepsilon$ .

We express  $A^{(q)}$ , generated after  $q$  applications of the CSP algorithm, in terms of the basis  $A$ ,

$$(4.3) \quad A^{(q)}(y, z, \varepsilon) = A(y, h_\varepsilon, \varepsilon)Q^{(q)}(y, z, \varepsilon), \quad q = 0, 1, \dots$$

Since  $B_{(q)}$  and  $B$  are the left inverses of  $A^{(q)}$  and  $A$ , respectively, we also have

$$(4.4) \quad B_{(q)}(y, z, \varepsilon) = R_{(q)}(y, z, \varepsilon)B(y, h_\varepsilon, \varepsilon), \quad q = 0, 1, \dots,$$

where  $R_{(q)} \equiv (Q^{(q)})^{-1}$ . Introducing the block structure of  $Q^{(q)}$  and  $R_{(q)}$ ,

$$(4.5) \quad Q^{(q)} = \begin{pmatrix} Q_{1f}^{(q)} & Q_{2f}^{(q)} \\ Q_{1s}^{(q)} & Q_{2s}^{(q)} \end{pmatrix}, \quad R_{(q)} = \begin{pmatrix} R_{(q)}^{1s\perp} & R_{(q)}^{1f\perp} \\ R_{(q)}^{2s\perp} & R_{(q)}^{2f\perp} \end{pmatrix},$$

we rewrite (4.3) and (4.4) as

$$(4.6) \quad A_1^{(q)} = A_f Q_{1f}^{(q)} + A_s Q_{1s}^{(q)}, \quad A_2^{(q)} = A_f Q_{2f}^{(q)} + A_s Q_{2s}^{(q)}$$

and

$$(4.7) \quad B_{(q)}^1 = R_{(q)}^{1s\perp} B^{s\perp} + R_{(q)}^{1f\perp} B^{f\perp}, \quad B_{(q)}^2 = R_{(q)}^{2s\perp} B^{s\perp} + R_{(q)}^{2f\perp} B^{f\perp}$$

for  $q = 0, 1, \dots$ .

Equation (4.7) shows that  $A_s Q_{1s}^{(q)}$  is the projection of  $A_1^{(q)}$  on  $\mathcal{TM}_\varepsilon$ . Thus, to establish Theorem 3.2, we need only to prove the asymptotic estimate  $Q_{1s}^{(q)} = \mathcal{O}(\varepsilon^{q+1})$ . The proof is by induction on  $q$ , where the induction hypothesis is

$$(4.8) \quad Q^{(q)}(\cdot, \psi_{(q)}, \varepsilon) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\varepsilon^q) \\ \mathcal{O}(\varepsilon^{q+1}) & \mathcal{O}(1) \end{pmatrix},$$

$$(4.9) \quad R_{(q)}(\cdot, \psi_{(q)}, \varepsilon) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\varepsilon^q) \\ \mathcal{O}(\varepsilon^{q+1}) & \mathcal{O}(1) \end{pmatrix}, \quad q = 0, 1, \dots$$

*Remark 4.1.* Although the estimate of  $Q_{1s}^{(q)}$  is sufficient to establish Theorem 3.2, we provide the estimates of all the blocks in (4.8)–(4.9) because they will be required in the induction step.

The validity of (4.8)–(4.9) for  $q = 0$  is shown in section 4.3. The induction step is carried out in section 4.4.

**4.3. Proof of Theorem 3.2 for  $q = 0$ .** We fix  $q = 0$  and verify the induction hypothesis for  $Q^{(0)}$  and  $R_{(0)}$ . By (4.3)

$$(4.10) \quad Q^{(0)} = BA^{(0)},$$

whence

$$(4.11) \quad Q^{(0)} = \begin{pmatrix} B^{s\perp} A_1^{(0)} & B^{s\perp} A_2^{(0)} \\ B^{f\perp} A_1^{(0)} & B^{f\perp} A_2^{(0)} \end{pmatrix}.$$

It suffices to show that the lower-left block is zero to leading order, since the other blocks are all  $\mathcal{O}(1)$ . We do this by showing that  $Q_{1s}^{(0,0)} = 0$ . By (4.11),

$$(4.12) \quad Q_{1s}^{(0,0)} = B_0^{f\perp} A_1^{(0,0)}.$$

$B_0^{f\perp}$  spans  $\mathcal{N}_p \mathcal{F}_0$  for every  $p \in \mathcal{K}_\varepsilon^{(0)}$ . Also,  $z$  is constant on  $\mathcal{N}_p \mathcal{F}_0$ , so  $B_0^{f\perp} = (B^{1f\perp}, 0)$ , where  $B^{1f\perp}$  is a full-rank matrix of size  $m$ . Last,  $A_1^{(0,0)} = A_1^{(0)} = \begin{pmatrix} 0 \\ A_{21}^{(0)} \end{pmatrix}$  by (3.22). Substituting these expressions for  $B_0^{f\perp}$  and  $A_1^{(0,0)}$  into (4.12), we obtain that  $Q_{1s}^{(0,0)} = 0$ .

The induction hypothesis on  $R_{(0)}$  can be verified either by a similar argument or by recalling that  $R_{(0)} = (Q^{(0)})^{-1}$ , where  $Q^{(0)}$  was shown above to be block-triangular to leading order.

**4.4. Proof of Theorem 3.2 for  $q = 1, 2, \dots$**  We assume that the induction hypothesis (4.8)–(4.9) holds for  $0, 1, \dots, q$  and show that it holds for  $q + 1$ . The proof proceeds in four steps. In Step 1, we derive explicit expressions for  $R_{(q+1)}$  and  $Q^{(q+1)}$  in terms of  $R_{(q)}$  and  $Q^{(q)}$ ; these expressions also involve  $U_{(q)}$  and  $L_{(q)}$ . In Step 2, we derive the leading-order asymptotics of  $U_{(q)}$  and in Step 3 the leading-order asymptotics of  $L_{(q)}$ . Then, in Step 4, we substitute these results into the expressions derived in Step 1 to complete the induction.

*Step 1.* We derive the expressions for  $Q^{(q+1)}$  and  $R_{(q+1)}$ . Equations (4.3) and (4.4), together with the update formulas (3.26) for  $A^{(q)}$  and (3.27) for  $B_{(q)}$ , yield

$$(4.13) \quad Q^{(q+1)} = Q^{(q)}(I - U_{(q)})(I + L_{(q)}),$$

$$(4.14) \quad R_{(q+1)} = (I - L_{(q)})(I + U_{(q)})R_{(q)}.$$

In terms of the constituent blocks, we have

$$(4.15) \quad Q_{1f}^{(q+1)} = Q_{1f}^{(q)} + Q_{2f}^{(q)}L_{(q)} - Q_{1f}^{(q)}U_{(q)}L_{(q)},$$

$$(4.16) \quad Q_{2f}^{(q+1)} = Q_{2f}^{(q)} - Q_{1f}^{(q)}U_{(q)},$$

$$(4.17) \quad Q_{1s}^{(q+1)} = Q_{1s}^{(q)} + Q_{2s}^{(q)}L_{(q)} - Q_{1s}^{(q)}U_{(q)}L_{(q)},$$

$$(4.18) \quad Q_{2s}^{(q+1)} = Q_{2s}^{(q)} - Q_{1s}^{(q)}U_{(q)}$$

and

$$(4.19) \quad R_{(q+1)}^{1s\perp} = R_{(q)}^{1s\perp} + U_{(q)}R_{(q)}^{2s\perp},$$

$$(4.20) \quad R_{(q+1)}^{1f\perp} = R_{(q)}^{1f\perp} + U_{(q)}R_{(q)}^{2f\perp},$$

$$(4.21) \quad R_{(q+1)}^{2s\perp} = R_{(q)}^{2s\perp} - L_{(q)}R_{(q)}^{1s\perp} - L_{(q)}U_{(q)}R_{(q)}^{2s\perp},$$

$$(4.22) \quad R_{(q+1)}^{2f\perp} = R_{(q)}^{2f\perp} - L_{(q)}R_{(q)}^{1f\perp} - L_{(q)}U_{(q)}R_{(q)}^{2f\perp}.$$

*Step 2.* We derive the leading-order asymptotics of the matrix  $U_{(q)}$ .

Recall that  $U_{(q)} = (\Lambda_{(q)}^{11})^{-1}\Lambda_{(q)}^{12}$ . Moreover,  $\Lambda_{(q)}^{11}$  is strictly  $\mathcal{O}(1)$  and  $\Lambda_{(q)}^{12}$  is strictly  $\mathcal{O}(\varepsilon^q)$  by Lemma 4.1. Hence,  $U_{(q)} = U_{(q,q)}\varepsilon^q + \mathcal{O}(\varepsilon^{q+1})$ , with  $U_{(q,q)} = (\Lambda_{(q,0)}^{11})^{-1}\Lambda_{(q,q)}^{12}$ . Therefore, it suffices to derive the leading-order asymptotics of these blocks of  $\Lambda$ .

By definition,  $\Lambda_{(q)} = B_{(q)}[A^{(q)}, g]$ . Therefore,

$$(4.23) \quad \Lambda_{(q)} = \begin{pmatrix} B_{(q)}^1[A_1^{(q)}, g] & B_{(q)}^1[A_2^{(q)}, g] \\ B_{(q)}^2[A_1^{(q)}, g] & B_{(q)}^2[A_2^{(q)}, g] \end{pmatrix}.$$

The individual blocks of  $\Lambda_{(q)}$  are obtained by substituting (4.6) and (4.7) into (4.23). We observe that one-half of all the terms would vanish, were they to be evaluated on  $\mathcal{M}_\varepsilon$ , by virtue of Lemma 3.1. Since they are evaluated on  $\mathcal{K}_\varepsilon^{(q+1)}$ , instead, which is  $\mathcal{O}(\varepsilon^{q+1})$ -close to  $\mathcal{M}_\varepsilon$ , these terms are  $\mathcal{O}(\varepsilon^{q+2})$  and therefore of higher order for each of the blocks; recall Lemma 4.1. Thus,

$$(4.24) \quad \Lambda_{(q)}^{11} = R_{(q)}^{1s\perp} B^{s\perp} [A_f Q_{1f}^{(q)}, g] + R_{(q)}^{1f\perp} B^{f\perp} [A_s Q_{1s}^{(q)}, g],$$

$$(4.25) \quad \Lambda_{(q)}^{12} = R_{(q)}^{1s\perp} B^{s\perp} [A_f Q_{2f}^{(q)}, g] + R_{(q)}^{1f\perp} B^{f\perp} [A_s Q_{2s}^{(q)}, g],$$

$$(4.26) \quad \Lambda_{(q)}^{21} = R_{(q)}^{2s\perp} B^{s\perp} [A_f Q_{1f}^{(q)}, g] + R_{(q)}^{2f\perp} B^{f\perp} [A_s Q_{1s}^{(q)}, g],$$

where the remainders of  $\mathcal{O}(\varepsilon^{q+2})$  have been omitted for brevity. Recalling the definition of the Lie bracket, we rewrite (4.24) as

$$(4.27) \quad \begin{aligned} \Lambda_{(q)}^{11} = & R_{(q)}^{1s\perp} B^{s\perp} \left( (Dg)A_f Q_{1f}^{(q)} - \frac{d}{dt} \left( A_f Q_{1f}^{(q)} \right) \right) \\ & + R_{(q)}^{1f\perp} B^{f\perp} \left( (Dg)A_s Q_{1s}^{(q)} - \frac{d}{dt} \left( A_s Q_{1s}^{(q)} \right) \right), \end{aligned}$$

where we recall that all of the quantities are evaluated at  $(y, \psi_{(q+1)}, \varepsilon)$ . Next,  $(Dg)A_s$  and the two time derivatives in (4.27) are zero to leading order by Lemma A.1 and [34, Lemma A.2], respectively. Therefore, to leading order (4.27) becomes

$$(4.28) \quad \Lambda_{(q,0)}^{11} = R_{(q,0)}^{1s\perp} B_0^{s\perp} (Dg)_0 A_f Q_{1f}^{(q,0)}.$$

Here,  $\Lambda_{(q,0)}^{11}$  stands for the leading-order term in the asymptotic expansion of  $\Lambda_{(q)}^{11}(y, \psi_{(q+1)}(y), \varepsilon)$ , and the right member is the leading order term in the asymptotic expansion of  $(R_{(q)}^{1s\perp} B^{s\perp} (Dg)A_f Q_{1f}^{(q)})(y, h_\varepsilon(y), \varepsilon)$ .

We derive a similar formula for  $\Lambda_{(q,q)}^{12}$ . First, we rewrite (4.25) as

$$(4.29) \quad \begin{aligned} \Lambda_{(q)}^{12} = & R_{(q)}^{1s\perp} B^{s\perp} \left( (Dg)A_f Q_{2f}^{(q)} - \frac{d}{dt} \left( A_f Q_{2f}^{(q)} \right) \right) \\ & + R_{(q)}^{1f\perp} B^{f\perp} \left( (Dg)A_s Q_{2s}^{(q)} - \frac{d}{dt} \left( A_s Q_{2s}^{(q)} \right) \right). \end{aligned}$$

Next,  $Q_{2f}^{(q)} = \mathcal{O}(\varepsilon^q)$ ,  $Q_{2s}^{(q)} = \mathcal{O}(1)$ ,  $R_{(q)}^{1s\perp} = \mathcal{O}(1)$ , and  $R_{(q)}^{1f\perp} = \mathcal{O}(\varepsilon^q)$  by the induction hypothesis (4.8)–(4.9). Thus, [34, Lemma A.2] implies that the two terms in (4.29) involving time derivatives are  $\mathcal{O}(\varepsilon^{q+1})$  and therefore of higher order. Also,  $(Dg)A_s$  is zero to leading order by Lemma A.1, and thus

$$(4.30) \quad \Lambda_{(q,q)}^{12} = R_{(q,0)}^{1s\perp} B_0^{s\perp} (Dg)_0 A_f Q_{2f}^{(q,q)}.$$

We now substitute  $\Lambda_{(q,0)}^{11}$  and  $\Lambda_{(q,q)}^{12}$  from (4.28) and (4.30) in the expression  $U_{(q,q)} = (\Lambda_{(q,0)}^{11})^{-1} \Lambda_{(q,q)}^{12}$  to find the desired expression for  $U_{(q,q)}$  in terms of  $Q^{(q)}$ ,

$$(4.31) \quad U_{(q,q)} = \left( Q_{1f}^{(q,0)} \right)^{-1} Q_{2f}^{(q,q)}.$$

We also need an expression for  $U_{(q,q)}$  in terms of blocks of  $R_{(q)}$ , which we will use in (4.19)–(4.22). Since  $R_{(q)}$  has the near block-diagonal structure given by the induction hypothesis (4.8)–(4.9) and  $Q^{(q)}$  is its inverse, we find that

$$(4.32) \quad Q^{(q)} = \begin{pmatrix} (R_{(q,0)}^{1s\perp})^{-1} & -\varepsilon^q (R_{(q,0)}^{1s\perp})^{-1} R_{(q,q)}^{1f\perp} (R_{(q,0)}^{2f\perp})^{-1} \\ -\varepsilon^{q+1} (R_{(q,0)}^{2f\perp})^{-1} R_{(q,q+1)}^{2s\perp} (R_{(q,0)}^{1s\perp})^{-1} & (R_{(q,0)}^{2f\perp})^{-1} \end{pmatrix}$$

to leading order for each of the blocks and for  $q = 1, 2, \dots$ . Equations (4.31) and (4.32) lead to the desired expression for  $U_{(q,q)}$  in terms of  $R_{(q)}$ ,

$$(4.33) \quad U_{(q,q)} = -R_{(q,q)}^{1f\perp} \left( R_{(q,0)}^{2f\perp} \right)^{-1}.$$

*Step 3.* We derive the leading-order asymptotics of the matrix  $L_{(q)}$ .

Recall that  $L_{(q)} = \Lambda_{(q)}^{21} (\Lambda_{(q)}^{11})^{-1}$ . Moreover, by Lemma 4.1,  $\Lambda_{(q)}^{11}$  is strictly  $\mathcal{O}(1)$  and  $\Lambda_{(q)}^{21}$  is strictly  $\mathcal{O}(\varepsilon^{q+1})$ . Hence,  $L_{(q)} = L_{(q,q+1)} \varepsilon^{q+1} + \mathcal{O}(\varepsilon^{q+2})$ , with  $L_{(q,q+1)} = \Lambda_{(q,q+1)}^{21} (\Lambda_{(q,0)}^{11})^{-1}$ . An expression for  $\Lambda_{(q,0)}^{11}$  was derived in (4.28), so here we focus on  $\Lambda_{(q,q+1)}^{21}$ .

Equation (4.26) and the definition of the Lie bracket imply that

$$(4.34) \quad \begin{aligned} \Lambda_{(q)}^{21} = & R_{(q)}^{2s\perp} B^{s\perp} \left( (Dg) A_f Q_{1f}^{(q)} - \frac{d}{dt} \left( A_f Q_{1f}^{(q)} \right) \right) \\ & + R_{(q)}^{2f\perp} B^{f\perp} \left( (Dg) A_s Q_{1s}^{(q)} - \frac{d}{dt} \left( A_s Q_{1s}^{(q)} \right) \right). \end{aligned}$$

Next,  $Q_{1f}^{(q)} = \mathcal{O}(1)$ ,  $Q_{1s}^{(q)} = \mathcal{O}(\varepsilon^{q+1})$ ,  $R_{(q)}^{2s\perp} = \mathcal{O}(\varepsilon^{q+1})$ , and  $R_{(q)}^{2f\perp} = \mathcal{O}(1)$  by the induction hypothesis. Also, the time derivatives are  $\mathcal{O}(\varepsilon)$  by [34, Lemma A.2], and thus the two terms in (4.34) that involve time derivatives are  $\mathcal{O}(\varepsilon^{q+2})$ . Last,  $(Dg)A_s = \mathcal{O}(\varepsilon)$  by Lemma A.1. Thus, we find that

$$(4.35) \quad \Lambda_{(q,q+1)}^{21} = R_{(q,q+1)}^{2s\perp} B_0^{s\perp} (Dg)_0 A_f^0 Q_{1f}^{(q,0)}.$$

Equations (4.28) and (4.35) yield the desired formula for  $L_{(q,q+1)}$  in terms of the blocks of  $R_{(q)}$ ,

$$(4.36) \quad L_{(q,q+1)} = \Lambda_{(q,q+1)}^{21} \left( \Lambda_{(q,0)}^{11} \right)^{-1} = R_{(q,q+1)}^{2s\perp} \left( R_{(q,0)}^{1s\perp} \right)^{-1}.$$

Next, we recast (4.36) in terms of blocks of  $Q^{(q)}$  in order to use it in (4.15)–(4.18). The matrix  $R_{(q)}$  is the inverse of  $Q^{(q)}$  and has the near block-diagonal form given in (4.9). Thus,

$$(4.37) \quad R_{(q)} = \begin{pmatrix} (Q_{1f}^{(q,0)})^{-1} & -\varepsilon^q (Q_{1f}^{(q,0)})^{-1} Q_{2f}^{(q,q)} (Q_{2s}^{(q,0)})^{-1} \\ -\varepsilon^{q+1} (Q_{2s}^{(q,0)})^{-1} Q_{1s}^{(q,q+1)} (Q_{1f}^{(q,0)})^{-1} & (Q_{2s}^{(q,0)})^{-1} \end{pmatrix}$$

to leading order for each block and for  $q = 1, 2, \dots$ . Equations (4.36) and (4.37) lead to the desired expression for  $L_{(q,q+1)}$  in terms of the blocks of  $Q^{(q)}$ ,

$$(4.38) \quad L_{(q,q+1)} = - \left( Q_{2s}^{(q,0)} \right)^{-1} Q_{1s}^{(q,q+1)}.$$

*Step 4.* We substitute the results obtained in Steps 2 and 3 into the formulas (4.15)–(4.22) derived in Step 1.

Equations (4.15) and (4.18), together with the induction hypothesis and the estimates  $U_{(q)} = \mathcal{O}(\varepsilon^q)$  and  $L_{(q)} = \mathcal{O}(\varepsilon^{q+1})$ , imply that  $Q_{1f}^{(q+1)}$  and  $Q_{2s}^{(q+1)}$  remain  $\mathcal{O}(1)$ . This concludes the estimation of these blocks.

Next, we show that  $Q_{2f}^{(q+1)} = \mathcal{O}(\varepsilon^{q+1})$ . First,  $Q_{2f}^{(q+1)}$  and  $Q_{2f}^{(q)}$  are equal up to and including terms of  $\mathcal{O}(\varepsilon^{q-1})$  by (4.16) and the estimate on  $U_{(q)}$ . Thus,  $Q_{2f}^{(q+1,i)} = 0$  for  $i = 0, 1, \dots, q - 1$  by the induction hypothesis on  $Q_{2f}^{(q)}$ . It suffices to show that  $Q_{2f}^{(q+1,q)} = 0$ . Equation (4.16) implies that

$$(4.39) \quad Q_{2f}^{(q+1,q)} = Q_{2f}^{(q,q)} - Q_{1f}^{(q,0)}U_{(q,q)}.$$

The right member of this equation is zero by (4.31), and the estimation of  $Q_{2f}^{(q+1)}$  is complete.

Finally, we show that  $Q_{1s}^{(q+1)} = \mathcal{O}(\varepsilon^{q+2})$  to complete the estimates on the blocks of  $Q^{(q+1)}$ . First,  $Q_{1s}^{(q+1)}$  and  $Q_{1s}^{(q)}$  are equal up to and including terms of  $\mathcal{O}(\varepsilon^q)$  by (4.17) and the order estimates on  $U_{(q)}$  and  $L_{(q)}$ . Thus,  $Q_{1s}^{(q+1,i)} = 0$  for  $i = 0, 1, \dots, q$  by the induction hypothesis on  $Q_{1s}^{(q)}$ . It suffices to show that  $Q_{1s}^{(q+1,q+1)} = 0$ . Equation (4.17) implies that

$$(4.40) \quad Q_{1s}^{(q+1,q+1)} = Q_{1s}^{(q,q+1)} + Q_{2s}^{(q,0)}L_{(q,q+1)},$$

where the right member of this equation is zero by (4.38). The estimation of  $Q_{1s}^{(q+1)}$  is complete.

The blocks of  $R_{(q)}$  may be estimated in an entirely similar manner, using (4.19)–(4.22) instead of (4.15)–(4.18) and (4.33) and (4.36) instead of (4.31) and (4.38). The proof of Theorem 3.2 is complete.

**5. The Michaelis–Menten–Henri model.** In this section, we illustrate Theorem 3.2 by applying the CSP method to the Michaelis–Menten–Henri (MMH) mechanism of enzyme kinetics [24, 25]. We consider the planar system of ordinary differential equations for a slow variable  $s$  and a fast variable  $c$ ,

$$(5.1) \quad s' = \varepsilon(-s + (s + \kappa - \lambda)c),$$

$$(5.2) \quad c' = s - (s + \kappa)c.$$

The parameters satisfy the inequalities  $0 < \varepsilon \ll 1$  and  $\kappa \gg \lambda > 0$ . Only nonnegative values of  $s$  and  $c$  are relevant. The system of equations (5.1)–(5.2) is of the form (2.2)–(2.3) with  $m = 1$ ,  $n = 1$ ,  $y = s$ ,  $z = c$ ,  $g_1 = -s + (s + \kappa - \lambda)c$ , and  $g_2 = s - (s + \kappa)c$ .

**5.1. Slow manifolds and fast fibers.** In the limit as  $\varepsilon \downarrow 0$ , the dynamics of the MMH equations are confined to the reduced slow manifold,

$$(5.3) \quad \mathcal{M}_0 = \left\{ (c, s) : c = \frac{s}{s + \kappa}, s \geq 0 \right\}.$$

The manifold  $\mathcal{M}_0$  is asymptotically stable, so there exists a locally invariant slow manifold  $\mathcal{M}_\varepsilon$  for all sufficiently small  $\varepsilon$  that is  $\mathcal{O}(\varepsilon)$  close to  $\mathcal{M}_0$  on any compact set. Moreover,  $\mathcal{M}_\varepsilon$  is the graph of a function  $h_\varepsilon$ ,

$$(5.4) \quad \mathcal{M}_\varepsilon = \{(c, s) : c = h_\varepsilon(s), s \geq 0\},$$

and  $h_\varepsilon$  admits an asymptotic expansion,  $h_\varepsilon = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \dots$ . The coefficients are found from the invariance equation,

$$(5.5) \quad s - (s + \kappa)h_\varepsilon(s) = \varepsilon h'_\varepsilon(s)(-s + (s + \kappa - \lambda)h_\varepsilon(s)).$$

The first few coefficients are

$$(5.6) \quad h_0(s) = \frac{s}{s + \kappa}, \quad h_1(s) = \frac{\kappa \lambda s}{(s + \kappa)^4}, \quad h_2(s) = \frac{\kappa \lambda s(2\kappa \lambda - 3\lambda s - \kappa s - \kappa^2)}{(s + \kappa)^7}.$$

In the limit as  $\varepsilon \downarrow 0$ , each line of constant  $s$  is trivially invariant under (5.1)–(5.2). These are the (one-dimensional) fast fibers  $\mathcal{F}_0^p$  with base point  $p = (s, h_0(s)) \in \mathcal{M}_0$ . All points on  $\mathcal{F}_0^p$  contract exponentially fast to  $p$  with rate constant  $-(s + \kappa)$ . The fast fiber  $\mathcal{F}_0^p$  perturbs to a curve  $\mathcal{F}_\varepsilon^p$  that is  $\mathcal{O}(\varepsilon)$ -close to  $\mathcal{F}_0^p$  in any compact neighborhood of  $\mathcal{M}_\varepsilon$ . The fast fibers  $\mathcal{F}_\varepsilon^p$ ,  $p \in \mathcal{M}_\varepsilon$ , form an invariant family.

**5.2. Asymptotic expansions of the fast fibers.** To derive asymptotic information about the fast fibers, we look for general solutions of (5.1)–(5.2) that are given by asymptotic expansions,

$$(5.7) \quad s(t; \varepsilon) = \sum_{i=0} \varepsilon^i s_i(t), \quad c(t; \varepsilon) = \sum_{i=0} \varepsilon^i c_i(t),$$

where the coefficients  $s_i$  and  $c_i$  are determined order by order.

Consider the fast fiber  $\mathcal{F}_\varepsilon^p$  with base point  $p = (s, h_\varepsilon(s))$ , and let  $(s^A, c^A)$  and  $(s^B, c^B)$  be two points on it; let  $\Delta s(t) = s^B(t) - s^A(t)$  and  $\Delta c(t) = c^B(t) - c^A(t)$ . The distance between any two points on the same fast fiber will contract exponentially fast towards zero at the  $\mathcal{O}(1)$  rate, as long as both points are chosen in a neighborhood of  $\mathcal{M}_\varepsilon$ . We may write

$$(5.8) \quad \Delta s(t; \varepsilon) = \sum_{i=0} \varepsilon^i \Delta s_i(t), \quad \Delta c(t; \varepsilon) = \sum_{i=0} \varepsilon^i \Delta c_i(t),$$

where  $\Delta s_i(t) = s_i^B(t) - s_i^A(t)$  and  $\Delta c_i(t) = c_i^B(t) - c_i^A(t)$ . The condition on fast exponential decay of  $\Delta s(t)$  and  $\Delta c(t)$  translates into

$$(5.9) \quad \Delta s_i(t) = \mathcal{O}(e^{-C_s t}), \quad \Delta c_i(t) = \mathcal{O}(e^{-C_c t}), \quad t \rightarrow \infty,$$

for some positive constants  $C_s$  and  $C_c$ . We let  $(s^A, c^A)$  and  $(s^B, c^B)$  be infinitesimally close, since we are interested in vectors tangent to the fast fiber.

**5.2.1.  $\mathcal{O}(1)$  fast fibers.** Substituting the expansions (5.7) into (5.1)–(5.2) and equating  $\mathcal{O}(1)$  terms, we find

$$(5.10) \quad s'_0 = 0,$$

$$(5.11) \quad c'_0 = s_0 - (s_0 + \kappa)c_0.$$

The equations can be integrated,

$$(5.12) \quad s_0(t) = s_0(0) = s_0,$$

$$(5.13) \quad c_0(t) = \frac{s_0}{s_0 + \kappa} + \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right) e^{-(s_0 + \kappa)t}.$$

Hence,

$$(5.14) \quad \Delta s_0(t) = \Delta s_0(0),$$

$$(5.15) \quad \Delta c_0(t) = \Delta c_0(0)e^{-(s_0+\kappa)t} + (\partial_{s_0}c_0(t))\Delta s_0(0) + \mathcal{O}((\Delta s_0(0))^2).$$

The points  $A$  and  $B$  lie on the same fiber if and only if

$$(5.16) \quad \Delta s_0(0) = 0.$$

Thus, (5.15) simplifies to

$$(5.17) \quad \Delta c_0(t) = \Delta c_0(0)e^{-(s_0+\kappa)t},$$

and  $\Delta c_0(t)$  decays exponentially towards zero, irrespective of the choice of  $\Delta c_0(0)$ . Hence,  $\Delta c_0(0)$  is a free parameter.

We conclude that, to  $\mathcal{O}(1)$ , any vector  $\begin{pmatrix} 0 \\ \alpha \end{pmatrix}$  with  $\alpha$  constant ( $\alpha \neq 0$ ) is tangent to every fast fiber at the base point.

**5.2.2.  $\mathcal{O}(\varepsilon)$  fast fibers.** At  $\mathcal{O}(\varepsilon)$ , we obtain the equations

$$(5.18) \quad s_1' = -s_0 + (s_0 + \kappa - \lambda)c_0,$$

$$(5.19) \quad c_1' = s_1 - (s_0 + \kappa)c_1 - s_1c_0.$$

Using (5.12) and (5.13), we integrate (5.18) to obtain

$$(5.20) \quad s_1(t) = s_1(0) - \frac{\lambda s_0}{s_0 + \kappa}t + \frac{s_0 + \kappa - \lambda}{s_0 + \kappa} \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right) (1 - e^{-(s_0+\kappa)t}).$$

Therefore, at  $\mathcal{O}(\varepsilon)$ ,

$$(5.21) \quad \Delta s_1(t) = \Delta s_1(0) + \frac{s_0 + \kappa - \lambda}{s_0 + \kappa} \Delta c_0(0) (1 - e^{-(s_0+\kappa)t}).$$

For the two points to have the same phase asymptotically, it is necessary that  $\lim_{t \rightarrow \infty} \Delta s_1(t) = 0$ . This condition is satisfied if and only if

$$(5.22) \quad \Delta s_1(0) = -\frac{s_0 + \kappa - \lambda}{s_0 + \kappa} \Delta c_0(0).$$

Next,  $c_1(t)$  follows upon integration of (5.19),

$$(5.23) \quad \begin{aligned} c_1(t) &= c_1(0)e^{-(s_0+\kappa)t} \\ &+ \frac{\kappa}{(s_0 + \kappa)^2} \left( s_1(0) + \frac{s_0 + \kappa - \lambda}{s_0 + \kappa} \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right) \right) (1 - e^{-(s_0+\kappa)t}) \\ &- \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right) \left( s_1(0) + \frac{s_0 + \kappa - \lambda}{s_0 + \kappa} \left( c_0(0) + \frac{\kappa - s_0}{s_0 + \kappa} \right) \right) te^{-(s_0+\kappa)t} \\ &- \frac{s_0 + \kappa - \lambda}{(s_0 + \kappa)^2} \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right)^2 (e^{-2(s_0+\kappa)t} - e^{-(s_0+\kappa)t}) \\ &+ \frac{\lambda s_0}{2(s_0 + \kappa)} \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right) t^2 e^{-(s_0+\kappa)t} \\ &- \frac{\kappa \lambda s_0}{(s_0 + \kappa)^4} (e^{-(s_0+\kappa)t} + (s_0 + \kappa)t - 1). \end{aligned}$$

We infer from this expression that  $\lim_{t \rightarrow \infty} \Delta c_1(t) = 0$ , as long as (5.22) and (5.16) hold. Hence,  $\Delta c_1(0)$  is a free parameter, just like  $\Delta c_0(0)$ , and the only condition that arises at  $\mathcal{O}(\varepsilon)$  is (5.22) on  $\Delta s_1(0)$ .

We conclude that any vector

$$(5.24) \quad \begin{pmatrix} 0 \\ \alpha \end{pmatrix} + \varepsilon \begin{pmatrix} -\left(1 - \frac{\lambda}{s_0 + \kappa}\right) \alpha \\ \beta \end{pmatrix},$$

with  $\alpha$  and  $\beta$  constant ( $\alpha \neq 0$ ), is tangent to every fast fiber at the base point up to and including terms of  $\mathcal{O}(\varepsilon)$ . Any such vector may be written as the product of a free parameter and a constant vector (fixed by  $s_0$ ),

$$(5.25) \quad (\alpha + \varepsilon\beta) \begin{pmatrix} -\varepsilon \left(1 - \frac{\lambda}{s_0 + \kappa}\right) \\ 1 \end{pmatrix} + \mathcal{O}(\varepsilon^2).$$

**5.2.3.  $\mathcal{O}(\varepsilon^2)$  fast fibers.** At  $\mathcal{O}(\varepsilon^2)$ , we obtain the equation

$$(5.26) \quad s_2' = s_1(c_0 - 1) + (s_0 + \kappa - \lambda)c_1.$$

Direct integration yields

$$(5.27) \quad \begin{aligned} s_2(t) = s_2(0) &+ \left[ \frac{\lambda}{(s_0 + \kappa)^2} \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right) - \frac{\kappa(s_0 + \kappa - \lambda)}{(s_0 + \kappa)^3} \right] s_1(0) \\ &- \left[ \frac{\kappa(s_0 + \kappa - \lambda)(s_0 + \kappa - 2\lambda) + \lambda^2 s_0}{(s_0 + \kappa)^4} \right] \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right) \\ &+ \frac{\lambda(s_0 + \kappa - \lambda)}{2(s_0 + \kappa)^3} \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right)^2 \\ &+ \left( 1 - \frac{\lambda}{s_0 + \kappa} \right) \left( c_1(0) - \frac{\kappa\lambda s_0}{(s_0 + \kappa)^4} \right) \\ &- \frac{\kappa\lambda}{(s_0 + \kappa)^2} \left[ s_1(0) + \frac{s_0 + \kappa - \lambda}{s_0 + \kappa} \left( c_0(0) - \frac{2s_0}{s_0 + \kappa} \right) \right] t \\ &+ \frac{\kappa\lambda^2 s_0}{2(s_0 + \kappa)^3} t^2 + \mathcal{R}(t), \end{aligned}$$

where the remainder  $\mathcal{R}(t)$  involves the functions  $e^{-(s_0 + \kappa)t}$ ,  $te^{-(s_0 + \kappa)t}$ ,  $t^2e^{-(s_0 + \kappa)t}$ , and  $e^{-2(s_0 + \kappa)t}$ . From this expression we find

$$(5.28) \quad \begin{aligned} \Delta s_2(t) = \Delta s_2(0) &+ (\partial_{s_0} s_2(t)) \Delta s_0(0) + (\partial_{c_0} s_2(t)) \Delta c_0(0) \\ &+ (\partial_{s_1} s_2(t)) \Delta s_1(0) + (\partial_{c_1} s_2(t)) \Delta c_1(0) + \mathcal{O}(2) + \mathcal{O}(e^{-Ct}) \end{aligned}$$

for some  $C > 0$ . Here,  $\partial_{c_0}$  is an abbreviation for the partial derivative  $\partial_{c_0(0)}$ , and so on, and  $\mathcal{O}(2)$  denotes quadratic terms in the multivariable Taylor expansion. First, we recall that  $\Delta s_0(0) = 0$  by (5.16). Next, we calculate the partial derivatives in each of the three remaining terms,

$$(5.29) \quad \begin{aligned} \partial_{c_0} s_2(t) = \frac{\lambda s_1(0)}{(s_0 + \kappa)^2} &- \frac{\kappa(s_0 + \kappa - \lambda)(s_0 + \kappa - 2\lambda) + \lambda^2 s_0}{(s_0 + \kappa)^4} \\ &+ \frac{\lambda(s_0 + \kappa - \lambda)}{(s_0 + \kappa)^3} \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right) - \frac{\kappa\lambda(s_0 + \kappa - \lambda)}{(s_0 + \kappa)^3} t, \end{aligned}$$

$$(5.30) \quad \partial_{s_1} s_2(t) = \frac{\lambda}{(s_0 + \kappa)^2} \left( c_0(0) - \frac{s_0}{s_0 + \kappa} \right) - \frac{\kappa(s_0 + \kappa - \lambda)}{(s_0 + \kappa)^3} - \frac{\kappa\lambda}{(s_0 + \kappa)^2} t,$$

$$(5.31) \quad \partial_{c_1} s_2(t) = 1 - \frac{\lambda}{s_0 + \kappa}.$$

We substitute these expressions into (5.28), recall (5.22), and carry out the algebra to obtain

$$(5.32) \quad \begin{aligned} \Delta s_2(t) &= \Delta s_2(0) + \left( 1 - \frac{\lambda}{s_0 + \kappa} \right) \Delta c_1(0) \\ &\quad + \frac{\lambda}{(s_0 + \kappa)^2} \left( s_1(0) + \frac{\kappa(s_0 + \kappa - \lambda) - \lambda s_0}{(s_0 + \kappa)^2} \right) \Delta c_0(0) \\ &\quad + \mathcal{O}(2) + \mathcal{O}(e^{-Ct}), \quad C > 0. \end{aligned}$$

In the limit  $t \rightarrow \infty$ , (5.32) yields the condition

$$(5.33) \quad \begin{aligned} \Delta s_2(0) &= - \left( 1 - \frac{\lambda}{s_0 + \kappa} \right) \Delta c_1(0) \\ &\quad - \frac{\lambda}{(s_0 + \kappa)^2} \left( s_1(0) + \frac{\kappa(s_0 + \kappa - \lambda) - \lambda s_0}{(s_0 + \kappa)^2} \right) \Delta c_0(0). \end{aligned}$$

Finally,  $\Delta c_2(t)$  vanishes exponentially, as follows directly from the conditions (5.22) and (5.33). Thus, no further conditions besides (5.33) arise at  $\mathcal{O}(\varepsilon^2)$ .

We conclude that any vector

$$(5.34) \quad \begin{aligned} &\begin{pmatrix} 0 \\ \alpha \end{pmatrix} + \varepsilon \begin{pmatrix} - \left( 1 - \frac{\lambda}{s_0 + \kappa} \right) \alpha \\ \beta \end{pmatrix} \\ &+ \varepsilon^2 \begin{pmatrix} - \left( 1 - \frac{\lambda}{s_0 + \kappa} \right) \beta - \frac{\lambda}{(s_0 + \kappa)^2} \left( s_1(0) + \frac{\kappa(s_0 + \kappa - \lambda) - \lambda s_0}{(s_0 + \kappa)^2} \right) \alpha \\ \gamma \end{pmatrix}, \end{aligned}$$

with  $\alpha$ ,  $\beta$ , and  $\gamma$  constant ( $\alpha \neq 0$ ), is tangent to every fiber at the base point up to and including terms of  $\mathcal{O}(\varepsilon^2)$ .

**5.3. CSP approximations of the fast fibers.** We choose the stoichiometric vectors as the basis vectors, so

$$(5.35) \quad A^{(0)} = (A_1^{(0)}, A_2^{(0)}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_{(0)} = \begin{pmatrix} B_{(0)}^1 \\ B_{(0)}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The CSP condition  $B_{(0)}^1 g = 0$  is satisfied if  $c = h_0(s)$ , so the CSP manifold  $\mathcal{K}_\varepsilon^{(0)}$  coincides with  $\mathcal{M}_0$ . With this choice of initial basis, we have

$$(5.36) \quad \Lambda_{(0)} = B_{(0)}(Dg)A^{(0)} = \begin{pmatrix} -(s + \kappa) & -(c - 1) \\ \varepsilon(s + \kappa - \lambda) & \varepsilon(c - 1) \end{pmatrix}.$$

**5.3.1. First iteration.** At any point  $(s, c)$ , we have

$$(5.37) \quad A_1^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon \frac{s + \kappa - \lambda}{s + \kappa} \begin{pmatrix} -1 \\ \frac{c-1}{s+\kappa} \end{pmatrix}, \quad A_2^{(1)} = \begin{pmatrix} 1 \\ -\frac{c-1}{s+\kappa} \end{pmatrix},$$

$$(5.38) \quad B_{(1)}^1 = \left(-A_{22}^{(1)}, A_{12}^{(1)}\right), \quad B_{(1)}^2 = \left(A_{21}^{(1)}, -A_{11}^{(1)}\right).$$

In the first step, we evaluate  $A_2^{(1)}$  and  $B_{(1)}^1$  on  $\mathcal{K}_\varepsilon^{(0)}$  to obtain

$$(5.39) \quad A_2^{(1)} = \begin{pmatrix} 1 \\ \frac{\kappa}{(s+\kappa)^2} \end{pmatrix}, \quad B_{(1)}^1 = \left(-\frac{\kappa}{(s + \kappa)^2}, 1\right).$$

Hence, the CSP condition,

$$(5.40) \quad B_{(1)}^1 g = s - (s + \kappa)c - \varepsilon \frac{\kappa(-s + (s + \kappa - \lambda)c)}{(s + \kappa)^2} = 0,$$

is satisfied if

$$(5.41) \quad c = \frac{s}{s + \kappa} + \varepsilon \frac{\kappa \lambda s}{(s + \kappa)^4} - \varepsilon^2 \frac{\kappa^2 \lambda s (s + \kappa - \lambda)}{(s + \kappa)^7} + \mathcal{O}(\varepsilon^3).$$

Equation (5.41) defines  $\mathcal{K}_\varepsilon^{(1)}$ , the CSPM of order one, which agrees with  $\mathcal{M}_\varepsilon$  up to and including terms of  $\mathcal{O}(\varepsilon)$ ; recall (5.6).

Then, in the second step, the new fast basis vector,  $A_1^{(1)}$ , and its complement,  $B_{(1)}^2$ , in the dual basis are evaluated on  $\mathcal{K}_\varepsilon^{(1)}$ ,

$$(5.42) \quad A_1^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \varepsilon \begin{pmatrix} 1 \\ \frac{\kappa(s+\kappa-\lambda)}{(s+\kappa)^3} \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 \\ \frac{\kappa \lambda s (s + \kappa - \lambda)}{(s + \kappa)^6} \end{pmatrix} + \mathcal{O}(\varepsilon^3),$$

$$(5.43) \quad B_{(1)}^2 = \left(A_{21}^{(1)}, -A_{11}^{(1)}\right).$$

Thus, we see that  $A_1^{(1)}$  is tangent to the fast fibers at their base points up to and including terms of  $\mathcal{O}(\varepsilon)$ , as (5.24) (with  $\alpha = 1, \beta = -\frac{\kappa(s+\kappa-\lambda)}{(s+\kappa)^3}$ ) implies. As a result,  $\mathcal{L}_\varepsilon^{(1)}$  approximates  $\mathcal{T}\mathcal{F}_\varepsilon$  also up to and including terms of  $\mathcal{O}(\varepsilon)$ .

*Remark 5.1.* If one evaluates  $A_1^{(1)}$  on  $\mathcal{K}_\varepsilon^{(0)}$ , as opposed to  $\mathcal{K}_\varepsilon^{(1)}$  as we did above, then the approximation of  $\mathcal{T}\mathcal{F}_\varepsilon$  is also accurate up to and including terms of  $\mathcal{O}(\varepsilon)$ . See also Appendix B.

**5.3.2. Second iteration.** The blocks of  $\Lambda_{(1)}$  are

$$(5.44) \quad \Lambda_{(1)}^{11} = -(s + \kappa) + \varepsilon \frac{(s + \kappa - \lambda)}{s + \kappa} \left[ (c - 1) + \left( c - \frac{s}{s + \kappa} \right) \right] + \varepsilon^2 \frac{(c - 1)(s + \kappa - \lambda)}{(s + \kappa)^3} \left[ -\lambda(c - 1) + [(s + \kappa - \lambda)c - s] \right],$$

$$(5.45) \quad \Lambda_{(1)}^{12} = \frac{s}{s + \kappa} - c + \varepsilon \frac{c - 1}{(s + \kappa)^2} \left[ \lambda(c - 1) - [(s + \kappa - \lambda)c - s] \right],$$

$$(5.46) \quad \Lambda_{(1)}^{21} = \frac{\varepsilon^2}{(s + \kappa)^2} \left[ (c - 1)(s + \kappa - \lambda)(s + \kappa - 2\lambda) + \lambda[(s + \kappa - \lambda)c - s] + (s + \kappa - \lambda)^2 \left( c - \frac{s}{s + \kappa} \right) \right],$$

$$(5.47) \quad \Lambda_{(1)}^{22} = \frac{\varepsilon}{s+\kappa} \left[ \lambda(c-1) + (s+\kappa-\lambda) \left( \frac{s}{s+\kappa} - c \right) \right] \\ + \varepsilon^2 \frac{(c-1)(s+\kappa-\lambda)}{(s+\kappa)^3} \left[ \lambda(c-1) - [(s+\kappa-\lambda)c-s] \right],$$

with remainders of  $\mathcal{O}(\varepsilon^3)$ .

In the first step, we update  $A_2^{(1)}$  and  $B_{(1)}^1$  and evaluate the updated quantities on  $\mathcal{K}_\varepsilon^{(1)}$  to obtain

$$(5.48) \quad A_{12}^{(2)} = 1 + \varepsilon^2 \frac{\kappa\lambda(2s-\kappa)(s+\kappa-\lambda)}{(s+\kappa)^6},$$

$$(5.49) \quad A_{22}^{(2)} = \frac{\kappa}{(s+\kappa)^2} + \varepsilon \frac{\kappa\lambda(\kappa-3s)}{(s+\kappa)^5} \\ + \varepsilon^2 \frac{\kappa^2\lambda(7s-2\kappa)(s+\kappa-\lambda) + \kappa\lambda^2s(s-2\kappa)}{(s+\kappa)^8},$$

$$(5.50) \quad B_{(2)}^1 = \left( -A_{22}^{(2)}, A_{12}^{(2)} \right)$$

up to and including terms of  $\mathcal{O}(\varepsilon^2)$ .

The CSP condition

$$(5.51) \quad B_{(2)}^1 g = s - (s+\kappa)c - \varepsilon \frac{\kappa(-s+(s+\kappa-\lambda)c)}{(s+\kappa)^2} \\ + \varepsilon^2 \kappa\lambda \left( \frac{(3s-\kappa)(-s+(s+\kappa-\lambda)c)}{(s+\kappa)^5} \right. \\ \left. + \frac{(2s-\kappa)(s+\kappa-\lambda)(s-(s+\kappa)c)}{(s+\kappa)^6} \right) + \mathcal{O}(\varepsilon^3) \\ = 0$$

is satisfied if

$$(5.52) \quad c = \frac{s}{s+\kappa} + \varepsilon \frac{\kappa\lambda s}{(s+\kappa)^4} + \varepsilon^2 \frac{\kappa\lambda s(2\kappa\lambda - 3\lambda s - \kappa s - \kappa^2)}{(s+\kappa)^7} + \mathcal{O}(\varepsilon^3).$$

Equation (5.52) defines  $\mathcal{K}_\varepsilon^{(2)}$ , the CSPM of order two, which agrees with  $\mathcal{M}_\varepsilon$  up to and including terms of  $\mathcal{O}(\varepsilon^2)$ ; recall (5.6).

Then, in the second step, we update  $A_1^{(1)}$  and  $B_{(1)}^2$  to obtain

$$(5.53) \quad A_{11}^{(2)} = -\varepsilon \frac{s+\kappa-\lambda}{s+\kappa} - \varepsilon^2 \frac{1}{(s+\kappa)^3} \left[ (s+\kappa-\lambda)(s+\kappa-2\lambda)(c-1) \right. \\ \left. + (s+\kappa-\lambda)^2 \left( c - \frac{s}{s+\kappa} \right) + \lambda[(s+\kappa-\lambda)c-s] \right].$$

$$(5.54) \quad A_{21}^{(2)} = 1 + \varepsilon \frac{(s+\kappa-\lambda)(c-1)}{(s+\kappa)^2} + \varepsilon^2 \frac{1}{(s+\kappa)^4} \left[ (s+\kappa-\lambda) \left[ (s+\kappa-2\lambda)(c-1) \right. \right. \\ \left. \left. + (s+\kappa-\lambda) \left( c - \frac{s}{s+\kappa} \right) + \lambda c \right] - \lambda s \right] \left( 2c - \frac{2s+\kappa}{s+\kappa} \right),$$

$$(5.55) \quad B_{(2)}^2 = \left( A_{21}^{(2)}, -A_{11}^{(2)} \right),$$

with remainders of  $\mathcal{O}(\varepsilon^3)$ . Evaluating these expressions on  $\mathcal{K}_\varepsilon^{(2)}$ , we obtain

$$(5.56) \quad A_{11}^{(2)} = -\varepsilon \frac{s + \kappa - \lambda}{s + \kappa} + \varepsilon^2 \frac{\kappa(s + \kappa - 2\lambda)(s + \kappa - \lambda) + \lambda^2 s}{(s + \kappa)^4},$$

$$(5.57) \quad \begin{aligned} A_{21}^{(2)} &= 1 - \varepsilon \frac{\kappa(s + \kappa - \lambda)}{(s + \kappa)^3} \\ &\quad + \varepsilon^2 \frac{(s + \kappa - \lambda)(\kappa^2(s + \kappa - 2\lambda) + \kappa\lambda s) + \kappa\lambda^2 s}{(s + \kappa)^6}, \end{aligned}$$

$$(5.58) \quad B_{(2)}^2 = \left( A_{21}^{(2)}, -A_{11}^{(2)} \right),$$

with remainders of  $\mathcal{O}(\varepsilon^3)$ . Therefore,  $A_1^{(2)}$  is tangent to the fast fibers at their base points up to and including terms of  $\mathcal{O}(\varepsilon^2)$ , according to (5.34) (with  $\alpha = 1$ ,  $\beta = -\frac{\kappa(s + \kappa - \lambda)}{(s + \kappa)^3}$ ,  $\gamma = \frac{(s + \kappa - \lambda)(\kappa^2(s + \kappa - 2\lambda) + \kappa\lambda s) + \kappa\lambda^2 s}{(s + \kappa)^6}$ ), and  $\mathcal{L}_\varepsilon^{(2)}$  is an  $\mathcal{O}(\varepsilon^2)$ -accurate approximation to  $\mathcal{T}\mathcal{F}_\varepsilon$ .

*Remark 5.2.* If one evaluates  $A_1^{(2)}$  on  $\mathcal{K}_\varepsilon^{(1)}$  instead of on  $\mathcal{K}_\varepsilon^{(2)}$ , as we did above, then the approximation of  $\mathcal{T}\mathcal{F}_\varepsilon$  is also accurate up to and including terms of  $\mathcal{O}(\varepsilon^2)$ .

**6. Linear projection of initial conditions.** The main result of this article, Theorem 3.2, states that after  $q$  iterations the CSP method successfully identifies  $\mathcal{T}\mathcal{F}_\varepsilon$  up to and including terms of  $\mathcal{O}(\varepsilon^{q+1})$ , where this approximation is given explicitly by  $A_1^{(q)}$ . This information is postprocessed to project the initial conditions on the CSPM of order  $q$ . In this section, we discuss the accuracy and limitations of this linear projection.

Geometrically, one knows from Fenichel’s theory that any given initial condition  $x_0$  sufficiently close to  $\mathcal{M}_\varepsilon$  lies on a (generally nonlinear) fiber  $\mathcal{F}_\varepsilon^p$  with base point  $p$  on  $\mathcal{M}_\varepsilon$ . Hence, the ideal projection would be  $\pi_F(x_0) = p$  (the subscript  $F$  stands for fiber or Fenichel), and this is, in general, a nonlinear projection.

Within the framework of an algorithm that yields only linearized information about the fast fibers, one must ask how best to approximate this ideal. A consistent approach is to identify a point on the slow manifold such that the approximate linearized fiber through it also goes through the given initial condition. This approach was used, for example, by Roberts [26] for systems with asymptotically stable center manifolds, where we note that a different method is first used to approximate the center manifold. Also, this approach is exact in the special case that the perturbed fast fibers are hyperplanes which need not be vertical. In general, if  $x_0$  lies on the linearized fiber  $\mathcal{L}_\varepsilon^{p_1}$  and if  $\pi_F(x_0) = p_2$ , then the error  $\|p_1 - p_2\|$  made by projecting linearly is  $\mathcal{O}(\varepsilon)$  and proportional to the curvature of the fiber (see also [26]).

For fast-slow systems, there is yet another way to linearly project initial conditions on the slow manifold. One projects along the approximate CSPF to the space  $\mathcal{T}_p\mathcal{F}_\varepsilon$ , where  $p$  is the point on the CSPM that lies on the same  $\varepsilon = 0$  fiber as the initial condition. This type of projection is also consistent, in the sense that it yields an exact result for  $\varepsilon = 0$  but has an error of  $\mathcal{O}(\varepsilon)$  for  $\varepsilon > 0$ . Moreover, it is algorithmically simpler, since it does not involve a search for the base point of the linearized fiber on which the initial conditions lie. However, it has the disadvantage that the projection is not exact in the special case that the fast fibers are (nonvertical) hyperplanes.

**Appendix A. The action of the  $\mathcal{O}(1)$  Jacobian on  $\mathcal{T}_p\mathcal{M}_0$ .** The spaces  $\mathcal{T}_p\mathcal{F}_\varepsilon$  and  $\mathcal{T}_p\mathcal{M}_\varepsilon$  depend, in general, on both the point  $p \in \mathcal{M}_\varepsilon$  and  $\varepsilon$ . As a result,

the basis  $A$  also depends on  $p$  and  $\varepsilon$ , and hence  $A_f$  and  $A_s$  possess formal asymptotic expansions in terms of  $\varepsilon$ ,

$$(A.1) \quad A_f = \sum_{i=0} \varepsilon^i A_f^i, \quad A_s = \sum_{i=0} \varepsilon^i A_s^i.$$

Next, we compute the action of the Jacobian on  $A_s$  to leading order.

LEMMA A.1.  $\text{Ker}(Dg(p))_0 = \mathcal{T}_p \mathcal{M}_0$  for  $p \in \mathcal{M}_0$ . In particular,  $(Dg)_0 A_s^0 = 0$ .

*Proof.* The Jacobian is a linear operator, so it suffices to show that every column vector of a basis for  $\mathcal{T}_p \mathcal{M}_0$  vanishes under the left action of the Jacobian. We choose this basis to be the matrix  $\begin{pmatrix} I_m \\ D_y h_0 \end{pmatrix}$ .

We compute

$$(A.2) \quad Dg_0 \begin{pmatrix} I_m \\ D_y h_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ D_y g_2 & D_z g_2 \end{pmatrix} \begin{pmatrix} I_m \\ D_y h_0 \end{pmatrix} = \begin{pmatrix} 0 \\ D_y g_2 + D_z g_2 D_y h_0 \end{pmatrix}.$$

Differentiating both members of the  $\mathcal{O}(1)$  invariance equation  $g_2(y, h_0(y), 0) = 0$  with respect to  $y$ , we obtain

$$(A.3) \quad D_y g_2(y, h_0(y), 0) + D_z g_2(y, h_0(y), 0) D_y h_0(y) = 0.$$

Equations (A.2) and (A.3) yield the desired result,

$$(A.4) \quad Dg_0 \begin{pmatrix} I_m \\ D_y h_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \mathcal{M}_0.$$

Finally, the identity  $(Dg)_0 A_s^0 = 0$  follows from the fact that  $A_s^0$  spans  $\mathcal{T}_p \mathcal{M}_0$ , since  $A_s^0 = A_s|_{\varepsilon=0}$  by (A.1).  $\square$

**Appendix B. The CSPFs in a variant of the CSP method.** In section 3.5, we emphasized that when we construct the CSPFs of order  $q$  at the second step of the  $q$ th iteration we use the information obtained in the first step of the same iteration. In particular, we evaluate  $A_1^{(q)}$  on  $\mathcal{K}_\varepsilon^{(q)}$  and define the CSPF of order  $q$  to be its span; see (3.32).

In this section we examine the asymptotic accuracy of a variant of the CSP method, where we evaluate  $A_1^{(q)}$  on  $\mathcal{K}_\varepsilon^{(q-1)}$ , so that the CSP quantities updated in the  $q$ th iteration are all evaluated on the same manifold, namely on the CSPM of order  $q-1$ . We show that this modification does not reduce the asymptotic accuracy of the CSPFs.

The proof is by induction. We assume that the variant of the CSP method described above yields, at the  $q$ th iteration, CSPFs of asymptotic accuracy  $\mathcal{O}(\varepsilon^q)$  for  $0, 1, \dots, q$ , and we show that the same is true for  $q+1$ . Recall that the CSPFs of order  $q+1$  are constructed at the second step of the  $(q+1)$ th iteration and that this step is carried out via the update matrix  $L_{(q)}$ . The idea behind the proof is to show that the modifications introduced in  $L_{(q)}$  by replacing  $\mathcal{K}_\varepsilon^{(q+1)}$  by  $\mathcal{K}_\varepsilon^{(q)}$  are of  $\mathcal{O}(\varepsilon^{q+2})$ , although  $\mathcal{K}_\varepsilon^{(q+1)}$  and  $\mathcal{K}_\varepsilon^{(q)}$  differ at terms of  $\mathcal{O}(\varepsilon^{q+1})$ . (This property can be attributed to the fact that, at each iteration,  $L_{(q)}$  is zero to leading order; see Lemma 4.1.) Since  $L_{(q)}$  is  $\mathcal{O}(\varepsilon^{q+1})$  on  $\mathcal{K}_\varepsilon^{(q+1)}$  by Lemma 4.1, we conclude that  $L_{(q)}$  stays unaltered to leading order when evaluated on  $\mathcal{K}_\varepsilon^{(q)}$ , instead of on  $\mathcal{K}_\varepsilon^{(q+1)}$ , and thus the CSPFs of order  $q+1$  retain their asymptotic accuracy of  $\mathcal{O}(\varepsilon^{q+1})$ . The full details of the calculation will be published in [33].

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