

CONVERGENCE OF A CLASS OF SEMI-IMPLICIT TIME-STEPPING SCHEMES FOR NONSMOOTH RIGID MULTIBODY DYNAMICS

BOGDAN I. GAVREA*, MIHAI ANITESCU†, AND FLORIAN A. POTRA‡

Abstract. In this work we present a framework for the convergence analysis in a measure differential inclusion sense of a class of time-stepping schemes for multibody dynamics with contacts, joints, and friction. This class of methods solves one linear complementarity problem per step and contains the semi-implicit Euler method, as well as trapezoidal-like methods for which second-order convergence was recently proved under certain conditions. By using the concept of a reduced friction cone, the analysis includes, for the first time, a convergence result for the case that includes joints. An unexpected intermediary result is that we are able to define a discrete velocity function of bounded variation, although the natural discrete velocity function produced by our algorithm may have unbounded variation.

Key words. rigid body, contact dynamics, friction, measure differential inclusion, complementarity problems

AMS subject classifications. 65K10, 90C33.

1. Introduction. The dynamic rigid multi body contact problem is concerned with predicting the motion of several rigid bodies in contact and it is one of the fundamental paradigms in modern computational science. It appears in the description of fuel motion in the pebble bed reactor [14], in the compaction of nanopowders [19, 8], and in the study of biological membranes [30, 18, 39, 15]. Such simulations are also used extensively in structural engineering [11], pedestrian evacuation dynamics [17], granular matter [29], robotics simulation and design [12], and virtual reality [2].

The problem of multi body rigid systems involving contact and friction is a *differential complementarity problem* (DCP). The DCP is part of a broader class of problems known as *differential variational inequalities* (DVI), which were recently introduced in [25], [26]. Approaches used in the past for the numerical approximation of rigid multibody dynamics with contact and friction include piecewise DAE approaches [16], acceleration-force linear complementarity problem (LCP) approaches [7, 27, 38], penalty approaches [10, 31, 32, 24], and velocity-impulse LCP-based time-stepping methods [37, 35, 4, 6].

The DCP gives rise to event-driven time-stepping schemes that are solved in an acceleration-force framework. These types of scheme will detect the discontinuity events; and, if these events are isolated, they will treat the dynamics as differential algebraic equations (DAEs) on each smooth piece. For the corresponding DAEs, numerical schemes of high accuracy may be used. This approach is natural and appealing because it leads to high-order time-stepping schemes. The major weakness of such an approach is that it excludes the presence of impulsive forces in the absence of an impact. One fairly simple example where such forces occur was pointed out in 1895 by P. Painlevé [23], who argued that the equations of classical rigid body dynamics are incompatible with the Coulomb friction model. Recently it was shown that a solution of Painlevé’s example exists in the sense of *measure differential inclusions* [33].

Time-stepping schemes that are not vulnerable to Painlevé-type examples integrate the dynamics at a velocity-impulse level, thereby allowing for impulsive forces at any time instant. Most of the time-stepping schemes that build a discrete model at a velocity-impulse level are based on Euler’s method for solving ordinary differential equations (ODEs). In this context, the methods of Anitescu and Potra [4] and Stewart and Trinkle [37] are based on a semi-implicit Euler scheme, while the model

*GRASP Laboratory, Department of Mechanical Engineering, University of Pennsylvania, Philadelphia, PA 19104, email gavrea@seas.upenn.edu. Part of the work was completed while the author was with the Department of Mathematics and Statistics, University of Maryland-Baltimore County, Baltimore, MD 21228

†Mathematics and Computer Science Division, Argonne National Laboratory, 9700 S. Cass Avenue, Argonne, IL 60439, email anitescu@mcs.anl.gov

‡Department of Mathematics and Statistics, University of Maryland-Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250 email potra@math.umbc.edu

of [5] is based on a linearly implicit Euler scheme. All three formulations require the solution of one linear complementarity problem (LCP) at each time step (see [9] for an extensive analysis of linear complementarity problems).

Recently, we proposed a new time-stepping scheme based on the trapezoidal method for solving ordinary differential equations [28]. The scheme solves one LCP at each non collisional integration step. We have shown that the numerical velocity is uniformly bounded as the integration step approaches 0 and that the scheme has global second order of convergence under additional restrictions. In order to globally achieve this convergence, events such as collision, take-off (contact deletion), and stick-slip transitions have to be detected with sufficient accuracy. To do so, we have proposed detection strategies that use information only at the position-velocity level, thereby remaining consistent with the solution concept of measure differential inclusions. We have shown that the scheme is stiffly stable when the stiffness originates in springs and dampers attached to pairs of bodies in the system. The scheme was implemented in UMBRA [13] and has proved successful in industrial-scale applications.

1.1. Our Contribution. The treatment of joints in time-stepping schemes is not new [4, 3]. What is novel in this work is that we prove that the solution produced by a class of time-stepping schemes that solves one linear complementarity problem per step and that includes the methods presented in [4] and [28] converges to a solution of a measure differential inclusion, in a sense to be defined in Section 6, *even when joint constraints are present*. The main conceptual novelty is the reduced friction cone which allows us to reduce the treatment of bilateral constraints to one of unilateral constraints, without altering the pointedness property. It is conceivable that a proof of convergence of linearized backward Euler schemes can be obtained from the one in the jointless case [34] for configurations with joints if the system is represented in relative coordinates which eliminate the joint constraints [16]. Nonetheless, this requires a nonlinear projection at every step which may be computationally costly [16] in addition to losing the property of having a constant mass matrix which presents several desirable numerical properties [5].

In addition, we prove for the first time that certain trapezoid-like methods [28] converge in the sense of measure differential inclusions. In doing that, we are able to define a discrete velocity function of bounded variation, although the natural discrete velocity function produced by our algorithm may have unbounded variation.

2. Notation and Model. In our analysis, we use the notation and framework from [34, 4]. We assume that the state of the system of rigid bodies can be described by a generalized position vector $q \in \mathbb{R}^s$ and a generalized velocity vector $v \in \mathbb{R}^s$. We assume that the system is subject to equality, nonpenetration, contact, and Coulomb friction.

The *equality constraints* arise usually from joint constraints [16] and can be described by equations of the form

$$\Theta^{(i)}(q) = 0, \quad i = 1, 2, \dots, m, \quad (2.1)$$

where $\Theta^{(i)}$ are sufficiently smooth functions. The force exerted by joint (i) on the system is $c_\nu^{(i)} \nu^{(i)}(q)$, where $\nu^{(i)}(q) = \nabla_q \Theta^{(i)}(q)$ is the gradient of $\Theta^{(i)}(q)$ and $c_\nu^{(i)}$ is the appropriate Lagrange multiplier [16].

The *nonpenetration constraints* are generated by the rigid body hypothesis according to which the bodies constituting the system cannot penetrate each other. We assume that for any pair of bodies we can define a signed distance function $\Phi^{(j)}(q)$ so that the noninterpenetration constraints can be written as

$$\Phi^{(j)}(q) \geq 0, \quad j = 1, 2, \dots, p, \quad (2.2)$$

where p is the number of pairs of bodies of the system that could get in contact, which in most applications is substantially smaller than the number of all possible pairs of bodies. Details of how the functions $\Phi^{(j)}$ can be defined and calculated are presented in [2].

The contact and frictional constraints may be introduced by means of the active set and the friction cone. If $\Phi^{(j)}(q) > 0$, then the j th constraint doesn't contribute to the dynamics of the system. When $\Phi^{(j)}(q) = 0$, however, the contact impulse generated by the j th noninterpenetration constraint must lie inside the *contact friction cone*:

$$FC^{(j)}(q) = \left\{ z_c = n^{(j)} c_n^{(j)} + \bar{D}^{(j)} \beta^{(j)} \mid c_n^{(j)} \geq 0, \|\beta^{(j)}\|_2 \leq \mu^{(j)} c_n^{(j)} \right\}. \quad (2.3)$$

Here the columns of $\bar{D}^{(j)} \in \mathbb{R}^{s \times 2}$ span the friction space, and $\beta^{(j)} \in \mathbb{R}^2$ is the corresponding tangential impulse due to friction. The parameter $\mu^{(j)} \geq 0$, which may be different for each contact, is the friction coefficient, and the second inequality that involves it in (2.3) is the first part of the Coulomb law. By including the joint forces in the above multivalued map, we obtain what we call the *constraint friction cone*, $\mathcal{FC}^{(j)}(q)$, corresponding to the j th contact. More precisely, we have

$$\mathcal{FC}^{(j)}(q) = \left\{ z = \tilde{\nu} c_\nu + n^{(j)} c_n^{(j)} + \bar{D}^{(j)} \beta^{(j)} \mid c_n^{(j)} \geq 0, \|\beta^{(j)}\|_2 \leq \mu^{(j)} c_n^{(j)} \right\}. \quad (2.4)$$

The total friction cones are then defined by

$$FC(q) = \sum_{\Phi^{(j)}(q)=0} FC^{(j)}(q) \quad (2.5)$$

for the *total contact friction cone* and by

$$\mathcal{FC}(q) = \sum_{\Phi^{(j)}(q)=0} \mathcal{FC}^{(j)}(q) \quad (2.6)$$

for the *total constraint friction cone*. To simplify terminology, we will refer, unless specified in advance, to the cone in (2.6) as the *total friction cone*. Note that the definition above implies that the set of active contact constraints, \mathcal{A} , is determined by

$$\mathcal{A} = \left\{ j \in \{1, \dots, p\} : \Phi^{(j)}(q) = 0 \right\}$$

for given position q . The total friction cone can be approximated by a polyhedral cone [37]. That is, $\mathcal{FC}^{(j)}(q)$ is replaced by

$$\widehat{\mathcal{FC}}^{(j)}(q) = \left\{ z = \tilde{\nu} c_\nu + n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)} \mid c_n^{(j)} \geq 0, \beta^{(j)} \geq 0, \|\beta^{(j)}\|_1 \leq \mu^{(j)} c_n^{(j)} \right\}, \quad (2.7)$$

where $D^{(j)} \in \mathbb{R}^{s \times m_C}$ is a balanced matrix in the sense that if $d_i^{(j)}$ is a column of $D^{(j)}$, then there is another index k such that $d_i^{(j)} = -d_k^{(j)}$. In this way we can represent the frictional impulses by using a nonnegative vector of multipliers $\beta^{(j)} = (\beta^{(j)})_i \geq 0$ with the 2-norm being replaced by the 1-norm. Here the nonnegative integer m_C represents the number of edges used in the approximation of the full cone. The polyhedral approximation of the friction cone is then given by

$$\widehat{\mathcal{FC}}(q) = \sum_{j \in \mathcal{A}} \widehat{\mathcal{FC}}^{(j)}(q) = \left\{ z = \tilde{\nu} c_\nu + \tilde{n} c_n + \tilde{D} \tilde{\beta} \mid \tilde{c}_n \geq 0, \tilde{\beta} \geq 0, \|\beta^{(j)}\|_1 \leq \mu^{(j)} c_n^{(j)}, \forall j \in \mathcal{A} \right\}, \quad (2.8)$$

where we define the block matrices $\tilde{\nu}$, \tilde{c}_ν , \tilde{n} , \tilde{c}_n , \tilde{D} , and $\tilde{\beta}$ as in (4.6).

3. Total Friction Cone and Regularity Assumptions. A regularity assumption, the pointedness of the friction cone, is used to obtain convergence results in the contact-only case [1, 34]. We present here an extension of the pointedness assumption to the case including bilateral constraints.

DEFINITION 3.1. *We say that*

$$\mathcal{FC}(q) \text{ is pointed} \Leftrightarrow \forall (\tilde{c}_\nu, \tilde{c}_n \geq 0, \tilde{\beta}) \neq 0 \text{ such that } \|\beta^{(j)}\|_2 \leq \mu^{(j)} c_n^{(j)}, \forall j \in \mathcal{A} \quad (3.1)$$

we must have that $\tilde{\nu}\tilde{c}_\nu + \tilde{n}\tilde{c}_n + \tilde{D}\tilde{\beta} \neq 0$.

$$\widehat{\mathcal{FC}}(q) \text{ is pointed} \Leftrightarrow \forall (\tilde{c}_\nu, \tilde{c}_n \geq 0, \tilde{\beta} \geq 0) \neq 0 \text{ such that } \|\beta^{(j)}\|_1 \leq \mu^{(j)} c_n^{(j)}, \forall j \in \mathcal{A} \quad (3.2)$$

we must have that $\tilde{\nu}\tilde{c}_\nu + \tilde{n}\tilde{c}_n + \tilde{D}\tilde{\beta} \neq 0$.

This definition clearly implies that the joint-constraint matrix $\tilde{\nu}$ is full rank. Moreover, the pointed friction cone assumption is weaker than the linear independence of the columns of the matrix $(\tilde{\nu}^T, \tilde{n}^T, \tilde{D}^T)^T$. Its name originates in the fact that, when there are no joint constraints, the condition is equivalent to the cone's not containing any proper linear subspace and thus being "pointed". An equivalent definition of the pointed friction cone assumption is given by the following condition [1]:

$$\mathcal{FC}(q) \text{ is pointed} \Leftrightarrow \text{there exists } c_{\mathcal{FC}} > 0, \text{ such that } \|(\tilde{c}_\nu, \tilde{c}_n, \tilde{\beta})\| \leq c_{\mathcal{FC}} \|z\| \quad (3.3)$$

with $z = \tilde{\nu}\tilde{c}_\nu + \tilde{n}\tilde{c}_n + \tilde{D}\tilde{\beta} \in \mathcal{FC}(q)$.

$$\widehat{\mathcal{FC}}(q) \text{ is pointed} \Leftrightarrow \text{there exists } c_{\widehat{\mathcal{FC}}} > 0, \text{ such that } \|(\tilde{c}_\nu, \tilde{c}_n, \tilde{\beta})\| \leq c_{\widehat{\mathcal{FC}}} \|z\| \quad (3.4)$$

with $z = \tilde{\nu}\tilde{c}_\nu + \tilde{n}\tilde{c}_n + \tilde{D}\tilde{\beta} \in \widehat{\mathcal{FC}}(q)$.

We say that the total friction cone $\mathcal{FC}(q)$ ($\widehat{\mathcal{FC}}(q)$) is **uniformly pointed** if the constant $c_{\mathcal{FC}}$ ($c_{\widehat{\mathcal{FC}}}$) can be taken the same for all possible configurations q . As noted in [3], the pointedness assumption is equivalent (in the frictionless case) to the existence of a force that will disassemble all contacts without breaking the joints.

LEMMA 3.2. *Assume that $\mathcal{FC}(q)$ is pointed. Let*

$$z = \tilde{n}\tilde{c}_n + \tilde{D}\tilde{\beta}, \text{ where } \tilde{c}_n \geq 0, \tilde{c}_n \neq 0, \|\beta^{(j)}\|_2 \leq \mu^{(j)} c_n^{(j)}, \forall j \in \mathcal{A}.$$

That is, z is an element of the (full) friction cone obtained by excluding the bilateral constraints, with the normal impulses not all equal to 0. Then,

$$z \neq 0 \text{ and } z \notin \text{Range}(\tilde{\nu}).$$

Proof. As suggested by the claim above, consider the set

$$FC(q) = \left\{ z_c = \tilde{n}\tilde{c}_n + \tilde{D}\tilde{\beta} \mid \tilde{c}_n \geq 0, \|\beta^{(j)}\|_2 \leq \mu^{(j)} c_n^{(j)}, \forall j \in \mathcal{A} \right\} \quad (3.5)$$

(note the difference in notation: caligraphic characters denote the friction cone that includes all constraint impulses, while roman letters denote the cone that contains only the contact impulses). Clearly the pointedness of $\mathcal{FC}(q)$ implies the pointedness of $FC(q)$. Therefore, by taking $z \in FC(q)$ with the normal impulses not all zero ($\tilde{c}_n \neq 0$), we obtain $z \neq 0$. If $z \in \text{Range}(\tilde{\nu})$ then $z = \tilde{\nu}u$, $u \neq 0$, and therefore by taking $\bar{z} = -\tilde{\nu}u + z$, we have $\bar{z} \in \mathcal{FC}(q)$, $\bar{z} = 0$ with a nonzero constraint impulse, a contradiction to the pointedness of $\mathcal{FC}(q)$. ■

We will use this Lemma to analyze the properties of the set

$$\tilde{\nu}_\perp^T \mathcal{FC}(q) = \{ \tilde{\nu}_\perp^T z : z \in \mathcal{FC}(q) \}.$$

Here $\tilde{\nu}_\perp$ denotes the orthogonal complement of $\tilde{\nu} \in \mathbb{R}^{s \times m}$. More precisely, $\tilde{\nu}_\perp \in \mathbb{R}^{s \times (s-m)}$ such that $\tilde{\nu}_\perp^T \tilde{\nu} = 0$ and $\tilde{\nu}_\perp^T \tilde{\nu}_\perp = I$. It follows that for any $x \in \mathbb{R}^s$, there exist unique vectors $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^{s-m}$ such that the decomposition

$$x = \tilde{\nu}u + \tilde{\nu}_\perp w \quad (3.6)$$

holds. We have the following simple results.

LEMMA 3.3. *Assume that $\mathcal{FC}(q)$ is pointed. Then for all $j \in \mathcal{A}$ we have*

$$\tilde{\nu}_\perp^T n^{(j)} \neq 0.$$

Proof. The proof follows immediately from Lemma 3.2. More precisely, assume that $\tilde{\nu}_\perp^T n^{(j)} = 0$ for some $j \in \mathcal{A}$. Take $z = c_n^{(j)} n^{(j)}$, with $c_n^{(j)} > 0$ (note that $n^{(j)} \neq 0$). It follows that $\tilde{\nu}_\perp^T z = 0$. Therefore, by the decomposition above, we must have $z \in \text{Range}(\tilde{\nu})$, which contradicts the conclusion of Lemma 3.2. \blacksquare

REMARK 3.4. *We cannot say the same thing about $\tilde{\nu}_\perp^T \bar{d}_i^{(j)}$, where $\bar{d}_i^{(j)}$ is a column of $\bar{D}^{(j)}$. Actually it is possible to have $\tilde{\nu}_\perp^T \bar{d}_i^{(j)} = 0$, which shows once again that the pointedness assumption is weaker than the linear independence of the active set (active set here includes all constraints). Let us define*

$$W_n^{(j)} = \tilde{\nu}_\perp^T n^{(j)} \quad \text{and} \quad \bar{W}_D^{(j)} = \tilde{\nu}_\perp^T \bar{D}^{(j)}.$$

As discussed above, all the $W_n^{(j)}$ are nonzero vectors; thus, by adjoining all of them, we obtain a matrix that we denote by \widetilde{W}_n . By taking only those $\bar{W}_D^{(j)}$ that are non-zero and adjoining we obtain, in a similar fashion, a matrix denoted by \widetilde{W}_D . Now let us define the *full reduced friction cone* $FC_r(q)$ by

$$FC_r(q) = \left\{ z_r = \widetilde{W}_n \tilde{c}_n + \widetilde{W}_D \tilde{\beta} \mid \tilde{c}_n \geq 0, \|\tilde{\beta}^{(j)}\|_2 \leq \mu^{(j)} \bar{c}_n^{(j)}, [\forall j \in \mathcal{A} \text{ s.t. } (\tilde{\nu}_\perp(q))^T \bar{D}^{(j)}(q) \neq 0] \right\}, \quad (3.7)$$

where the matrix \widetilde{W}_D is assumed to have only nonzero columns. In a similar fashion we introduce the *polyhedral reduced friction cone* $\widehat{FC}_r(q)$:

$$\widehat{FC}_r(q) = \left\{ z_r = \widetilde{W}_n \tilde{c}_n + \widetilde{W}_D \tilde{\beta} \mid \tilde{c}_n \geq 0, \tilde{\beta} \geq 0, \|\tilde{\beta}^{(j)}\|_1 \leq \mu^{(j)} \bar{c}_n^{(j)}, [\forall j \in \mathcal{A} \text{ s.t. } (\tilde{\nu}_\perp(q))^T \bar{D}^{(j)}(q) \neq 0] \right\}. \quad (3.8)$$

The active set used for the reduced friction cone is the same the one used for the nonreduced one. However, the number of frictional contacts in the reduced cone may be smaller than the number in the nonreduced cone. We have the following result.

LEMMA 3.5. *If $\mathcal{FC}(q)$ ($\widehat{\mathcal{FC}}(q)$) is pointed for all q , then the full (polyhedral) reduced friction cone $FC_r(q)$ ($\widehat{FC}_r(q)$) is pointed for all q .*

Proof. Let q be any possible system configuration, and let z_r be an arbitrary element of $FC_r(q)$. Then z_r can be written as

$$z_r = \tilde{\nu}_\perp^T \tilde{n} \tilde{c}_n + \tilde{\nu}_\perp^T \widetilde{W}_D \tilde{\beta},$$

where we have already eliminated those columns of \widetilde{D} that are in the range of \widetilde{v} . This new matrix is denoted by $\widetilde{\widetilde{D}}$, and the corresponding frictional impulses are given by the vector $\widetilde{\widetilde{\beta}}$. Note that, as shown above, all the columns of \widetilde{n} have nonzero components outside the range of \widetilde{v} . The normal and tangential impulses $(\widetilde{c}_n, \widetilde{\beta})$ satisfy $c_n^{(j)} \geq 0$ for all $j \in \mathcal{A}$ and $\|\widetilde{\beta}^{(j)}\|_2 \leq \mu^{(j)} c_n^{(j)}$ for all $j \in \mathcal{A}$ such that $(\widetilde{v}_\perp(q))^T \widetilde{D}^{(j)}(q) \neq 0$.

Assume now that $z_r = 0$ with $\widetilde{c}_n \neq 0$ (a necessary condition for $(\widetilde{c}_n, \widetilde{\beta}) \neq 0$). We want to reach a contradiction to the pointedness of the nonreduced cone. This immediately follows from the fact that

$$z_r = 0, \widetilde{c}_n \neq 0 \Rightarrow z_c = \widetilde{n}\widetilde{c}_n + \widetilde{\widetilde{D}}\widetilde{\beta} \in FC(q) \text{ satisfies } z_c = \widetilde{v}u, u \neq 0.$$

Here $\widetilde{\beta}$ is obtained from $\widetilde{\widetilde{\beta}}$ by adding zeros to the columns of $\widetilde{\widetilde{D}}$ missing in \widetilde{D} . By taking $\widetilde{c}_\nu = -u$, we obtain that $z := -\widetilde{v}u + z_c \in \mathcal{FC}(q)$ is zero, but $(\widetilde{c}_\nu, \widetilde{c}_n, \widetilde{\beta}) \neq 0$, which contradicts the pointedness of $\mathcal{FC}(q)$. Given that the argument can be carried out for any q for which $\mathcal{FC}(q)$ is pointed, we obtained the pointedness for $FC_r(q)$. Following the same argument, one proves the pointedness of the polyhedral reduced friction cone $\widehat{FC}_r(q)$. \blacksquare

REMARK 3.6. *The pointedness of the reduced cones is equivalent to the usual notion of pointedness, that is, "a cone K is pointed $\Leftrightarrow K \cap (-K) = \{0\}$ ".*

Now let $z = \widetilde{v}\widetilde{c}_\nu + \widetilde{n}\widetilde{c}_n + \widetilde{\widetilde{D}}\widetilde{\beta} \in \mathcal{FC}(q)$. From the pointedness of the reduced cone there exists [34] a unitary vector, $u_0 := u_0(q)$, such that for any $z = \widetilde{v}\widetilde{c}_\nu + \widetilde{n}\widetilde{c}_n + \widetilde{\widetilde{D}}\widetilde{\beta} \in \mathcal{FC}(q)$ we have

$$u_0^T \widetilde{v}_\perp^T z \geq C_2 \|z_r\| \geq C_3 \|\widetilde{c}_n\|, \quad (3.9)$$

where $z_r = \widetilde{v}_\perp^T z$. This estimate is one of the main ingredients that will be later used in proving the uniform bound on the variation of the velocities.

We can visualize the friction cones $\mathcal{FC}(q)$ and $\widehat{\mathcal{FC}}(q)$ as mappings from \mathbb{R}^s to the subsets of \mathbb{R}^s , that is, $\mathcal{FC}(q), \widehat{\mathcal{FC}}(q) : \mathbb{R}^s \rightarrow \mathcal{P}(\mathbb{R}^s)$. The graph of $\mathcal{FC}(\cdot)$ is defined by

$$\text{graph}(\mathcal{FC}) = \{(q, z(q)) \mid z(q) \in \mathcal{FC}(q)\}, \quad (3.10)$$

and similarly for $\widehat{\mathcal{FC}}(q)$. Clearly, from the constructions above, $\mathcal{FC}(q)$ and its approximation $\widehat{\mathcal{FC}}(q)$ are convex sets for each fixed q . Under the uniform pointedness assumption we obtain that these mappings have closed graphs.

LEMMA 3.7 (**Closed Graph Property of the Friction Cones**). *Assume that $\mathcal{FC}(q)$ ($\widehat{\mathcal{FC}}(q)$) is uniformly pointed. Then the graph of $\mathcal{FC}(\cdot)$ ($\widehat{\mathcal{FC}}(\cdot)$) is closed.*

Proof. We will prove the result for $\mathcal{FC}(\cdot)$. A similar argument is used for the polyhedral approximation $\widehat{\mathcal{FC}}(\cdot)$. Consider a sequence $(q^n, z^n) \in \text{graph}(\mathcal{FC}(q^n))$, where z^n has the form: $z^n = \widetilde{v}(q^n)c_\nu^n + \widetilde{n}(q^n)\widetilde{c}_n^n + \widetilde{\widetilde{D}}(q^n)\widetilde{\beta}^n$. Assume that $q^n \rightarrow q$ and $z^n \rightarrow z$ as $n \rightarrow \infty$. We want to show that $z \in \mathcal{FC}(q)$. From the uniform pointedness of $\mathcal{FC}(\cdot)$ we obtain that

$$\|(\widetilde{c}_\nu^n, \widetilde{c}_n^n, \widetilde{\beta}^n)\| \leq C_{\mathcal{FC}} \|z^n\|,$$

where $C_{\mathcal{FC}}$ is independent of q^n . Given that $z^n \rightarrow z$, it follows from the above inequality that $\|(\widetilde{c}_\nu^n, \widetilde{c}_n^n, \widetilde{\beta}^n)\|$ is bounded, and therefore we can extract convergent subsequences $\widetilde{c}_\nu^{n_k} \rightarrow \widetilde{c}_\nu^*$, $\widetilde{c}_n^{n_k} \rightarrow \widetilde{c}_n^*$, and $\widetilde{\beta}^{n_k} \rightarrow \widetilde{\beta}^*$, where $\widetilde{c}_n^* \geq 0$ and $\|\beta^{(j),*}\|_2 \leq \mu^{(j)} \|c_n^{(j),*}\|$ due to the similar inequalities satisfied by the corresponding subsequence. Using the fact that $\widetilde{v}(\cdot)$, $\widetilde{n}(\cdot)$ and $\widetilde{\widetilde{D}}(\cdot)$ are continuous, we have that the subsequence z^{n_k} converges to $z^* = \widetilde{v}(q)c_\nu^* + \widetilde{n}(q)\widetilde{c}_n^* + \widetilde{\widetilde{D}}(q)\widetilde{\beta}^* \in \mathcal{FC}(q)$. Given that z^{n_k} , with $z^{n_k} \rightarrow z^*$,

is a subsequence of the convergent sequence z^n , with $z^n \rightarrow z$, we conclude that $z = z^* \in \mathcal{FC}(q)$, which proves our claim. \blacksquare

Note that the closed graph property implies that the values of the multivalued mappings are closed. This can be easily seen by taking $q^n = q$ and $z^n \in \mathcal{FC}(q)$. An immediate consequence of the above lemma is the following corollary.

COROLLARY 3.8. *Assume that $\mathcal{FC}(q) \left(\widehat{\mathcal{FC}}(q) \right)$ is uniformly pointed. Then the graph of $FC_r(\cdot) \left(\widehat{FC}_r(\cdot) \right)$ is closed.*

Proof. Consider a sequence (q^n, z_r^n) such that $z_r^n \in FC_r(q^n)$ and $q^n \rightarrow q$, $z_r^n \rightarrow z_r$, as $n \rightarrow \infty$. By definition $z_r^n = (\tilde{\nu}_\perp(q^n))^T z^n$, for some $z^n \in \mathcal{FC}(q^n)$. Taking the limit, as $n \rightarrow \infty$, we conclude that $z^n \rightarrow z$. The fact that $\mathcal{FC}(\cdot)$ has a closed graph implies that $z \in \mathcal{FC}(q)$, which immediately leads to $z_r = (\tilde{\nu}_\perp(q))^T z \in FC_r(q)$. \blacksquare

4. The Time-Stepping Scheme. We are interested in convergence properties for a family of linearly implicit time-stepping schemes that accommodate methods based on semi-implicit Euler methods [4, 33] as well as various instances of the trapezoidal method from [28]. The time-stepping scheme solves at each integration step a linear complementarity problem. We will assume that only inelastic collisions are solved. In terms of the collision rule given in [4], which in general involves a compression phase followed by a decompression phase, for inelastic collisions only the former LCP needs to be solved, and therefore the algorithm will solve only one LCP per time-step. The main difference between a noncollisional and a collisional integration step is that the latter uses a zero time-step to get out of the compression phase.

To write the integration step as a *mixed linear complementarity* problem, we use the following approximations. The joint constraints are written at the velocity level and approximated by

$$\left(\nu^{(i)}(q^l) \right)^T (\alpha v^{l+1} + (1 - \alpha)v^l) = 0, \quad i = 1, \dots, m,$$

where α is a scalar parameter, $\alpha \in (0, 1]$.

The nonpenetration and frictional constraints are approximated in the same fashion. We can write these as the following complementarity conditions

$$\begin{aligned} 0 \leq \rho^{(j),l+1} := (n^{(j)}(q^l))^T (\alpha v^{l+1} + (1 - \alpha)v^l) &\perp c_n^{(j),l+1} \geq 0, & j \in \mathcal{A}, \\ 0 \leq \sigma^{(j),l+1} := \lambda^{(j),l+1} e^{(j)} + (D^{(j)}(q^l))^T (\alpha v^{l+1} + (1 - \alpha)v^l) &\perp \beta^{(j),l+1} \geq 0, & j \in \mathcal{A}, \\ 0 \leq \zeta^{(j),l+1} := \mu^{(j)} c_n^{(j),l+1} - e^{(j)T} \beta^{(j),l+1} &\perp \lambda^{(j),l+1} \geq 0, & j \in \mathcal{A}. \end{aligned}$$

Here $e^{(j)}$ is a vector, of dimension $m_C^{(j)}$, whose every entry is 1. The equations of motion in implicit form can be written as

$$M (v^{l+1} - v^l) - z^{l+1} = hk(t_{l+1}, q^{l+1}, v^{l+1}). \quad (4.1)$$

Here M is the mass matrix, which is assumed to be a constant symmetric positive definite matrix, z^{l+1} represent the contact and joint impulses, and $k(t_{l+1}, q^{l+1}, v^{l+1})$ are the inertial and applied forces acting at time t_{l+1} . Since the goal is to formulate the integration step as a linear complementarity problem, we will linearize (4.1) as follows. The term

$$z^{l+1} = \tilde{\nu}(q^{l+1}) \tilde{c}_\nu^{l+1} + \tilde{n}(q^{l+1}) \tilde{c}_n^{l+1} + \tilde{D}(q^{l+1}) \tilde{\beta}^{l+1}$$

is replaced by

$$z^{l+1} = \tilde{\nu}^l \tilde{c}_\nu^{l+1} + \tilde{n}^l \tilde{c}_n^{l+1} + \tilde{D}^l \tilde{\beta}^{l+1},$$

where $\tilde{v}^l = \tilde{\nu}(q^l)$, $\tilde{n}^l = \tilde{n}(q^l)$ and $\tilde{D}^l = \tilde{D}(q^l)$. To linearize the term $k(t_{l+1}, q^{l+1}, v^{l+1})$ in (4.1), we first introduce the position update formula. Given a parameter $\gamma \in [0, 1]$ (fixed at the beginning of the simulation), we obtain the position at time t_{l+1} by the formula

$$q^{l+1} = q^l + h((1 - \gamma)v^l + \gamma v^{l+1}).$$

For the term $k(t_{l+1}, q^{l+1}, v^{l+1})$ we have

$$k(t_{l+1}, q^{l+1}, v^{l+1}) = f_C(v^{l+1}) + k_1(t_{l+1}, q^{l+1}, v^{l+1}) = F(v^{l+1})v^{l+1} + k_1(t_{l+1}, q^{l+1}, v^{l+1}),$$

where $f_C(v^{l+1}) = F(v^{l+1})v^{l+1}$ are the Coriolis forces and $k_1(t_{l+1}, q^{l+1}, v^{l+1})$ are the external forces. A discussion related to this representation of the Coriolis forces is given at the end of this section. We replace the Coriolis term by

$$F(v^{l+1})v^{l+1} \approx F(v^l)((1 - \alpha)v^l + \alpha v^{l+1}) = F(v^l)v^l + \alpha F(v^l)(v^{l+1} - v^l). \quad (4.2)$$

The term $k_1(t_{l+1}, q^{l+1}, v^{l+1})$ is approximated as follows:

$$\begin{aligned} k_1(t_{l+1}, q^{l+1}, v^{l+1}) &\approx (1 - \alpha)k_1(t_l, q^l, v^l) + \alpha k_1(t_{l+1}, q^{l+1}, v^{l+1}), \\ &\approx (1 - \alpha)k_1(t_l, q^l, v^l) + \alpha k_1(t_{l+1}, q^l, v^l) + \alpha \left(\tilde{k}_{1q}(q^{l+1} - q^l) + \tilde{k}_{1v}^l(v^{l+1} - v^l) \right), \\ &\approx (1 - \alpha)k_1(t_l, q^l, v^l) + \alpha k_1(t_{l+1}, q^l, v^l) + \alpha h \tilde{k}_{1q}^l v^l + \alpha \left(\tilde{k}_{1v}^l + \gamma h \tilde{k}_{1q}^l \right) (v^{l+1} - v^l), \end{aligned} \quad (4.3)$$

where

$$\tilde{k}_{1q}^l \approx k_{1q}(t_{l+1}, q^l, v^l) \quad \tilde{k}_{1v}^l \approx k_{1v}(t_{l+1}, q^l, v^l)$$

are approximations of the Jacobians k_{1q} and k_{1v} . Combining the equations of motion with the joint constraints described at the velocity level and the frictional contact constraints, we obtain the following time-stepping scheme:

$$q^{l+1} = q^l + h((1 - \gamma)v^l + \gamma v^{l+1}) \quad (4.4a)$$

$$\tilde{M}^l v^{l+1} - \sum_{i=1}^m \nu^{(i),l} c_\nu^{(i),l+1} - \sum_{j \in \mathcal{A}} (n^{(j),l} c_n^{(j),l+1} + D^{(j),l} \beta^{(j),l+1}) = \tilde{M}^l v^l + \tilde{k}^l \quad (4.4b)$$

$$\left(\nu^{(i),l} \right)^T (\alpha v^{l+1} + (1 - \alpha)v^l) = 0, \quad i = 1, 2, \dots, m \quad (4.4c)$$

$$0 \leq \rho^{(j),l+1} := \left(n^{(j),l} \right)^T (\alpha v^{l+1} + (1 - \alpha)v^l) \perp c_n^{(j),l+1} \geq 0, \quad j \in \mathcal{A} \quad (4.4d)$$

$$0 \leq \sigma^{(j),l+1} := \lambda^{(j),l+1} e^{(j)} + \left(D^{(j),l} \right)^T (\alpha v^{l+1} + (1 - \alpha)v^l) \perp \beta^{(j),l+1} \geq 0, \quad j \in \mathcal{A} \quad (4.4e)$$

$$0 \leq \zeta^{(j),l+1} := \mu^{(j)} c_n^{(j),l+1} - e^{(j)T} \beta^{(j),l+1} \perp \lambda^{(j),l+1} \geq 0, \quad j \in \mathcal{A}, \quad (4.4f)$$

where $\nu^{(i),l} = \nu^{(i)}(q^l)$, $n^{(j),l} = n^{(j)}(q^l)$, $D^{(j),l} = D^{(j)}(q^l)$ and

$$\tilde{M}^l = \left(M - \alpha h \left(F(v^l) + \tilde{k}_{1v}^l \right) - \alpha \gamma h^2 \tilde{k}_{1q}^l \right), \quad (4.5)$$

$$\tilde{k}^l = h \left((1 - \alpha)k_1(t_l, q^l, v^l) + \alpha k_1(t_{l+1}, q^l, v^l) \right) + (1 - \alpha)h F(v^l)v^l + \alpha h^2 \tilde{k}_{1q}^l v^l.$$

We note that the equations (4.4) represent a mixed linear complementarity problem (MLCP). If at time-step l the index set of active contact constraints is given by $\mathcal{A} = \{j_1, j_2, \dots, j_a\}$, and if we denote

$$\begin{aligned} \tilde{\nu} &= [\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(m)}], & \tilde{c}_\nu &= [c_\nu^{(1)}, c_\nu^{(2)}, \dots, c_\nu^{(m)}], & \tilde{D} &= [D^{(j_1)}, D^{(j_2)}, \dots, D^{(j_a)}] \\ \tilde{c}_n &= [c_n^{(j_1)}, c_n^{(j_2)}, \dots, c_n^{(j_a)}], & \tilde{n} &= [n^{(j_1)}, n^{(j_2)}, \dots, n^{(j_a)}], & \tilde{\beta} &= [\beta^{(j_1)}, \beta^{(j_2)}, \dots, \beta^{(j_a)}]^T \\ \tilde{\lambda} &= [\lambda^{(j_1)}, \lambda^{(j_2)}, \dots, \lambda^{(j_a)}], & \tilde{\zeta} &= [\zeta^{(j_1)}, \zeta^{(j_2)}, \dots, \zeta^{(j_a)}], & \tilde{E} &= \text{diag}(e^{(j_1)}, e^{(j_2)}, \dots, e^{(j_a)}) \\ \tilde{\sigma} &= [\sigma^{(j_1)}, \sigma^{(j_2)}, \dots, \sigma^{(j_a)}], & \tilde{\rho} &= [\rho^{(j_1)}, \rho^{(j_2)}, \dots, \rho^{(j_a)}], & \tilde{\mu} &= \text{diag}(\mu^{(j_1)}, \mu^{(j_2)}, \dots, \mu^{(j_a)}), \end{aligned} \quad (4.6)$$

then the matrix form of the integration step is given by

$$\begin{bmatrix} \tilde{M}^l & -\tilde{\nu}^l & -\tilde{n}^l & -\tilde{D}^l & 0 \\ (\tilde{\nu}^l)^T & 0 & 0 & 0 & 0 \\ (\tilde{n}^l)^T & 0 & 0 & 0 & 0 \\ (\tilde{D}^l)^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{l+1} \\ \tilde{c}^{l+1} \\ \tilde{c}_\nu^{l+1} \\ \tilde{\beta}^{l+1} \\ \tilde{\lambda}^{l+1} \end{bmatrix} - \begin{bmatrix} \tilde{M}^l v^l + \tilde{k}^l \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho}^{l+1} \\ \tilde{\sigma}^{l+1} \\ \tilde{\zeta}^{l+1} \end{bmatrix} \quad (4.7)$$

$$0 \leq [\tilde{c}_n^{l+1}, \tilde{\beta}^{l+1}, \tilde{\lambda}^{l+1}] \perp [\tilde{\rho}^{l+1}, \tilde{\sigma}^{l+1}, \tilde{\zeta}^{l+1}] \geq 0. \quad (4.8)$$

Here we have used $[\cdot, \cdot, \dots, \cdot]$ to denote a block matrix with the same number of rows as its blocks and $[\cdot, \cdot, \dots, \cdot]$ to denote a block matrix with the same number of columns as its blocks. We denote by $\mathcal{L}(q^l, v^l, k, h, \alpha, \gamma)$ the solution set of the MLCP (4.7).

We note that the choice $\alpha = \gamma = 1$ results in the scheme from [4, 5] (the former is obtained once we choose $\tilde{k}_{1q} = 0$ and $\tilde{k}_{1v} = 0$) and the choice $\alpha = \gamma = \frac{1}{2}$ results in a variant of the scheme from [28]. The time-stepping scheme in [28] detects (behind collisions) other type of events such as stick-slip transitions, take-off transitions and changes in the active friction components. If the number of such changes is uniformly bounded as $h \rightarrow 0$, these transitions could be resolved in the same fashion in which collisions are here dealt with. For simplicity we restrict ourselves to collision detection only.

A collision occurs in the interval $(lh, (l+1)h]$ if $\Phi^{(j)}(q^l) > 0$ and $\Phi^{(j)}(\bar{q}^{l+1}) \leq 0$, where \bar{q}^{l+1} is the position computed by the time-stepping scheme without including j in the active set at time t_{l+1} . The active set at t_{l+1} is taken as

$$\mathcal{A}(t_{l+1}) = \mathcal{A}^{l+1} = \{j : \Phi^{(j)}(q^l) \leq 0\}.$$

Whenever a collision is encountered, cubic interpolation is used to determine the precollision velocity and the position at which the collision occurs [28]. The detected position q^- and the precollision velocity v^- are used in the compression phase to obtain the new velocity. A mixed linear complementarity problem of the same type as (4.7) is solved in the compression phase. More precisely, the solution set of the MLCP modeling the compression phase is $\mathcal{L}(q^-, v^-, 0, 0, 1, 1)$.

Collision detection may result in a nonuniform partition of the simulation interval $[0, T]$. More precisely, a collision may be detected at time t^* such that, for a given time-step h , $t^* \neq lh$ for any integer l . To make the upcoming proofs easier to follow, we enforce a uniform partition of $[0, T]$. When collision is detected at time $t^{*,l+1} \in ((lh, (l+1)h]$, the collision is solved, resulting in the collision position $q^{-,l+1}$ and postcollision velocity $v^{+,l+1} \in \mathcal{L}(q^{-,l+1}, v^{-,l+1}, 0, 0, 1, 1)$. Instead of introducing the collision time t^* in the time partition of $[0, T]$ or solving another MLCP in the interval $(t^*, (l+1)h]$, we take

$$t_{l+1} = (l+1)h, \quad q^{l+1} = q^{-,l+1} \text{ and } v^{l+1} = v^{+,l+1}.$$

Assuming that we do this for every collision and that the first integration step is not a collisional one, we have $t_l = lh$, for all l , and the scheme will keep a fixed time-step throughout the integration process.

Note that, while this simplification possibly affects the accuracy of the scheme, our choice essentially represents only a notation convention. Indeed, if the external force k_1 does not depend on time, then the sequence of velocities and positions is identical to the one with the normal convention (where the collision time is considered one of the time points). If the force does depend on time, then the change in its value is only order $O(h)$, since from Assumption **(H7)**, which will be defined shortly, the number of collisions is bounded above and does not affect the convergence proofs.

We extend the numerical solution to time instants different from the ones given by the discrete solution, as follows. The velocity sequence $v^{h,\alpha}(t)$ is defined by

$$v^{h,\alpha}(t) = \begin{cases} v^{l+1,\alpha} & \text{if } t \in (lh, (l+1)h] \text{ and no collision in } (lh, (l+1)h], \\ v^{l+1} := v^{+,l+1} & \text{if } t \in (lh, (l+1)h] \text{ and collision detected in } (lh, (l+1)h], \end{cases} \quad (4.9)$$

where $v^{+,l+1}$ denotes the velocity at the end of the compression phase and where

$$v^{l+1,\alpha} = (1 - \alpha)v^l + \alpha v^{l+1}. \quad (4.10)$$

The velocity function that uses no weighting is denoted by $v^h(\cdot)$ and defined in a similar fashion:

$$v^h(t) = \begin{cases} v^{l+1} & \text{if } t \in (lh, (l+1)h] \text{ and no collision in } (lh, (l+1)h], \\ v^{l+1} := v^{+,l+1} & \text{if } t \in (lh, (l+1)h] \text{ and collision detected in } (lh, (l+1)h]. \end{cases} \quad (4.11)$$

For the position sequence, we take $q^{h,\alpha}(t)$ to be

$$q^{h,\alpha}(t) = \frac{1}{h} \left((t - t_l)q^{(l+1)} + (t_{l+1} - t)q^l \right), \text{ whenever } t \in (t_l = lh, t_{l+1} = (l+1)h], \quad (4.12a)$$

$$\text{where } q^{l+1} = \begin{cases} q^{(l)} + hv^{l+1,\alpha} & \text{if } t \in (lh, (l+1)h] \text{ and no collision in } (lh, (l+1)h], \\ q^{-,l+1} & \text{if } t \in (lh, (l+1)h] \text{ and collision detected in } (lh, (l+1)h]. \end{cases} \quad (4.12b)$$

Here q^{l+1} is computed by the position update formula (4.4a), except for collisional instants (that is, a collision occurred in the $(lh, (l+1)h]$ interval), in which case $q^{l+1} := q^{-,l+1}$, where $q^{-,l+1}$ is the estimated collision position. Since the collision time $t^{*,l+1}$ is detected by solving

$$\Phi^{(j)}(\tilde{q}(t)) = 0,$$

where $\tilde{q} : [lh, (l+1)h] \rightarrow \mathbb{R}^s$ is the cubic interpolant of the data $\tilde{q}(lh) = q^l$, $\frac{d\tilde{q}}{dt}(lh) = v^l$, $\tilde{q}((l+1)h) = \bar{q}^{l+1}$, $\frac{d\tilde{q}}{dt}((l+1)h) = \bar{v}^{l+1}$ (\bar{q}^{l+1} and \bar{v}^{l+1} are obtained by applying a regular step with $j \notin \mathcal{A}$) and $q^{l+1} = q^{-,l+1} = \tilde{q}(t^{*,l+1})$, we can guarantee that

$$\Phi^{(j)}(q^{l+1}) = \Phi^{(j)}(q^{-,l+1}) \geq -C_c h^2. \quad (4.13)$$

for a fixed constant C_c .

To obtain the convergence results, we use the following assumptions.

(H1) The nonpenetration constraints are twice-continuously differentiable, and there exists B_H such that

$$\|\nabla_{qq} \Phi^{(j)}(q)\| \leq B_H, \text{ for all } q \text{ and } j = 1, \dots, p. \quad (4.14)$$

(H2) The functions $\Theta^{(i)}(q)$, $i = 1, \dots, m$ are sufficiently smooth functions.

- (H3) The generalized mass matrix, M , is constant, symmetric, and positive definite.
 (H4) The total friction cone $\mathcal{FC}(q)$ is uniformly pointed with respect to all configurations q .
 (H5) The norm of the external force increases at most linearly with the position and the velocity.
 That is,

$$\|k_1(t, q, v)\| \leq c_1 + c_2\|q\| + c_3\|v\|. \quad (4.15)$$

Here $k_1(q, v)$ denotes the external and inertial forces.

The Coriolis force is given by a bilinear operator

$$[f_C(v)]_i = \sum_{jk} f_{ijk} v_j v_k.$$

This is certainly true if the system is described by Newton-Euler equations in body coordinates [21, Section 2.4], where the matrix $F(v)$ of entries

$$[F(v)]_{ij} = \sum_k f_{ijk} v_k$$

is antisymmetric in the sense that

$$u^T F(v) u = 0, \quad \forall u.$$

We also assume that the approximations \tilde{k}_{1q} and \tilde{k}_{1v} are bounded. More precisely,

$$\|\tilde{k}_{1q}\| \leq c_4, \quad \|\tilde{k}_{1v}\| \leq c_5. \quad (4.16)$$

- (H6) The contact data given by $\tilde{n}(q)$, $\tilde{D}(q)$ are globally Lipschitz continuous functions.
 (H7) The number of collisions solved by the algorithm is uniformly upper bounded as $h \rightarrow 0$.
 (H8) The external forces $k_1(t, q, v)$ are linear in v , and the approximation \tilde{k}_{1v} is constant.

REMARK 4.1.

- Assumption (H4) implies that $\tilde{v}(q)$ has uniform full rank. That is, there exists a constant $\kappa > 0$ such that

$$\sigma_{\min}(\tilde{v}(q)) \geq \kappa \quad \forall q,$$

where $\sigma_{\min}(A)$ denotes the smallest singular value of the matrix A .

- Assumption (H8) is not needed to prove all the results. More precisely, uniform boundedness of the numerical velocities as well as a uniform bound on the variation of the numerical velocities can be obtained without this assumption. We note that this assumption is satisfied when external forces include linear damping terms, by far the prevailing type of external velocity-dependent passive force.

The MLCP (4.7) has the same structure as the ones in [4, 28], and therefore the same solvability results can be used to show that the solution set $\mathcal{L}(q^l, v^l, \tilde{k}, h, \alpha, \gamma)$ is not empty whenever the matrix \tilde{M} is positive definite. Since the mass matrix M is positive definite, the matrix $F(\cdot)$ is antisymmetric, and the approximations used are bounded, it follows from (4.5) that \tilde{M} will be positive definite for sufficiently small values of h and for any value of the velocity.

Note also that in the presence of stiff forces the use of exact Jacobians k_{1q} and k_{1v} may force the simulation to choose a very small time-step h in order to ensure the positive definiteness of the matrix

\widetilde{M} . In order to allow the simulation to proceed by using moderate values of the time-step h , appropriate negative semi-definite Jacobian approximations \widetilde{k}_{1q} and \widetilde{k}_{1v} may be used [28].

It is convenient for the proofs of the upcoming sections to separate the terms involving Coriolis forces in (4.4b). To this end, we introduce the following notation:

$$\begin{aligned}\overline{M}^l &= \left(M - \alpha h \widetilde{k}_{1v}^l - \alpha \gamma h^2 \widetilde{k}_{1q}^l \right), \\ \overline{k}^l &= h \left((1 - \alpha) k_1(t_l, q^l, v^l) + \alpha k_1(t_{l+1}, q^l, v^l) \right) + \alpha h^2 \widetilde{k}_{1q}^l v^l.\end{aligned}\quad (4.17)$$

In terms of the new notation, equation (4.4b) is rewritten as

$$\overline{M}^l v^{l+1} - \sum_{i=1}^m \nu^{(i),l} c_\nu^{(i),l+1} - \sum_{j \in \mathcal{A}} (n^{(j),l} c_n^{(j),l+1} + D^{(j),l} \beta^{(j),l+1}) = \overline{M} v^l + \overline{k}^l + h F(v^l) v^{l+1, \alpha}. \quad (4.18)$$

5. Kinetic Energy Estimates. The following result establishes a uniform bound for the numerical velocities, as $h \rightarrow 0$. Since we are dealing only with inelastic collisions and the friction cone is uniformly pointed, the compression phase guarantees that the postcollision kinetic energy will be less than the precollision kinetic energy. Therefore we restrict the proof of the next result to the noncollisional case.

THEOREM 5.1. *If (H1)–(H8) are satisfied and $\frac{1}{2} \leq \alpha \leq 1$, then there is a constant c such that*

$$(v^l)^T M v^l \leq \max \{ (v^0)^T M v^0, \|q^0\| + 1 \} e^{ct_l}, \quad l = 0, 1, \dots, \lfloor T/h \rfloor,$$

for all sufficiently small h .

Proof. Suppose that no collisions are detected in the interval $[t_l, t_{l+1}]$. The new velocity v^{l+1} will be determined by solving the LCP (4.4b)–(4.4f).

Left multiplying (4.18) by $(v^{l+1, \alpha})^T$ and using the fact that $F(v^l)$ is a skew-symmetric matrix, we get that

$$\begin{aligned}(v^{l+1, \alpha})^T \overline{M}^l v^{l+1} &= \sum_{i=1}^m c_\nu^{(i),l+1} (\nu^{(i),l})^T v^{l+1, \alpha} + \sum_{j \in \mathcal{A}} \left\{ c_n^{(j),l+1} (n^{(j),l})^T v^{l+1, \alpha} \right. \\ &\quad \left. + (\beta^{(j),l+1})^T (D^{(j),l})^T v^{l+1, \alpha} \right\} + (v^{l+1, \alpha})^T \overline{k}^l + (v^{l+1, \alpha})^T \overline{M}^l v^l.\end{aligned}\quad (5.1)$$

Using (4.4c), we deduce that $(\nu^{(i),l})^T v^{l+1, \alpha} = 0$, $i = 1, 2, \dots, m$. Also, using the contact constraints (4.4d), we obtain $c_n^{(j),l+1} (n^{(j),l})^T v^{l+1, \alpha} = 0$, $j \in \mathcal{A}$. Finally, from the frictional constraints (4.4e) and (4.4f) we get that

$$(\beta^{(j),l+1})^T (D^{(j),l})^T v^{l+1, \alpha} = -\lambda^{(j),l+1} (\beta^{(j),l+1})^T e^{(j)} = -\mu^{(j)} c_n^{(j),l+1} \lambda^{(j),l+1} \leq 0, \quad \forall j \in \mathcal{A}.$$

Then (5.1) implies

$$(v^{l+1, \alpha})^T \overline{M}^l v^{l+1} \leq (v^{l+1, \alpha})^T \overline{M}^l v^l + (v^{l+1, \alpha})^T \overline{k}^l. \quad (5.2)$$

By expanding the left- and right-hand sides of the above inequality, we obtain

$$\begin{aligned} (v^{l+1,\alpha})^T \overline{M}^l v^{l+1} &= \alpha v^{l+1T} M v^{l+1} + (1-\alpha) v^{lT} M v^{l+1} \\ &\quad - h\alpha^2 v^{l+1T} \left(\tilde{k}_{1v}^l + \gamma h \tilde{k}_{1q}^l \right) v^{l+1} \\ &\quad - h\alpha(1-\alpha) v^{lT} \left(\tilde{k}_{1v}^l + \gamma h \tilde{k}_{1q}^l \right) v^{l+1}, \end{aligned} \quad (5.3)$$

$$\begin{aligned} (v^{l+1,\alpha})^T \left(\overline{M}^l v^l + \overline{k}^l \right) &= (1-\alpha) v^{lT} M v^l + \alpha v^{l+1T} M v^l \\ &\quad - h\alpha^2 v^{l+1T} \left(\tilde{k}_{1v}^l + (\gamma-1) h \tilde{k}_{1q}^l \right) v^l \\ &\quad - h\alpha(1-\alpha) v^{lT} \left(\tilde{k}_{1v}^l + (\gamma-1) h \tilde{k}_{1q}^l \right) v^l \\ &\quad + h(v^{l+1,\alpha})^T \left((1-\alpha) k_1(t_{l+1}, q^l, v^l) + \alpha k_1(t_l, q^l, v^l) \right). \end{aligned} \quad (5.4)$$

Using Assumption (4.15) **(H5)** we are led to

$$\begin{aligned} (v^{l+1,\alpha})^T \overline{M}^l v^{l+1} &\geq \alpha(1-C_6h) \|M^{1/2} v^{l+1}\|^2 - C_7h \|M^{1/2} v^{l+1}\| \|M^{1/2} v^l\| \\ &\quad + (1-\alpha) v^{lT} M v^{l+1}, \end{aligned} \quad (5.5)$$

$$\begin{aligned} (v^{l+1,\alpha})^T \left(\overline{M}^l v^l + \overline{k}^l \right) &\leq \alpha \left(-1 + \frac{1}{\alpha} + C_8h \right) \|M^{1/2} v^l\|^2 + C_9h \|M^{1/2} v^l\| \|M^{1/2} v^{l+1}\| \\ &\quad + C_{10}h \left(\alpha \|M^{1/2} v^{l+1}\| + (1-\alpha) \|M^{1/2} v^l\| \right) \left(\|M^{1/2} v^l\| + \|q^l\| + 1 \right) \\ &\quad + \alpha v^{l+1T} M v^l \end{aligned} \quad (5.6)$$

Let us denote

$$\rho_l = \|M^{1/2} v^l\|, \sigma_l = \|q^l\| + 1.$$

Note that $\alpha \geq \frac{1}{2}$ gives

$$2\alpha - 1 \geq 0 \Rightarrow (2\alpha - 1) (v^{l+1})^T M v^l \leq (2\alpha - 1) \rho_{l+1} \rho_l. \quad (5.7)$$

Dividing by α both sides of the inequality (5.2) and using the symmetry of the matrix M , the estimates (5.5)-(5.6), the implication (5.7), as well as the notation above, implies that

$$(1 - C_{11}h) \rho_{l+1}^2 \leq \left(-1 + \frac{1}{\alpha} + C_{11}h \right) \rho_l^2 + C_{11}h \sigma_l (\rho_l + \rho_{l+1}) + \left(2 - \frac{1}{\alpha} \right) \rho_l \rho_{l+1} \quad (5.8)$$

for an appropriately defined constant C_{11} .

Consider now the case for which $\rho_l < \rho_{l+1}$. Dividing by ρ_{l+1} in (5.8) and using that $\rho_l/\rho_{l+1} < 1$ gives

$$(1 - C_{12}h) \rho_{l+1} \leq (1 + C_{12}h) \rho_l + C_{12}h \sigma_l, \quad (5.9)$$

for some constant $C_{12} \geq 0$ and all sufficiently small h . We can rewrite (5.9) in the form

$$\rho_{l+1} \leq (1 + C_{13}h) \rho_l + C_{13}h \sigma_l, \quad (5.10)$$

with C_{13} appropriately chosen. It is straightforward to see that for the remaining case, $\rho_{l+1} \leq \rho_l$, inequality (5.10) immediately follows. On the other hand, from (4.4a), we have

$$\|q^{l+1}\| \leq \|q^l\| + \|M^{-1/2}\| \left((1-\gamma) \|M^{1/2} v^l\| + \gamma \|M^{1/2} v^{l+1}\| \right). \quad (5.11)$$

Substituting the overestimate for ρ_{l+1} , (5.10), into (5.11) gives

$$\sigma_{l+1} \leq h \|M^{-1/2}\| (1 + \gamma C_{13}h) \rho_l + (1 + \gamma C_{13}h^2 \|M^{-1/2}\|) \sigma_l. \quad (5.12)$$

It follows that there is a constant C_{14} such that

$$\begin{aligned} \rho_{l+1} &\leq (1 + C_{14}h) \rho_l + C_{14}h \sigma_l \\ \sigma_{l+1} &\leq C_{14}h \rho_l + (1 + C_{14}h) \sigma_l. \end{aligned}$$

By taking $c = 2C_{14}$, we have that for all sufficiently small h , the following holds:

$$\left\| \begin{bmatrix} \rho_l \\ \sigma_l \end{bmatrix} \right\|_\infty \leq \left\| \begin{bmatrix} 1 + C_{14}h & C_{14}h \\ C_{14}h & 1 + C_{14}h \end{bmatrix} \right\|_\infty^l \left\| \begin{bmatrix} \rho_0 \\ \sigma_0 \end{bmatrix} \right\|_\infty = e^{ct_l} \left\| \begin{bmatrix} \rho_0 \\ \sigma_0 \end{bmatrix} \right\|_\infty,$$

which concludes the proof of our theorem. \blacksquare

REMARK 5.2. *The conclusion of Theorem 5.1 implies that both $v^h(\cdot)$ and $v^{h,\alpha}(\cdot)$ are uniformly bounded on $[0, T]$, as $h \rightarrow 0$.*

6. Measure Differential Inclusions. In the following we use the setup and some of the results of [34]. Formally, we are looking at complementarity systems of the following form.

$$\frac{dq}{dt} = v \quad (6.1)$$

$$M \frac{dv}{dt} = k(q, v) + \rho \quad (6.2)$$

$$\Theta^{(i)}(q) = 0, \quad i = 1, 2, \dots, m \quad (6.3)$$

$$\Phi^{(j)}(q) \geq 0, \quad j = 1, \dots, p \quad (6.4)$$

$$\rho(t) = \bar{\rho}(t) + \sum_{j=1}^p \rho^{(j)}(t) \in \mathcal{FC}(q) \quad (6.5)$$

$$\bar{\rho}(t) \in \text{span}\{\nu^{(i)}(q(t)) : i = 1, \dots, m\} \quad (6.6)$$

$$\|\rho^{(j)}\| \Phi^{(j)}(q) = 0, \quad j = 1, 2, \dots, p \quad (6.7)$$

The differences between the above formulation and the one corresponding to the contact-only case consists in a different friction cone being used and the additional bilateral constraints enforced by (6.3). Here $\mathcal{FC}(q)$ is the total friction cone (it includes all constraint forces, bilateral and unilateral) as defined in the previous section. In what follows, we specify what we mean by a solution of (6.1)-(6.7). This is motivated by the fact that a strong solution may not exist in general [33].

In contact mechanics, measures appear as a result of the presence of impulsive forces, while inclusions appear as a result of the presence of Coulomb friction. Because of possible impulsive forces the velocity of the system is no longer required to be an absolutely continuous function, but rather a function of bounded variation.

We are going to replace the forces, as they are understood in general, by vector measures. A vector measure is defined in terms of its action on a continuous function. Assume now that $v : [0, T] \rightarrow \mathbb{R}^s$ is a function of bounded variation. That is, the total variation of v , $\bigvee_0^T v(\cdot)$, is finite. Here $\bigvee_0^T v(\cdot)$ is the supremum of the sums $\sum_{i=0}^{N-1} \|v(t_{i+1}) - v(t_i)\|$ over all finite partitions $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$. We denote this by $v \in BV([0, T])$. It follows that the measure induced by v can be understood as a linear and continuous operator defined from $C([0, T])$ with values in \mathbb{R}^s . More precisely,

$$\langle dv, \phi \rangle = \int_0^T \phi(t) dv(t), \quad (6.8)$$

where $\phi : [0, T] \rightarrow \mathbb{R}$ is continuous. The Riemann-Stieljes integral in (6.8), which exists because of $v(\cdot)$ being of bounded variation, can be approximated by finite Riemann sums:

$$\sum_{i=0}^{N-1} \phi(\tau_i)[v(t_{i+1}) - v(t_i)],$$

where $a = t_0 < \tau_1 < t_1 < \dots < \tau_{N-1} < t_N = b$. Discontinuities in the velocity may lead to atoms of the measure dv . Therefore dv is not in general absolutely continuous with respect to the Lebesgue measure dt , and thus $\frac{dv}{dt}(\cdot)$ cannot be defined, in the usual sense, as a Radon–Nykodim derivative. To give a meaning to inclusions of the form

$$\frac{dv}{dt}(t) \in K(t), \quad \text{for } t \in [0, T], \quad (6.9)$$

we adopt the following definition [34].

DEFINITION 6.1 (Measure Differential Inclusion). *If $v \in BV([0, T])$ and $K(\cdot)$ is a convex-set valued mapping we say that (6.9) holds if, for all continuous $\phi : [0, T] \rightarrow \mathbb{R}$, $\phi \geq 0$ and ϕ not identically zero, we have that*

$$\frac{\int_0^T \phi(t) dv(t)}{\int_0^T \phi(t) dt} \in \bigcup_{\tau: \phi(\tau) \neq 0} K(\tau).$$

DEFINITION 6.2 (Weak Solution of (6.1)-(6.7)). *We say that $q(t), v(t)$ is a weak solution of (6.1)-(6.7) on $[0, T]$ if*

1. $v(\cdot)$ is a function of bounded variation on $[0, T]$.
2. $q(\cdot)$ is an absolutely continuous function that satisfies

$$q(t) = q(0) + \int_0^t v(\tau) d\tau, \quad \text{for } t \in [0, T]. \quad (6.10)$$

3. The measure $dv(t)$ must satisfy

$$M \frac{dv}{dt} - k(q, v) \in \mathcal{FC}(q). \quad (6.11)$$

4. $\Theta^{(i)}(q) = 0$, $i = 1, \dots, m$
5. $\Phi^{(j)}(q) \geq 0$, $j = 1, \dots, p$.

7. Uniform Bound in Variations. For the rest of the paper we consider (γ, α) satisfying:

$$\gamma = \alpha \in \left[\frac{1}{2}, 1 \right]. \quad (7.1)$$

Since $\gamma = \alpha$ and the number of collisions solved is uniformly upper bounded as $h \rightarrow 0$, we have, from (4.4a), that

$$q^{h,\alpha}(t) = q^{h,\alpha}(0) + \int_0^t v^{h,\alpha}(\tau) d\tau.$$

The uniform boundedness of the velocities implies that the sequence $\{q^{h,\alpha}(\cdot)\}$ is equicontinuous and equibounded. Therefore by the Arzela-Ascoli theorem, there exists a uniformly convergent subsequence, which we also denote by $q^{h,\alpha}(\cdot)$, that converges $q^{h,\alpha}(\cdot) \rightarrow q(\cdot)$ uniformly in $[0, T]$.

THEOREM 7.1. $\int_0^T v^{h,\alpha}(\cdot)$ is uniformly bounded as $h \rightarrow 0$, and there exists $v^*(\cdot)$ of bounded variation such that $v^{h,\alpha} \rightarrow v^*$ pointwise and $dv^{h,\alpha} \rightarrow dv^*$ weakly.

We break the proof in five subsections, along the lines given in [34], with some modifications due to the presence of joint constraints.

7.1. Use the regularity assumption on the reduced friction cone to obtain a bound on the sums $\sum_1 \|\tilde{c}_n^{l,h}\|$. Let $q(\cdot)$ be the limit of a uniformly convergent subsequence $q^{h,\alpha}(\cdot)$. Let t be a time instant in the interval $(0, T]$. From (3.9) it follows that there exist a unit vector $u_0(t)$ and a scalar $\zeta(t) > 0$, such that for any $z = \tilde{v}(q(t))\tilde{c}_v + \tilde{n}(q(t))\tilde{c}_n + \tilde{D}(q(t))\tilde{\beta} \in \mathcal{FC}(q(t))$ we have

$$u_0^T(t)\tilde{v}_\perp^T(q(t))z \geq \zeta(t)\|\tilde{c}_n\|. \quad (7.2)$$

By the closed-graph property of the $\mathcal{FC}(q(t))$ it follows that there is $\eta(t) > 0$ and $h_0 > 0$ such that, for any t'' satisfying $|t'' - t| \leq \eta(t)$ and any $h \leq h_0$, we have

$$u_0^T(t)\tilde{v}_\perp^T(q^h(t''))z \geq \frac{1}{2}\zeta(t)\|\tilde{c}_n\|, \quad (7.3)$$

for any $z \in \mathcal{FC}(q^h(t''))$. Provided that both lh and $(l+1)h$ lie in the interval $[t - \eta(t), t + \eta(t)]$, the numerical scheme gives

$$\left(M - \alpha h \tilde{k}_v^{l,h} - \alpha \gamma h^2 \tilde{k}_q^{l,h}\right) (v^{l+1,h} - v^{l,h}) = \tilde{k}^{l,h} + z^{l+1,h}, \quad (7.4)$$

with $z^{l+1,h} \in \widehat{\mathcal{FC}}(q^{(l),h})$. Let us denote

$$\tilde{v}_\perp^{l,h} := \tilde{v}_\perp(q^{l,h}) \text{ and } \tilde{v}^{l,h} := \tilde{v}(q^{l,h}).$$

From the joint constraint enforced at the velocity level, we have $(\tilde{v}^{l,h})^T (\alpha v^{l+1,h} + (1-\alpha)v^{l,h}) = 0$, for all l . By using the orthogonal decomposition we are led to

$$\begin{aligned} v^{l+1,h} &= \tilde{v}_\perp^{l,h} w^{l+1,h} + \tilde{v}^{l,h} u^{l+1,h} \\ v^{l,h} &= \tilde{v}_\perp^{l-1,h} w^{l,h} + \tilde{v}^{l-1,h} u^{l,h}. \end{aligned} \quad (7.5)$$

Multiplying both equations in (7.5) on the left by $(\tilde{v}_\perp^{l,h})^T M$ gives

$$\begin{aligned} (\tilde{v}_\perp^{l,h})^T M v^{l+1,h} &= \left((\tilde{v}_\perp^{l,h})^T M \tilde{v}_\perp^{l,h} \right) w^{l+1,h} + (\tilde{v}_\perp^{l,h})^T M \tilde{v}^{l,h} u^{l+1,h} \\ (\tilde{v}_\perp^{l,h})^T M v^{l,h} &= \left((\tilde{v}_\perp^{l-1,h})^T M \tilde{v}_\perp^{l-1,h} \right) w^{l,h} + (\tilde{v}_\perp^{l-1,h})^T M \tilde{v}^{l-1,h} u^{l,h} + \mathcal{O}(h). \end{aligned} \quad (7.6)$$

For the last equation in (7.6) we have used that $\tilde{v}_\perp^{l,h} = \tilde{v}_\perp^{l-1,h} + \mathcal{O}(h)$, which holds because of sufficient smoothness of the joint gradients and the uniform boundness of the velocities. Thus, by using $\omega^{i+1,h} := \left((\tilde{v}_\perp^{i,h})^T M \tilde{v}_\perp^{i,h} \right) w^{i+1,h}$ and $\omega^{i+1,h,\perp} := \left(\tilde{v}_\perp^{i,h} \right)^T M \tilde{v}^{i,h} u^{i+1,h}$, we have, with respect to the new notation,

$$\left(\tilde{v}_\perp^{l,h} \right)^T M v^{l+1,h} = \omega^{l+1,h} + \omega^{l+1,h,\perp} \text{ and } \left(\tilde{v}_\perp^{l,h} \right)^T M v^{l,h} = \omega^{l,h} + \omega^{l,h,\perp} + \mathcal{O}(h). \quad (7.7)$$

We multiply (7.4) on the left by $\tilde{v}_\perp^{l,h}$ to obtain

$$\omega^{l+1,h} - \omega^{l,h} + \omega^{l+1,h,\perp} - \omega^{l,h,\perp} = \tilde{v}_\perp^{l,h} z^{l+1,h} + \mathcal{O}(h), \quad (7.8)$$

where we have used the fact that $\tilde{k}_q^{l,h}$, $\tilde{k}_v^{l,h}$, $\frac{1}{h}\tilde{k}^{l,h}$ are uniformly bounded. It follows from (7.3) that

$$u_0^T(t) (\omega^{l+1,h} - \omega^{l,h} + \omega^{l+1,h,\perp} - \omega^{l,h,\perp}) + \mathcal{O}(h) \geq \frac{1}{2}\zeta(t) \|\tilde{c}_n^{l+1,h}\|. \quad (7.9)$$

Set $l_{min} = \lceil (t - \eta(t))/h \rceil$ and $l_{max} = \lfloor (t + \eta(t))/h \rfloor$. Then

$$\sum_{l_{min}}^{l_{max}-1} u_0^T(t) (\omega^{l+1,h} - \omega^{l,h} + \omega^{l+1,h,\perp} - \omega^{l,h,\perp}) + \mathcal{O}(h(l_{max} - l_{min})) \geq \frac{1}{2}\zeta(t) \sum_{l_{min}}^{l_{max}-1} \|\tilde{c}_n^{l+1,h}\|.$$

The sum on the left-hand side in the above inequality telescopes to

$$\begin{aligned} & \sum_{l_{min}}^{l_{max}-1} u_0^T(t) (\omega^{l+1,h} - \omega^{l,h} + \omega^{l+1,h,\perp} - \omega^{l,h,\perp}) = \\ & u_0^T(t) (\omega^{l_{max},h} - \omega^{l_{min},h} + \omega^{l_{max},h,\perp} - \omega^{l_{min},h,\perp}) \leq \\ & \|\omega^{l_{max},h}\| + \|\omega^{l_{min},h}\| + \|\omega^{l_{max},h,\perp}\| + \|\omega^{l_{min},h,\perp}\|. \end{aligned}$$

Using that $h(l_{max} - l_{min}) \leq \eta(t)$ and that $\omega^{l,h}$, $\omega^{l,h,\perp}$ are uniformly bounded (the uniform boundedness of the ω components results from the uniform boundedness of the velocities and the uniform linear independence of the columns of \tilde{v}) by a constant B_ω , we obtain

$$\sum_{l_{min}}^{l_{max}-1} \|\tilde{c}_n^{l+1,h}\| \leq \frac{2}{\zeta(t)} (2B_\omega + C_1\eta(t)) \quad \text{uniformly as } h \rightarrow 0, \quad (7.10)$$

where the constant C_1 above corresponds to the term $\mathcal{O}(h(l_{max} - l_{min}))$.

7.2. Show that all the other constraint impulses are bounded by the normal contact impulses. A bound on the tangential impulses $\tilde{\beta}^{l+1,h}$ is immediately obtained from the conic constraint:

$$\|\beta^{(j);l+1,h}\|_1 \leq \mu^{(j)} c_n^{(j);l+1,h}.$$

Thus, for the combined frictional impulses $f_T^{l+1,h} := \tilde{D}^{l,h} \tilde{\beta}^{l+1,h}$, we obtain

$$\sum_{l_{min}}^{l_{max}-1} \|f_T^{l+1,h}\| \leq C_2 \sum_{l_{min}}^{l_{max}-1} \|\tilde{c}_n^{l+1,h}\| \leq \frac{2C_2}{\zeta(t)} (2B_\omega + C_1\eta(t)), \quad (7.11)$$

with the last estimate holding uniformly as $h \rightarrow 0$. The constant C_2 above depends on the bounds on the frictional directions $d_i^{(j)}(q(\cdot))$, the friction coefficients, and the number of generators used in the polyhedral approximation of the friction cone.

To obtain a bound on $\sum_{l_{min}}^{l_{max}-1} \|f_J^{l+1,h,\alpha}\| := \sum_{l_{min}}^{l_{max}-1} \|\tilde{v}^{l,h} (\alpha \tilde{c}_n^{l+1,h} + (1-\alpha)\tilde{c}_n^{l,h})\|$, we go back to

$$\left(M - \frac{h}{2}\tilde{k}_v^{l,h} - \frac{h^2}{4}\tilde{k}_q^{l,h} \right) (v^{l+1,h} - v^{l,h}) = \tilde{k}^{l,h} + \tilde{v}^{l,h} \tilde{c}_v^{l+1,h} + \tilde{n}^{l,h} \tilde{c}_n^{l+1,h} + \tilde{D}^{l,h} \tilde{\beta}^{l+1,h},$$

which, together with the uniform bounds we have so far, implies

$$v^{l+1,h} - v^{l,h} = M^{-1}\tilde{\nu}^{l,h}\tilde{c}_\nu^{l+1,h} + M^{-1}\tilde{n}^{l,h}\tilde{c}_n^{l+1,h} + M^{-1}\tilde{D}^{l,h}\tilde{\beta}^{l+1,h} + \mathcal{O}(h). \quad (7.12)$$

Equation (7.12), together with the uniform boundedness of the velocity sequence and the uniformly pointed friction cone assumption, implies that the impulse multipliers $\tilde{c}_\nu^{l+1,h}$, $\tilde{c}_n^{l+1,h}$, and $\tilde{\beta}^{l+1,h}$ are bounded uniformly with respect to l . We define, for all indices l for which it makes sense, the following quantities.

$$\begin{aligned} v^{l+1,h,\alpha} &:= \alpha v^{l+1,h} + (1-\alpha)v^{l,h}, \\ \tilde{c}_\nu^{l+1,h,\alpha} &:= \alpha\tilde{c}_\nu^{l+1,h} + (1-\alpha)\tilde{c}_\nu^{l,h}, \\ \tilde{c}_n^{l+1,h,\alpha} &:= \alpha\tilde{c}_n^{l+1,h} + (1-\alpha)\tilde{c}_n^{l,h}, \\ \tilde{\beta}^{l+1,h,\alpha} &:= \alpha\tilde{\beta}^{l+1,h} + (1-\alpha)\tilde{\beta}^{l,h} \end{aligned}$$

We note that the definition of our time-stepping scheme (4.4c) implies that

$$(\tilde{\nu}^{l,h})^T v^{l+1,h,\alpha} = 0 \quad (7.13)$$

and that the triangle inequality implies that

$$\|\tilde{c}_n^{l,h,\alpha}\| \leq \alpha\|\tilde{c}_n^{l+1,h}\| + (1-\alpha)\|\tilde{c}_n^{l,h}\|, \quad \|\tilde{\beta}^{l+1,h,\alpha}\| \leq \alpha\|\tilde{\beta}^{l+1,h}\| + (1-\alpha)\|\tilde{\beta}^{l,h}\|. \quad (7.14)$$

We multiply (7.12) by α and (7.12) with $l \rightarrow l-1$ by $(1-\alpha)$. We add the results and obtain, by the uniform boundedness of the force multipliers and the uniform Lipschitz continuity of $\tilde{\nu}(q)$, $\tilde{n}(q)$, and $\tilde{D}(q)$, that

$$v^{l+1,h,\alpha} - v^{l,h,\alpha} = M^{-1}\tilde{\nu}^{l,h}\tilde{c}_\nu^{l+1,h,\alpha} + M^{-1}\tilde{n}^{l,h}\tilde{c}_n^{l+1,h,\alpha} + M^{-1}\tilde{D}^{l,h}\tilde{\beta}^{l+1,h,\alpha} + \mathcal{O}(h). \quad (7.15)$$

We multiply equation (7.15) on the left by $(\tilde{\nu}^{l,h})^T$ and use equation (7.13) at steps $l+1$ and l together with the fact that $\tilde{\nu}^{l-1,h} = \tilde{\nu}^{l,h} + \mathcal{O}(h)$. The result is

$$\mathcal{O}(h) = (\tilde{\nu}^{l,h})^T M^{-1}\tilde{\nu}^{l,h}\tilde{c}_\nu^{l+1,h,\alpha} + M^{-1}\tilde{n}^{l,h}\tilde{c}_n^{l+1,h,\alpha} + M^{-1}\tilde{D}^{l,h}\tilde{\beta}^{l+1,h,\alpha}.$$

Using the fact that the matrix $(\tilde{\nu}^{l,h})^T M^{-1}\tilde{\nu}^{l,h}$ is uniformly positive definite in the sense that its eigenvalues are bounded away from 0 uniformly with respect to $q^{l,h}$ as well as (7.14), we obtain a bound for the joint multipliers in terms of the normal contact impulses. More precisely, we have

$$\|\tilde{c}_\nu^{l+1,h,\alpha}\| \leq C_3\|\tilde{c}_n^{l+1,h,\alpha}\| + \mathcal{O}(h) \leq C_3(\alpha\|\tilde{c}_n^{l+1,h,\alpha}\| + \|\tilde{c}_n^{l,h,\alpha}\|) + \mathcal{O}(h),$$

where the constant C_3 can be chosen independent of l and h . Adding the above inequalities, we obtain

$$\sum_{l_{min}}^{l_{max}-1} \|f_J^{l+1,h,\alpha}\| := \sum_{l_{min}}^{l_{max}-1} \|\tilde{\nu}^{l,h}\tilde{c}_\nu^{l+1,h,\alpha}\| \leq C_4 \sum_{l_{min}}^{l_{max}-1} \|\tilde{c}_n^{l+1,h,\alpha}\| \leq \frac{2C_4}{\zeta(t)} (2B_\omega + C_1\eta(t)) + C_5\eta(t). \quad (7.16)$$

7.3. Obtain the bound for the variation of velocities on $[t-\eta(t), t+\eta(t)]$. As we have done in equation (7.4), denote by $z^{l+1,h,\alpha}$ the total constraint weighted impulse (combine total joint, normal, and tangential impulses) corresponding to step $l+1$, that is, $z^{l+1,h,\alpha} = \tilde{\nu}^{l,h}\tilde{c}_\nu^{l+1,h,\alpha} + \tilde{n}^{l,h}\tilde{c}_n^{l+1,h,\alpha} + \tilde{D}^{l,h}\tilde{\beta}^{l+1,h,\alpha}$. From the derivations above we have that

$$\sum_{l_{min}}^{l_{max}-1} \|z^{l+1,h,\alpha}\| \leq \frac{2C_6}{\zeta(t)} (2B_\omega + C_1\eta(t)) + C_7\eta(t), \quad (7.17)$$

where the constants above can be chosen independent of h . Now from (7.12) we have

$$\|v^{l+1,h,\alpha} - v^{l,h,\alpha}\| \leq \|M^{-1}z^{l+1,h,\alpha}\| + \mathcal{O}(h),$$

and by adding, we obtain

$$\sum_{l_{min}}^{l_{max}-1} \|v^{l+1,h,\alpha} - v^{l,h,\alpha}\| \leq \frac{2C_6}{\zeta(t)} (2B_\omega + C_1\eta(t)) + C_8\eta(t),$$

which shows that

$$\int_{t-\eta(t)/2}^{t+\eta(t)/2} v^{h,\alpha}(\cdot) \text{ is uniformly bounded as } h \rightarrow 0.$$

7.4. Obtain the bound for the variation velocities on the entire time interval. Since $(t - \eta(t)/2, t + \eta(t)/2)$ is a covering of $[0, T]$, there is a finite subcovering

$$\{(t_i - \eta(t_i)/2, t_i + \eta(t_i)/2 \mid i = 1, \dots, m_T)\}.$$

Therefore, by summing the contributions corresponding to this finite set of subintervals, we obtain a

uniform bound on $\int_0^T v^{h,\alpha}(\cdot)$ as $h \rightarrow 0$. If we use the fact that $v^{h,\alpha}(\cdot)$ has bounded variation, then,

by Helly's selection theorem, there exists a subsequence of $v^{h_k,\alpha}(\cdot)$ of $v^{h,\alpha}(\cdot)$ that converges pointwise to $v(\cdot)$ and has bounded variation. Since the limiting velocity $v(t)$ may not be well defined for every $t \in [0, T]$, we assume without loss of generality, [34], that $v(\cdot)$ is right-continuous, i.e. $v(t) = v^+(t)$ for all $t \in [0, T]$. The corresponding functions $q^{h_k,\alpha}(\cdot)$ converge to the indefinite integral of $v(\cdot)$ by the pointwise convergence theorem for Lebesgue integrals. We assume for simplicity that this is the entire sequence and therefore $q^{h,\alpha}(\cdot) \rightarrow q(\cdot)$ and $v^{h,\alpha}(\cdot) \rightarrow v(\cdot)$.

7.5. Weak * convergence. . Since $\int_0^T v^{h,\alpha}(\cdot)$ are uniformly bounded as $h \rightarrow 0$ and $v^{h,\alpha}(0) = v(0)$ and since $v^{h,\alpha}(\cdot) \rightarrow v(\cdot)$ pointwise, it follows that $dv^{h,\alpha} \rightarrow dv$ weakly *, that is,

$$\int_0^T \phi(t)^T dv^{h,\alpha}(t) \rightarrow \int_0^T \phi(t)^T dv(t)$$

for all continuous functions $\phi(t)$. Therefore, $dv^{h,\alpha}(\cdot) \rightarrow dv(\cdot)$ weak * as Borel measures. The proof of Theorem 7.1 is complete.

8. Limits are Solutions to the Measure Differential Inclusion. In this section we will use Assumptions **(H1)**–**(H8)** to prove that the limits are solutions to the rigid body MDI. Assume (q, v) is a solution of the measure differential inclusion of Definition (6.2). We write

$$v = \tilde{v}(q)u + \tilde{v}_\perp(q)w. \tag{8.1}$$

It follows that the Borel measure dv (which is well defined since v is a function of bounded variation on $[0, T]$) can be written as $dv = d(\tilde{v}(q)u) + d(\tilde{v}_\perp(q)w)$. Since from the joint constraints the velocity v satisfies $(\tilde{v}(q))^T v = 0$, we must have $u = 0$ which implies $d(\tilde{v}(q)u) = 0$. This leaves $dv = d(\tilde{v}_\perp(q)w)$. We can expand further to obtain, as detailed in Appendix A, that

$$\begin{aligned} dv &= d(\tilde{v}_\perp(q)w) = \tilde{v}_\perp(q)dw + \frac{\partial}{\partial q}(\tilde{v}_\perp(q)w) dq \\ &= \tilde{v}_\perp(q)dw + \frac{\partial}{\partial q}(\tilde{v}_\perp(q)w) v dt = \tilde{v}_\perp(q)dw + \left(\frac{\partial}{\partial q}(\tilde{v}_\perp(q)w) \right) \tilde{v}_\perp(q)w dt \end{aligned} \tag{8.2}$$

where for the second last equality we have used (6.10) and for the last one we have used the fact that u from (8.1) is zero. Note that the second term in the last equality of (8.2) is a measure which is absolutely continuous with respect to the Lebesgue measure dt . Motivated by the analysis above we introduce the following definition which gives the measure differential inclusion on the reduced cone.

DEFINITION 8.1 (Reduced Weak Solution of (6.1–6.7)). *We say that $q(t)$, $w(t)$ is a reduced weak solution of (6.1–6.7) on $[0, T]$ if*

1. $w(\cdot)$ is a function of bounded variation on $[0, T]$.
2. $q(\cdot)$ is an absolutely continuous function that satisfies

$$q(t) = q(0) + \int_0^t \tilde{\nu}_\perp(q(\tau))w(\tau)d\tau, \text{ for } t \in [0, T]. \quad (8.3)$$

3. The measure $dw(t)$ must satisfy

$$\left((\tilde{\nu}_\perp(q))^T M\tilde{\nu}_\perp(q) \right) \frac{dw}{dt} - k_{w,\perp}(t, q, w) \in \mathcal{FC}_r(q), \quad (8.4)$$

where

$$k_{w,\perp}(t, q, w) = (\tilde{\nu}_\perp(q))^T k_w(t, q, w) \quad (8.5)$$

and

$$k_w(t, q, w) = k(t, q, \tilde{\nu}_\perp(q)w) - M \left(\left(\frac{\partial}{\partial q} (\tilde{\nu}_\perp(q)w) \right) \tilde{\nu}_\perp(q)w \right) \quad (8.6)$$

4. $\Phi^{(j)}(q) \geq 0$, $j = 1, \dots, p$.

LEMMA 8.2. *If (q, w) is a reduced weak solution of (6.1–6.7) on $[0, T]$ in the sense of Definition 8.1 and $\Theta(q(0)) = 0$, then $(q, v) = (q, \tilde{\nu}_\perp(q)w)$ is a weak solution of (6.1–6.7) on $[0, T]$ in the sense of Definition 6.2*

Proof. By construction $(q, v) = (q, \tilde{\nu}_\perp(q)w)$ and from conditions 1, 2 and 4 of Definition 8.1 it immediately follows that conditions 1, 2 and 5 of Definition 6.2 are satisfied. To prove that condition 4 of Definition 6.2 is satisfied, we use (8.3) to obtain

$$\begin{aligned} \Theta(q(t)) &= \Theta(q(0)) + \int_0^t (\tilde{\nu}(q))^T v(\tau)d\tau \\ &= \Theta(q(0)) + \int_0^t \left((\tilde{\nu}(q))^T \tilde{\nu}_\perp(q(\tau)) \right) w(\tau)d\tau \\ &= \Theta(q(0)) \end{aligned}$$

Since $\Theta(q(0)) = 0$, we have $\Theta(q(t)) = 0$ for all $t \in [0, T]$ and therefore condition 4 of Definition 6.2 is satisfied.

8.1. The MDI for the Limit. To prove that (6.11) holds we mainly reverse the derivations in (8.2). That is, if (q, w) satisfies (8.4), then there exist $\bar{z} \in \mathcal{FC}(q)$ and a vector measure $\tilde{d}_\nu \in \mathbb{R}^m$ such that

$$M\tilde{\nu}_\perp(q) \frac{dw}{dt} - k_w(t, q, w) = \bar{z} + \tilde{\nu}\tilde{d}_\nu.$$

Since $\bar{z} \in \mathcal{FC}(q)$ implies that $\bar{z} + \tilde{\nu}\tilde{d}_\nu \in \mathcal{FC}(q)$ for any $\tilde{d}_\nu \in \mathbb{R}^m$, we can write

$$M\tilde{\nu}_\perp(q) \frac{dw}{dt} - k_w(t, q, w) \in \mathcal{FC}(q).$$

Using (8.2) together with (8.6) in the above inclusion gives

$$M \frac{dv}{dt} - k(t, q, w) \in \mathcal{FC}(q)$$

and therefore also condition 4 of Definition 6.2 is satisfied. This completes the proof. \blacksquare

We start by writing (4.18) at step $(l+1)$ and step (l) as follows:

$$\overline{M}^l (v^{l+1} - v^l) - \left(\overline{k}^l + hF(v^l)v^{l+1,\alpha} \right) = z^{l+1}, \quad (8.7)$$

$$\overline{M}^{l-1} (v^l - v^{l-1}) - \left(\overline{k}^{l-1} + hF(v^{l-1})v^{l,\alpha} \right) = z^l, \quad (8.8)$$

where $z^{k+1} \in \widehat{\mathcal{FC}}(q^k)$ and $\overline{M}^l, \overline{k}^l$ are given by (4.17).

Since the approximation \widetilde{k}_{1q} , is uniformly bounded, we have that

$$\overline{M}^k = M - \alpha h \widetilde{k}_{1v} + O(h^2).$$

We now multiply (8.7) by α and (8.8) by $(1-\alpha)$ and add them up. We obtain

$$\begin{aligned} & M (v^{l+1,\alpha} - v^{l,\alpha}) - \left(\alpha \overline{k}^l + (1-\alpha) \overline{k}^{l-1} \right) \\ & - h \alpha \widetilde{k}_{1v} (v^{l+1,\alpha} - v^{l,\alpha}) - h (\alpha F(v^l)v^{l+1,\alpha} + (1-\alpha)F(v^{l-1})v^{l,\alpha}) + O(h^2) = z^{l+1,\alpha}, \end{aligned} \quad (8.9)$$

where we have used the fact that by Assumption **(H8)** \widetilde{k}_{1v}^l is constant, i.e., $\widetilde{k}_{1v}^l = \widetilde{k}_{1v}$ for all l . Using the fact that $F(\cdot)$ is a linear map we have that

$$\begin{aligned} (1-\alpha)F(v^{l-1})v^{l,\alpha} &= F((1-\alpha)v^{l-1})v^{l,\alpha} \\ &= F(v^{l,\alpha})v^{l,\alpha} - \alpha F(v^l)v^{l,\alpha}. \end{aligned}$$

Then the Coriolis terms in (8.9) become

$$\alpha F(v^l)v^{l+1,\alpha} + (1-\alpha)F(v^{l-1})v^{l,\alpha} = F(v^{l,\alpha})v^{l,\alpha} + \alpha F(v^l) (v^{l+1,\alpha} - v^{l,\alpha}) \quad (8.10)$$

Since the sequence $v^h(\cdot)$ is uniformly bounded, $v^{h,\alpha}(t+h) \rightarrow v^+(t)$ and $v^{h,\alpha}(t) \rightarrow v(t) = v^+(t)$ a.e. on $[0, T]$, it follows that

$$F(v^h(t)) (v^{h,\alpha}(t+h) - v^{h,\alpha}(t)) \rightarrow 0 \text{ as } h \rightarrow 0,$$

for $t \in [0, T] - N$, where N is a set of Lebesgue measure zero. The same reasoning applies for the term $\widetilde{k}_{1v} (v^{l+1,\alpha} - v^{l,\alpha})$, giving

$$\widetilde{k}_{1v} (v^{h,\alpha}(t+h) - v^{h,\alpha}(t)) \rightarrow 0 \text{ pointwise a.e. in } [0, T]. \quad (8.11)$$

Now by using the fact that $k_1(t, q, v)$ is linear in v , as well as the fact that \widetilde{k}_{1q} is bounded and $q^l = q^{l-1} + \mathcal{O}(h)$ we get that

$$\overline{k}^h(t) := \overline{k}^h(t, q^{h,\alpha}(t), v^{h,\alpha}(t)) \rightarrow k_1(t, q(t), v(t)) \text{ pointwise a.e. in } [0, T]. \quad (8.12)$$

Here

$$\overline{k}^h(t) := \overline{k}^h(t, q^{h,\alpha}(t), v^{h,\alpha}(t)) = (1-\alpha)k_1(t, q^{h,\alpha}(t), v^{h,\alpha}(t)) + \alpha k_1(t+h, q^{h,\alpha}(t), v^{h,\alpha}(t)) + \alpha \widetilde{k}_{1q}(t, q^{h,\alpha}(t), v^{h,\alpha}(t))$$

is the function equivalent to the quantity $\frac{1}{h}\bar{k}^l$ from (4.17). Equation (8.12) implies that

$$\alpha\bar{k}^h(t) + (1-\alpha)\bar{k}^h(t-h) \rightarrow k_1(t, q^{h,\alpha}(t), v^{h,\alpha}(t)) \text{ pointwise a.e. in } [0, T]. \quad (8.13)$$

Using Assumptions **(H2)** and **(H6)** we can write $z^{l+1,\alpha}$ in (8.9) as

$$z^{l+1,\alpha} = \bar{z}^{l+1} + \mathcal{O}(h\|v^{l+1,\alpha} - v^{l,\alpha}\|)$$

with $\bar{z}^{l+1} \in \widehat{\mathcal{FC}}(q^l)$. This implies that for all h sufficiently small

$$M \frac{dv^{h,\alpha}}{dt} - \widehat{k}^h(t) \in \widehat{\mathcal{FC}}(q^{h,\alpha}(t)) \subset \mathcal{FC}(q^{h,\alpha}(t)), \quad (8.14)$$

where

$$\begin{aligned} \widehat{k}^h(t) &= \alpha\bar{k}^h(t) + (1-\alpha)\bar{k}^h(t-h) + \left(\widetilde{k}_{1v} + F(v^h(t)) \right) (v^{h,\alpha}(t+h) - v^{h,\alpha}(t)) \\ &\quad + F(v^{h,\alpha}(t))v^{h,\alpha}(t) + \mathcal{O}(h) + \mathcal{O}(h\|v^{l+1,\alpha} - v^{l,\alpha}\|). \end{aligned}$$

From (8.11–8.13) we can easily see that

$$\widehat{k}^h(t) \rightarrow k(t, q(t), v(t) = F(v(t))v(t) + k_1(t, q(t), v(t)) \text{ pointwise a.e. in } [0, T].$$

We now write:

$$v^{h,\alpha}(t) = \widetilde{v}_\perp(q^{h,\alpha}(t)) w^{h,\alpha}(t) + \widetilde{v}(q^{h,\alpha}(t)) u^{h,\alpha}(t),$$

which gives

$$u^{h,\alpha}(t) = \left((\widetilde{v}(q^{h,\alpha}(t)))^T \widetilde{v}(q^{h,\alpha}(t)) \right)^{-1} (\widetilde{v}(q^{h,\alpha}(t)))^T v^{h,\alpha}(t). \quad (8.15)$$

Using a Taylor expansion together with Assumption **(H2)** we obtain

$$\begin{aligned} (\widetilde{v}(q^{h,\alpha}(t)))^T v^{h,\alpha}(t) &= (\widetilde{v}(q^{h,\alpha}(t_l)))^T v^{h,\alpha}(t) + \left(\frac{\partial}{\partial q} (\widetilde{v}^T(q)v^{h,\alpha}(t)) \Big|_{q=q^{h,\alpha}(t_l)} \right) (q^{h,\alpha}(t) - q^{h,\alpha}(t_l)) \\ &\quad + \mathcal{O}(\|q^{h,\alpha}(t) - q^{h,\alpha}(t_l)\|^2) \end{aligned} \quad (8.16)$$

Since the definition of the time-stepping scheme enforces $(\widetilde{v}(q^{h,\alpha}(t_l)))^T v^{h,\alpha}(t) = 0$ for all $t \in (t_l, t_{l+1}]$ and since $q^{h,\alpha}(t) - q^{h,\alpha}(t_l) = (t - t_l)v^{h,\alpha}(t)$ for all $t \in [t_l, t_{l+1}]$, we have

$$(\widetilde{v}(q^{h,\alpha}(t)))^T v^{h,\alpha}(t) = (t - t_l) \left(\frac{\partial}{\partial q} (\widetilde{v}^T(q)v^{h,\alpha}(t)) \Big|_{q=q^{h,\alpha}(t_l)} \right) v^{h,\alpha}(t) + \mathcal{O}(h^2) \quad (8.17)$$

Combining (8.15) and (8.17) gives

$$\begin{aligned} (\widetilde{v}_\perp(q^{h,\alpha}(t)))^T M (\widetilde{v}(q^{h,\alpha}(t))u^{h,\alpha}(t) - \widetilde{v}(q^{h,\alpha}(t-h))u^{h,\alpha}(t-h)) &= \mathcal{O}(h\|v^{h,\alpha}(t) - v^{h,\alpha}(t-h)\|) \\ &\quad + \mathcal{O}(h^2) \end{aligned} \quad (8.18)$$

Using a similar methodology one also gets

$$\begin{aligned}
\tilde{\nu}_\perp(q^{h,\alpha}(t))w^{h,\alpha}(t) &= \tilde{\nu}_\perp(q^{h,\alpha}(t-h))w^{h,\alpha}(t-h) \\
&= \tilde{\nu}_\perp(q^{h,\alpha}(t)) (w^{h,\alpha}(t) - w^{h,\alpha}(t-h)) \\
&\quad - \left(\frac{\partial}{\partial q} (\tilde{\nu}^T(q)w^{h,\alpha}(t-h)) \Big|_{q=q^{h,\alpha}(t)} \right) (q^{h,\alpha}(t) - q^{h,\alpha}(t-h)) + \mathcal{O}(h^2) \\
&= \tilde{\nu}_\perp(q^{h,\alpha}(t)) (w^{h,\alpha}(t) - w^{h,\alpha}(t-h)) \\
&\quad + h \left(\frac{\partial}{\partial q} (\tilde{\nu}^T(q)w^{h,\alpha}(t-h)) \Big|_{q=q^{h,\alpha}(t)} \right) v^{h,\alpha}(t) + \mathcal{O}(h^2) \\
&= \tilde{\nu}_\perp(q^{h,\alpha}(t)) (w^{h,\alpha}(t) - w^{h,\alpha}(t-h)) \\
&\quad + h \left(\frac{\partial}{\partial q} (\tilde{\nu}_\perp^T(q)w^{h,\alpha}(t-h)) \Big|_{q=q^{h,\alpha}(t)} \right) \tilde{\nu}_\perp(q^{h,\alpha}(t))w^{h,\alpha}(t) + \mathcal{O}(h^2),
\end{aligned} \tag{8.19}$$

where for the last equality we have used (8.17) which give $h\tilde{\nu}(q^{h,\alpha}(t))u^{h,\alpha}(t) = \mathcal{O}(h^2)$. We use (8.19) to write

$$\begin{aligned}
(\tilde{\nu}_\perp(q^{h,\alpha}(t)))^T M &\left(\tilde{\nu}_\perp(q^{h,\alpha}(t))w^{h,\alpha}(t) - \tilde{\nu}_\perp(q^{h,\alpha}(t-h))w^{h,\alpha}(t-h) \right) \\
&= (\tilde{\nu}_\perp^T M \tilde{\nu}_\perp)(q^{h,\alpha}(t)) (w^{h,\alpha}(t) - w^{h,\alpha}(t-h)) \\
&\quad + h (\tilde{\nu}_\perp(q^{h,\alpha}(t)))^T M \left(\frac{\partial}{\partial q} (\tilde{\nu}_\perp^T(q)w^{h,\alpha}(t-h)) \Big|_{q=q^{h,\alpha}(t)} \right) \tilde{\nu}_\perp(q^{h,\alpha}(t))w^{h,\alpha}(t) \\
&\quad + \mathcal{O}(h^2).
\end{aligned} \tag{8.20}$$

Multiplying equation (8.14) on the left by $(\tilde{\nu}_\perp(q^{h,\alpha}(t)))^T$ and using (8.18), (8.20) we obtain

$$\begin{aligned}
(\tilde{\nu}_\perp(q^{h,\alpha}(t)))^T M \tilde{\nu}_\perp(q^{h,\alpha}(t)) \frac{dw^{h,\alpha}}{dt} - (\widehat{k}_{w,\perp}^h(t) + \mathcal{O}(\|v^{h,\alpha}(t) - v^{h,\alpha}(t-h)\|) + \mathcal{O}(h)) &\in \widehat{\mathcal{FC}}_r(q^{h,\alpha}(t)) \\
&\subset \mathcal{FC}_r(q^{h,\alpha}(t)),
\end{aligned} \tag{8.21}$$

where

$$\widehat{k}_{w,\perp}^h(t) = (\tilde{\nu}_\perp(q^{h,\alpha}(t)))^T \left(\widehat{k}^h(t) - M \left(\frac{\partial}{\partial q} (\tilde{\nu}_\perp^T(q)w^{h,\alpha}(t-h)) \Big|_{q=q^{h,\alpha}(t)} \right) \tilde{\nu}_\perp(q^{h,\alpha}(t))w^{h,\alpha}(t) \right) \tag{8.22}$$

Given that $q^{h,\alpha}(\cdot) \rightarrow q(\cdot)$ uniformly on $[0, T]$, $v^{h,\alpha}(\cdot) \rightarrow v(\cdot)$ a.e. on $[0, T]$ and $u^{h,\alpha}(\cdot) \rightarrow 0$ on $[0, T]$ we have

$$(\tilde{\nu}_\perp(q^{h,\alpha}(t)))^T M \tilde{\nu}_\perp(q^{h,\alpha}(t)) \rightarrow (\tilde{\nu}_\perp(q(t)))^T M \tilde{\nu}_\perp(q(t)) \text{ uniformly in } [0, T] \tag{8.23}$$

$$\widehat{k}_{w,\perp}^h(t) \rightarrow k_{w,\perp}(t, q(t), w(t)) \text{ pointwise a.e. on } [0, T] \tag{8.24}$$

To obtain the measure differential inclusion for the limits (q, w) we invoke [36, Theorem 4], stated in Appendix B, taking into account that (8.21), (8.23) and (8.24) are satisfied. In our case, the requirement of [36, Theorem 4] that $\min\{\|z\| \mid z \in K(w)\}$ is uniformly bounded is immediately satisfied because $K(w)$ are cones and always contain the zero element. Given also (8.23–8.24) as well as the fact that, from Lemma 3.5, $\mathcal{FC}_r(q)$ is uniformly pointed, we can apply the above result directly to obtain that the limits (q, w) satisfy the inclusion (8.4).

To complete this subsection, we note that for any $t \in [0, T]$, we have that

$$\begin{aligned}
q^{h,\alpha}(t) - q^{h,\alpha}(0) &= \int_{t_1}^{t_2} v^{h,\alpha}(\tau) d\tau \\
&= \int_{t_1}^{t_2} \tilde{\nu}_\perp(q^{h,\alpha}(\tau))w^{h,\alpha}(\tau) + \tilde{\nu}(q^{h,\alpha}(\tau))u^{h,\alpha}(\tau) d\tau
\end{aligned}$$

Since $u^{h,\alpha}(\cdot) \rightarrow 0 = u(\cdot)$ (this results from equation (8.15) and (8.16) together with $(\tilde{\nu}(q^{h,\alpha}(t_l)))^T v^{h,\alpha}(t) = 0$) as $h \rightarrow 0$ pointwise on $[0, T]$ and $q^{h,\alpha}(0) = q(0)$ we obtain that

$$q(t) = q(0) + \int_0^t \tilde{\nu}_\perp(q(\tau))w(\tau)$$

as required by (8.3).

8.2. Feasibility of the Limiting Trajectories.

LEMMA 8.3. *Assume that*

$$\Theta^{(i)}(q^0) = 0, \quad \Phi^{(j)}(q^0) \geq 0, \quad i = 1, \dots, m, \quad j = 1, \dots, p.$$

Then the limit $q(\cdot)$ is feasible in the sense that

$$\Theta^{(i)}(q(t)) = 0, \quad \Phi^{(j)}(q(t)) \geq 0, \quad i = 1, \dots, m, \quad j = 1, \dots, p, \quad \text{for all } t \in [0, T].$$

Proof. To prove the first part we note that by using the definition of the time-stepping scheme, the fact that the numerical velocities v^l are uniformly bounded as well as the fact that the algorithm solves a finite number of collisions in $[0, T]$, we obtain:

$$\left\| \left(\nu^{(i)}(q^{h,\alpha}(t)) \right)^T v^{h,\alpha}(t) \right\| \leq C_1 h, \quad \text{almost everywhere in } [0, T].$$

Taking the limit as $h \rightarrow 0$ gives:

$$\left(\nu^{(i)}(q(t)) \right)^T v(t) = 0 \quad \text{almost everywhere in } [0, T].$$

The last statement implies that for all $t \in [0, T]$ and all $i = 1, \dots, m$, we have

$$\begin{aligned} \Theta^{(i)}(q(t)) &= \Theta^{(i)}(q^0) + \int_0^t \left(\nu^{(i)}(q(\tau)) \right)^T v(\tau) d\tau \\ &= \Theta^{(i)}(q^0) \\ &= 0. \end{aligned}$$

To prove the second part, assume first that $\Phi^{(j)}(q^0) = 0$, for some $j \in \{1, \dots, m\}$. This implies that $j \in \mathcal{A}$ and therefore

$$\left(n^{(j)}(q^0) \right)^T (\alpha v^1 + (1 - \alpha)v^0) = 0.$$

Using this we obtain that $\Phi^{(j)}(q^1) = \Phi^{(j)}(q^0) + \mathcal{O}(h^2)$ which implies, by assumption **(H1)**, that

$$\Phi^{(j)}(q^1) \geq -C_2 h^2,$$

where the constant C_2 depends on the uniform bound for the velocities and the constant B_H in (4.14). Assuming $\Phi^{(j)}(q^1) \leq 0$, i.e., $j \in \mathcal{A}$ at step 2, we can bound (in the same fashion as we did above) the negative part of $\Phi^{(j)}(\cdot)$ at the next step by $\Phi^{(j)}(q^2) \geq -2C_2 h^2$. We can continue this process until the first k for which $\Phi^{(j)}(q^k) \leq 0$ and $\Phi^{(j)}(q^{k+1}) > 0$. We obtain the estimate:

$$\Phi^{(j)}(q^l) \geq -lC_2 h^2 \geq -(C_2 \cdot T)h, \quad l = 0, \dots, k, \tag{8.25}$$

where to obtain the last inequality we have used that $k \leq l_{\max} = \left\lfloor \frac{T}{h} \right\rfloor$.

If $\Phi^{(j)}(q^0) > 0$, the only way to obtain $\Phi^{(j)}(q^k) < 0$, for some k is to have at least one collision occurring. Assume that this k -th time-step is the first collisional time-step. We can guarantee by the collision-detection algorithm that $\Phi^{(j)}(q^k) \geq -C_3 h^2$ (for a fixed constant C_3), where q^k is the detected position for the collision. When computing the solution at step $(k+1)$, the index j is a component of the active set. We have two possibilities, for step $(k+1)$:

- The non-penetration constraint (j) leaves the active set, i.e., $\Phi^j(q^{k+1}) > 0$, in which case we can restart recursively, or,
- The non-penetration constraint (j) remains in the active set, i.e., $\Phi^{(j)}(q^{k+1}) \leq 0$. In this case, we have $\Phi^{(j)}(q^{k+1}) \geq -(C_3 + C_2)h^2$. Continuing like this until step $(k+r+1)$ where either take-off occurs or $(k+r+1) \geq \frac{T}{h}$, we obtain the estimate:

$$\Phi^{(j)}(q^{k+l}) \geq -(C_3 + lC_2)h^2 \geq -C_4 h, \quad l = 0, \dots, r, \quad (8.26)$$

Since the number of changes in the active set is uniformly upper bounded as $h \rightarrow 0$, we can separate the two cases above and combine equations (8.25–8.26) to obtain

$$\Phi^{(j)}(q^l) \geq -Ch, \quad 0 \leq l \leq \left\lfloor \frac{T}{h} \right\rfloor.$$

It follows that $\Phi^{(j)}(q^{h,\alpha}(t)) \geq -Ch$ for h sufficiently small and all $t \in [0, T]$. Taking the limit as $h \rightarrow 0$ we obtain $\Phi^{(j)}(q(t)) \geq 0$, $t \in [0, T]$. ■

We summarize the analysis above in the following result

THEOREM 8.4. *Assume that $\gamma = \alpha \in \left[\frac{1}{2}, 1\right]$ and conditions **(H1)**–**(H8)** hold. Then there exists a subsequence $h_k \rightarrow 0$ such that*

1. $q^{h_k, \alpha}(\cdot) \rightarrow q(\cdot)$ uniformly.
2. $v^{h_k, \alpha}(\cdot) \rightarrow v(\cdot)$ pointwise a.e.
3. $dv^{h_k, \alpha}(\cdot) \rightarrow dv(\cdot)$ weak * as Borel measures in $[0, T]$, and every such subsequence converges to a solution $(q(\cdot), v(\cdot))$ of the measure differential inclusion (6.10–6.11).

Therefore, $q(t), v(t)$ is a weak solution of our model.

9. Examples. In this section we present two numerical examples that illustrate some of the theoretical points made in this work.

9.1. A simple joint example. As an introductory example, consider the dynamics of the system $\ddot{q} = 0$, subject to the joint constraint $q = 0$, to which we apply the scheme (4.4) with parameters $\alpha = \frac{1}{2}$ and $\gamma = \frac{1}{2}$. If the initial conditions are $q = 0$ and $\dot{q} = 0$ then the exact solution satisfies $q(t) = 0$.

To model the effect of errors on initial conditions, we start with $q = 0$, $\dot{q} = \epsilon$. Our scheme produces $q^{l,\alpha} = 0$ and $v^l = (-1)^l \epsilon$. The total variation of the velocity for the time interval T is $\frac{2\epsilon T}{h}$, where h is the time step. Therefore, no matter how small the initial error, the total variation is unbounded, and the resulting velocity function does not converge pointwise as $h \rightarrow 0$. On the other hand, we can immediately see that $v^{l,\alpha} = 0$ and that the velocity function defined in our main result has bounded variation and is convergent pointwise. This also validates the fact that our bounded variation for $v^{l,\alpha}$ result holds irrespective of the initial error in constraint satisfaction, and that the same result cannot be proved for v^l (though for the case with exact satisfaction of the initial constraints we could neither prove nor disprove bounded variation of the velocity sequence).

Of course, this difficulty will disappear if we make $\epsilon = 0$. But on one hand, in practical examples exact satisfaction of the constraints is difficult to guarantee. And on the other hand, this example is indicative of the fact that $v^{l,\alpha}$ has a more stable behavior than v^l .

9.2. An example with stick-slip behavior. We want to further motivate our choice for the velocity sequence by looking at a very simple example, [28], with stick-slip behavior. In that example, a block of mass $m = 1$ is subjected to an exterior force $k(t) = 8 \cos(t)$ and is sliding on a flat table with friction coefficient $\mu = 0.8$. The initial position of the block is $q_0 = (3, 0)^T$ and the initial velocity is $v_0 = (0, 0)^T$. The gravity $G = (0, -mg)^T$ is calculated with $g = 9.81$. We compare the weighted numerical velocity sequence $v^{h,\alpha}(t)$ to the sequence $v^h(t)$, for $\alpha = \gamma = \frac{1}{2}$. The positions $q^{h,\alpha}(t_l)$ and velocities $v^h(t_l), v^{h,\alpha}(t_l)$ with $\alpha = \gamma = \frac{1}{2}$ are shown in Figure 9.2, and they indicate a typical stick-slip behavior. We note that the numerical velocities exhibit a quite different behavior, in line with our

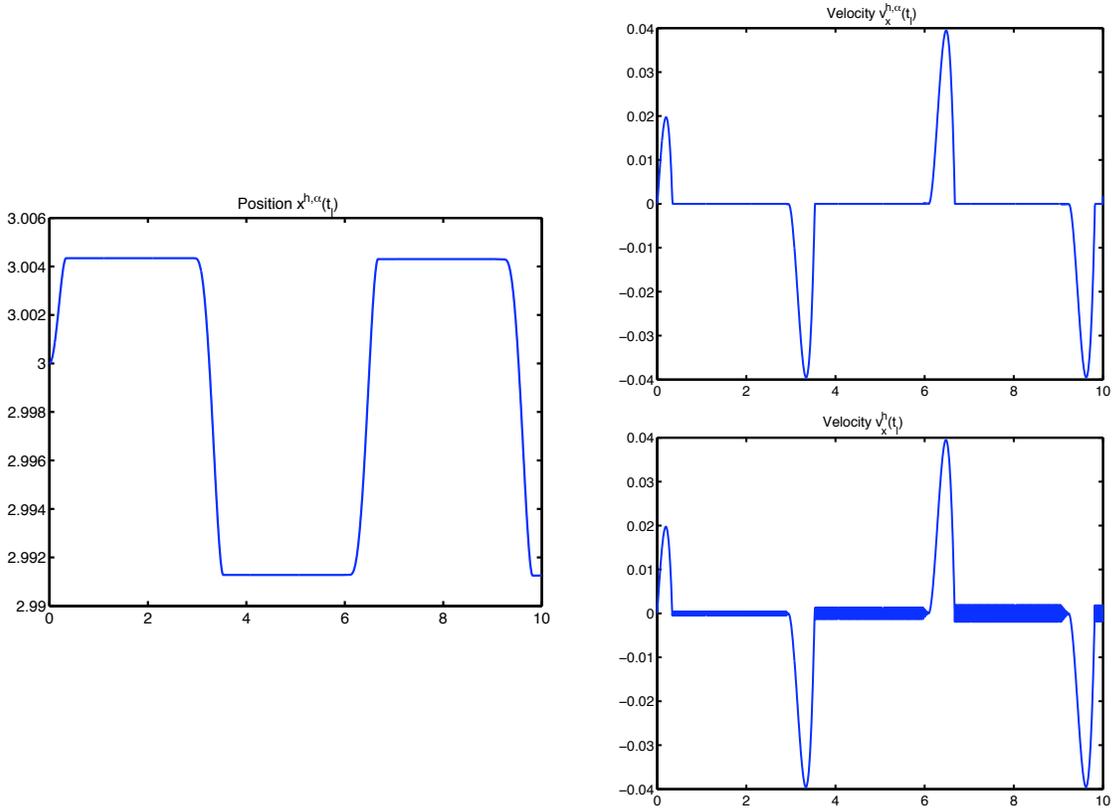


FIG. 9.1. Numerical position and velocities for $T = 10$ (s), $\alpha = \gamma = 1/2$ and $h = 0.01$. The plot on the left shows positions $q_1^{h,\alpha}(t_l)$, while the two plots on the right show the velocity sequences $v_1^{h,\alpha}(t_l)$ at the top and $v_1^h(t_l)$ at the bottom.

observations from the preceding sections. We see that, starting with the onset of sticking, the velocity sequence v^l exhibits oscillations that are not present in the sequence $v^{l,\alpha}$, which has the value 0 during the sticking phase. As opposed to the previous example, we do not obtain unbounded variation, though the total variation of the two velocity solutions is different. Nonetheless, the example illustrates the difficulty in obtaining a good behavior of the total variation of the velocity solution v^l , as opposed to $v^{l,\alpha}$, and justifies our choice of the latter for our convergence result.

10. Conclusions. In this work, we have defined a convergence framework for a class of time-stepping schemes for multi-rigid-body dynamics with joints, contact, and friction. In our framework the numerical solution is shown to converge to the solution of a measure differential inclusion. *The novelty of our approach resides in the fact that convergence in an MDI sense of an LCP time-stepping scheme is proved, for the first time, for the case that involves joint constraints as well.* We note that such a proof does not directly follow from representing a joint constraint (an equality constraint) as two opposite inequality constraints (contact constraints) and applying previous convergence results [33, 34], because the resulting system cannot possibly have a pointed friction cone, since any action can be realized with infinite multipliers by cancellation. The situation is analogous to the loss of the Mangasarian-Fromovitz constraint qualification in nonlinear programming when one equality constraint is represented as two inequality constraints [22]. In this work, results for cases involving joints are proved by defining the measure differential inclusion with respect to an appropriately defined reduced friction cone.

The convergence framework presented here accommodates time-stepping methods based on semi-explicit Euler methods [4, 33] as well as various instances of the trapezoidal method that have been shown to have second-order convergence under certain assumptions [28]. An important step in the convergence proof, following the technique developed in [34], is the proof of the bounded variation of the discrete velocity sequence. We show that, although this may not hold for most trapezoidal-like methods for the natural discrete velocity sequence ($v(t) = v^{(l+1)}$, for $t \in (t_l, t_{l+1}]$, which is the one used in the seminal work [34]), it does hold for the modified velocity sequence $v(t) = \alpha v^{(l+1)} + (1 - \alpha)v^{(l)}$ for $t \in (t_l, t_{l+1}]$ where α is the parameter used in the enforcement of the linearization of the geometrical constraints (contact and joint constraints). This point is reinforced by numerical examples.

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Appendix A. The Details in the Derivation of (8.2). In this section we present the details of obtaining equation (8.2). The main result that we use can be found in [20], page 9 and it is listed below:

LEMMA A.1 ([20], pp. 9). *If $u_1, u_2 \in \text{BV}([0, T], \mathbb{R}^k)$ then $d(u_1^T u_2)$ is a real Borel measure on $[0, T]$, which we write $d(u_1^T u_2) \in \mathcal{B}([0, T], \mathbb{R})$ and*

$$\begin{aligned} d(u_1^T u_2) &= (u_2^-)^T du_1 + (u_1^+)^T du_2 \\ &= (u_2^+)^T du_1 + (u_1^-)^T du_2, \end{aligned} \quad (1.1)$$

where for a function $f \in \text{BV}([0, T], \mathbb{R}^k)$, f^+ (f^-) denotes the right-limit (left-limit) of f . More precisely $f^+(t) = \lim_{s \rightarrow t, s > t} f(s)$ ($f^-(t) = \lim_{s \rightarrow t, s < t} f(s)$) with the convention that if t is the right (left) endpoint of $[0, T]$ we take $f^+(t) = f(t)$ ($f^-(t) = f(t)$). Note that since f is of bounded variation these limits exist for all t in $[0, T]$.

Proving (8.2). We recall that $q : [0, T] \rightarrow \mathbb{R}^s$ is a Lipschitz continuous function, $v = \tilde{v}_\perp(q)w \in \text{BV}([0, T], \mathbb{R}^s)$ and $\tilde{v}_\perp : \mathbb{R}^s \rightarrow \mathbb{R}^{s \times (s-m)}$ is sufficiently smooth. We further assume that $v(\cdot) = v^+(\cdot)$ (Note that since $q(\cdot)$ is continuous and $\tilde{v}_\perp(\cdot)$ is uniformly full column rank this also implies that $w(\cdot)$ is equal to its right limit). To prove (8.2) the steps itemized below are followed.

- **Chain rule:** $d(\tilde{v}_\perp(q)w) = \tilde{v}_\perp(q)dw + \frac{\partial}{\partial q}(\tilde{v}_\perp(q)w) dq$

For every $i \in \{1, \dots, s\}$ we apply (1.1) with

$$u_1 = (\tilde{v}_\perp(q))_i \text{ and } u_2 = w.$$

Here if A is a given matrix A_i denotes its i -th row written in column format. Since $q(\cdot)$ is Lipschitz continuous and $\tilde{v}_\perp(\cdot)$ is sufficiently smooth it follows that $u_1 \in \text{BV}([0, T], \mathbb{R}^{s-m})$ and $u_1^+(t) = u_1^-(t) = u_1(t)$, for all $t \in [0, T]$. We also have $u_2 = w \in \text{BV}([0, T], \mathbb{R}^{s-m})$. Using (1.1) we obtain

$$(dv)_i = (d(\tilde{v}_\perp(q))_i)^T w + ((\tilde{v}_\perp(q))_i)^T dw, \quad (1.2)$$

where we have used the continuity of u_1 and right continuity of u_2 . Since $\tilde{v}_\perp(\cdot)$ is sufficiently smooth we can write

$$d(\tilde{v}_\perp(q))_i = \left(\frac{\partial}{\partial q} ((\tilde{v}_\perp)_i)(q) \right) dq, \quad (1.3)$$

where the $(s-m) \times s$ matrix in the right-hand side is the Jacobian of $(\tilde{v}_\perp(q))_i$. Using (1.3) in (1.2) for all i gives the desired result, i.e.,

$$d(\tilde{v}_\perp(q)w) = \tilde{v}_\perp(q)dw + \frac{\partial}{\partial q}(\tilde{v}_\perp(q)w) dq. \quad (1.4)$$

- **The differential vector measure induced by q :** $dq = vdt$.

Since

$$q(t) = q(0) + \int_0^t v(\tau) d\tau,$$

for all $t \in [0, T]$ and v is bounded on $[0, T]$ it follows that dq is absolutely continuous w.r.t. the Lebesgue measure dt and the Radon-Nicodym derivative is

$$v = \frac{dq}{dt}.$$

and therefore we may write $dq = vdt$. Note that the Radon-Nicodym derivative (with respect to the Lebesgue measure) is uniquely determined up to a set of (Lebesgue) measure 0.

Appendix B. Theorem 4, [36]. *Suppose that $q_{\hat{n}}(\cdot)$ are continuous, $v_{\hat{n}}(\cdot)$ have uniformly bounded variation and $k_{\hat{n}}(\cdot)$ are uniformly bounded, all on $[0, T]$, and $q_{\hat{n}}(\cdot) \rightarrow q(\cdot)$ uniformly, $v_{\hat{n}}(\cdot) \rightarrow v(\cdot)$ pointwise a.e. and $k_{\hat{n}}(\cdot) \rightarrow k(\cdot)$ pointwise a.e. Suppose also that $K : \mathbb{R}^n \rightrightarrows \mathcal{C}(\mathbb{R}^n)$ has closed graph, $\min \{\|z\| \mid z \in K(w)\}$ is uniformly bounded and $K(w)$ is pointed for all $w \in \mathbb{R}^n$. Then if*

$$\frac{dv_{\hat{n}}}{dt}(t) \in K(q_{\hat{n}}(t)) - k_{\hat{n}}(t)$$

for all \hat{n} , the limit satisfies

$$\frac{dv}{dt}(t) \in K(q(t)) - k(t).$$

Here $\mathcal{C}(\mathbb{R}^n)$ denotes all the closed and convex subsets of \mathbb{R}^n .

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