Acyclic and Star Colorings of Joins of Graphs and an Algorithm for Cographs (Extended Abstract)

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An *acyclic coloring* of a graph is a proper vertex coloring such that the subgraph induced by the union of any two color classes is a disjoint collection of trees. The more restricted notion of *star coloring* requires that the union of any two color classes induces a disjoint collection of stars. The acyclic and star chromatic numbers of a graph G are defined analogously to the chromatic number $\chi(G)$ and are denoted by $\chi_a(G)$ and $\chi_s(G)$, respectively. In this paper, we consider acyclic and star colorings of graphs that are decomposable with respect to the join operation, which builds a new graph from a collection of two or more disjoint graphs by adding all possible edges between them. In particular, we present a recursive formula for the acyclic chromatic number of joins of graphs and show that a similar formula holds for the star chromatic number. We also demonstrate the algorithmic implications of our results for the cographs, which have the unique property that they are recursively decomposable with respect to the join and disjoint union operations.

1 Introduction

Both acyclic and star colorings have applications in the field of combinatorial scientific computing, where they model two different schemes for the evaluation of sparse Hessian matrices. The general idea behind the use of coloring in computing derivative matrices is the identification of entities that are essentially independent and thus may be computed concurrently; see [5] for a survey.

A number of results exist for acyclic and star colorings of graphs formed by certain graph operations. Results have been obtained for Cartesian products of paths [4], trees [9], cycles [7], and complete graphs [8]. In Section 2, we describe the acyclic and star chromatic numbers of graphs formed by the join operation. The *join* of a collection $\{G_i = (V_i, E_i)\}_{i\in\mathcal{I}}$ of pairwise disjoint graphs,

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denoted $\oplus$, is the graph $G = (V, E)$, where $V = \bigcup_{i \in \mathcal{I}} V_i$ and $E = \{ab \mid ab \in E_i, i \in \mathcal{I}\} \cup \{ab \mid a \in V_i, b \in V_j, i, j \in \mathcal{I}, i \neq j\}$. Here and throughout this paper, $\mathcal{I}$ denotes a finite index set.

The problems of finding optimal acyclic and star colorings are both **NP**-hard and remain so even for bipartite graphs [2, 1]. It was shown recently [6] that every coloring of a chordal graph is also an acyclic coloring. Since recognizing and optimally coloring chordal graphs can be done in linear time, this result immediately implies a linear time algorithm for the acyclic coloring problem on chordal graphs. A generalization of this result and other related results can be found in [10], where it is shown that the graphs for which every acyclic coloring is also a star coloring are exactly the cographs. In Section 3, we show that our results imply a linear time algorithm for finding optimal acyclic and star colorings of cographs.

2 Joins of graphs

In this section, we outline a proof of the following theorem.

Theorem 1. *Let* $\{G_i = (V_i, E_i)\}_{i \in \mathcal{I}}$ *be a finite collection of graphs. Then*

(i) $$\chi_a\left(\bigoplus_{i \in \mathcal{I}} G_i\right) = \sum_{i \in \mathcal{I}} \chi_a(G_i) + \min_{j \in \mathcal{I}} \left\{ \sum_{i \in \mathcal{I}, i \neq j} (|V_i| - \chi_a(G_i)) \right\};$$

(ii) $$\chi_s\left(\bigoplus_{i \in \mathcal{I}} G_i\right) = \sum_{i \in \mathcal{I}} \chi_s(G_i) + \min_{j \in \mathcal{I}} \left\{ \sum_{i \in \mathcal{I}, i \neq j} (|V_i| - \chi_s(G_i)) \right\}.$$

For ease of exposition, we will focus on the case where G is the join of exactly two graphs as in the following lemma. To see that these results generalize to joins of arbitrarily large collections of graphs, first observe that the join operation is commutative and associative; the result is then obtained by using induction on $|\mathcal{I}|$.

Lemma 2. *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be graphs. Then*

(i) $\chi_a(G_1 \oplus G_2) = \chi_a(G_1) + \chi_a(G_2) + \min\{|V_1| - \chi_a(G_1), |V_2| - \chi_a(G_2)\}$;

(ii) $\chi_s(G_1 \oplus G_2) = \chi_s(G_1) + \chi_s(G_2) + \min\{|V_1| - \chi_s(G_1), |V_2| - \chi_s(G_2)\}$.

We now sketch the idea behind the proof of this lemma. Suppose we are given graphs G_1 and G_2 and we wish to find an optimal acyclic or star coloring of their join. Since every vertex in V_1 is adjacent to every vertex in V_2, no color can occur in V_1 and V_2 simultaneously. Moreover, the desired coloring must also be valid for the subgraphs induced by each $V_i, i \in \{1, 2\}$, where the lower bound will be $\chi_a(G_i)$ or $\chi_s(G_i)$ depending on the type of coloring that is sought. The key observation is that at least one V_i must be *saturated*, meaning that each vertex receives a unique color. It can be shown that G will otherwise contain a bichromatic cycle – a violation of the conditions of acyclic coloring. Furthermore, such a bichromatic cycle implies a bichromatic path on four vertices, which cannot occur in a star coloring. Thus, given disjoint optimal acyclic colorings of G_1 and G_2, an optimal acyclic coloring of their join can be constructed by saturating the graph G_i that minimizes $|V_i| - \chi_a(G_i)$. It is easy to see that the same procedure can be used in the context of star coloring.

3 Cographs

In this section, we outline a linear time algorithm for finding optimal acyclic and star colorings of cographs. The algorithm works on the cotree — defined below — in a way that is typical for algorithms on cographs. We begin with some definitions. The *disjoint union* of a collection $\{G_i = (V_i, E_i)\}_{i\in\mathcal{I}}$ of pairwise disjoint graphs, denoted $\cup$, is the graph $G = (V, E)$, where $V = \bigcup_{i\in\mathcal{I}} V_i$ and $E = \bigcup_{i\in\mathcal{I}} E_i$. A graph $G = (V, E)$ is a *cograph* if and only if one of the following is true:

(i) $|V| = 1$;

(ii) there exists a collection $\{G_i\}_{i\in\mathcal{I}}$ of cographs such that $G = \bigcup_{i\in\mathcal{I}} G_i$;

(iii) there exists a collection $\{G_i\}_{i\in\mathcal{I}}$ of cographs such that $G = \bigoplus_{i\in\mathcal{I}} G_i$.

Cographs can be recognized in linear time [3], where most recognition algorithms also produce a special decomposition structure when the input graph G is a cograph. We associate with a cograph G a tree T_G called a *cotree*, whose leaves correspond to the vertices of G and whose internal nodes are labeled either 0 or 1. The 0-nodes correspond to the disjoint union of their children, and the 1-nodes correspond to the join of their children.

As in Section 2, we describe the binary case, which can be appropriately generalized. The algorithm proceeds by traversing the cotree starting with the leaves, such that no node is visited before both of its children have been visited. We do the following when we visit a node $t \in T_G$ with children t_1 and t_2. If t is a 0-node, we construct a coloring that uses $\chi_a(t) = \max\{\chi_a(t_1), \chi_a(t_2)\}$ colors in the obvious way. If t is a 1-node, we use the process described in Section 2 to construct a coloring that is optimal by Theorem 1. Since the algorithm produces an optimal acyclic coloring for every node in the cotree, the last step will produce an optimal acyclic coloring of G itself. Our final theorem follows from the fact that every acyclic coloring of a cograph is also a star coloring and vice versa.

Theorem 3. *An optimal acyclic coloring of a cograph can be found in linear time. Furthermore, the obtained coloring is also an optimal star coloring.*

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