

# Convergence Analysis of a Parallel Newton Scheme for Dynamic Power Grid Simulations

Brett A. Robbins, *Student Member, IEEE*, Victor M. Zavala, *Member, IEEE*

**Abstract**—In this work, we analyze the convergence properties of a parallel Newton scheme for differential systems. The scheme concurrently solves the time-coupled nonlinear systems arising from the application of implicit discretization schemes. We have found that the scheme acts as a tracking algorithm that converges to the manifold given by the solution of the nonlinear system at the current time step parameterized in the "moving" iterating solution at the previous step. This property explains why the method can significantly reduce the number of iterations compared to sequential Newton methods. We have also found, however, that the method exhibits a theoretical lower bound on the number of iterations equal to the number of discretization points. A numerical study using a detailed dynamic power grid model is provided to demonstrate the developments.

**Index Terms**—Parallel Processing, Dynamic Simulation, Power Systems, Newton's Method, Convergence

## I. INTRODUCTION

THE U.S. power grid is expected to sustain highly volatile environments as a result of large adoptions of intermittent renewable power and price-responsive demands. The understanding of transient phenomena arising under these environments has been hindered by the computational complexity of the associated dynamic models. For instance, in the 2003 blackout report produced by the U.S. Department of Energy, it was highlighted that limited computational resources prevented a more detailed analysis of the dynamic phenomena that triggered cascading events. This indicates that, while high-performance computing systems have evolved significantly during recent years, they have not been fully exploited to address existing needs in power grid simulation. Motivated by these limitations, we analyze the convergence properties of a parallel method for large-scale dynamic simulation.

Parallel methods for dynamic power grid simulations date back to the 1970's [1] and can be broadly classified as (i) parallel in time, (ii) parallel in space, and (iii) combinations of both. An extensive review can be found in [2].

The majority of the proposed methods can be classified as parallel in space strategies where weak coupling between buses is exploited to partition the network [3]–[5]. Alternatively, the approaches in [6]–[9] decouple the network equations from the differential equations.

Parallel in time simulation strategies have been reported in [10]–[13]. Similar to space partitioning, these relaxation

(e.g., Gauss-Jacobi) methods allow the time horizon to be partitioned and solved concurrently by dropping the coupling terms in the Newton system between neighboring time steps. This type of methods are also referred to as parallel block Newton or multi-splitting methods [14]. Empirically, it has been observed that parallel in time Newton methods can significantly reduce the number of iterations compared to sequential Newton methods. In addition, parallel method allow to overcome memory bottlenecks associated to full-space Newton methods. An existing limitation, however, is that the convergence properties of parallel block Newton methods cannot be easily analyzed using traditional techniques.

This work provides alternative insights on the convergence properties of a parallel in time block Newton method and on efficiency gains over the sequential method. We have found that the parallel Newton scheme acts as a manifold tracking algorithm that converges rapidly to a manifold given by the solution of the nonlinear system of current time step parameterized by the iterating solution at the previous step. This property explains why the method can significantly reduce the number of iterations compared to sequential methods.

The paper is structured as follows. Sections II and III describe the dynamic simulation setting. Section IV provides convergence results for the parallel Newton method. A differential algebraic model for power systems is presented in Section V. This model is used to demonstrate the performance of parallel method in Section VI. Section VII concludes this paper and provides directions for future work.

## II. SETTING

We consider the differential-algebraic (DAE) system,

$$\dot{z}(t) = f(z(t), y(t)), \quad z(0) = z_0 \quad (2a)$$

$$0 = g(z(t), y(t)), \quad t \in [0, T] \quad (2b)$$

where  $z(\cdot) \in \mathbb{R}^{n_z}$  are the differential states with initial conditions  $z_0$ ,  $y(\cdot) \in \mathbb{R}^{n_y}$  are the algebraic states,  $t$  is the scalar time dimension, and  $T$  is the final time. The mappings  $f : \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_z}$  and  $g : \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_y}$  are assumed to be at least twice continuously differentiable. The DAE system is assumed to be index one.

We seek to solve the above system using an implicit discretization approach such as implicit Euler, Gauss, and Radau collocation [15]. In the case of implicit Euler, for instance, we

V. M. Zavala is with the Mathematics and Computer Science Division at Argonne National Laboratory, Argonne, IL 60439. e-mail: zavala@mcs.anl.gov

B. A. Robbins is with the Department of Electrical Engineering, University of Illinois at Urbana-Champaign, Urbana, IL, 61801. e-mail: robins3@illinois.edu.

$$\Phi'(x) = \begin{bmatrix} \nabla_{x_1} \Phi(x_1, x_0) \\ \nabla_{x_1} \Phi(x_2, x_1) & \nabla_{x_2} \Phi(x_2, x_1) \\ & \ddots \\ \nabla_{x_{N-1}} \Phi(x_{N-1}, x_{N-2}) \\ \nabla_{x_{N-1}} \Phi(x_N, x_{N-1}) & \nabla_{x_N} \Phi(x_N, x_{N-1}) \end{bmatrix} \quad (1)$$

obtain a set of nonlinear equations of the following form:

$$\begin{aligned} \Phi_z(z_{k+1}, y_{k+1}, z_k) &= z_{k+1} - z_k - h_k \cdot f(z_{k+1}, y_{k+1}) \\ &= 0, \end{aligned} \quad (3a)$$

$$\begin{aligned} \Phi_y(z_{k+1}, y_{k+1}, z_k) &= g(z_{k+1}, y_{k+1}) \\ &= 0, \end{aligned} \quad (3b)$$

for  $k = 0, \dots, N-1$ . Here,  $N$  is the number of discretization steps of length  $h_k$  and we have  $\sum_{k=0}^{N-1} h_k = T$ . If we group the differential and algebraic states into a single vector  $x_k \in \mathbb{R}^{n_z+n_y}$ ,  $k = 0, \dots, N$  we can write the above system in the following general form:

$$\Phi(x_{k+1}, x_k) = 0, \quad k = 0, \dots, N-1 \quad (4)$$

We will refer to the above system as the *full space system*. Its solution is denoted as  $x_k^*$ ,  $k = 0, \dots, N$ . The Jacobian of the mapping  $\Phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x}$  with respect to the first argument ( $x_{k+1}$ ) is denoted as  $\Phi' := \nabla_{x_{k+1}} \Phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_x}$ . If an implicit Euler discretization scheme is used, the Jacobian terms of the full-space system have the following structure,

$$\begin{aligned} \Phi'_z(z_{k+1}, y_{k+1}, z_k) &= \mathbb{I}_{n_z} - h_k (\nabla_z f(z_{k+1}, y_{k+1}) + \nabla_y f(z_{k+1}, y_{k+1})) \end{aligned} \quad (5a)$$

$$\begin{aligned} \Phi'_y(z_{k+1}, y_{k+1}, z_k) &= \nabla_z g(z_{k+1}, y_{k+1}) + \nabla_y g(z_{k+1}, y_{k+1}) \end{aligned} \quad (5b)$$

for  $k = 0, \dots, N-1$ .

### III. PARALLEL NEWTON METHOD

To solve the full-space system (4), we make use of a *parallel in time Newton scheme*. The iteration  $j$  takes the form:

$$\Phi'(x_{k+1}^j, x_k^j) \Delta x_{k+1}^j = -\Phi(x_{k+1}^j, x_k^j), \quad k = 0, \dots, N-1 \quad (6)$$

where,

$$x_{k+1}^{j+1} = x_{k+1}^j + \Delta x_{k+1}^j, \quad k = 0, \dots, N-1. \quad (7)$$

Here,  $x_0$  is the fixed initial condition. The stopping criterion is the convergence of the *global* error,

$$\left\| \begin{bmatrix} \Phi(x_1^j, x_0^j) \\ \vdots \\ \Phi(x_N^j, x_{N-1}^j) \end{bmatrix} \right\|_{\infty} \leq \epsilon. \quad (8)$$

We note that the Newton step for all time steps  $k = 0, \dots, N-1$  can be computed in parallel. This method is also often called block-Newton or multi-splitting method [14], [16], [17]. The method can be interpreted as a Newton scheme applied to the

full-space system (4) that drops the time coupling terms from the Jacobian matrix. To see this, we define the full-space vector  $x^T = [x_1^T, \dots, x_N^T]^T$  and system,

$$\Phi(x) = \begin{bmatrix} \Phi(x_1, x_0) \\ \Phi(x_2, x_1) \\ \vdots \\ \Phi(x_N, x_{N-1}) \end{bmatrix}. \quad (9)$$

The Jacobian for the full-space system is given in equation (1). The full-space Newton iteration is  $x^{j+1} = x^j + \Phi'(x^j)^{-1} \Phi(x^j)$ . By dropping the off-diagonal terms from the Jacobian and by noticing that  $\Phi'(x_{k+1}, x_k) = \nabla_{x_{k+1}} \Phi(x_{k+1}, x_k)$  we recover the parallel Newton scheme (6). The advantage of the full-space Newton method is that it gives a fast convergence rate. A key limitation, however, are the increasing computational times and memory requirements as the number of discretization points is extended.

We also highlight the difference of the parallel Newton scheme with that of the *sequential* Newton scheme that marches forward in time. This scheme solves the system  $\Phi(x_1, x_0, h_0)$  starting at  $k = 0$  to obtain  $x_1^*$  and then solves  $\Phi(x_2, x_1^*, h_1)$  to obtain  $x_2^*$  and so on. In other words, the system iterations on,

$$\Phi'(x_{k+1}^j, x_k^*) \Delta x_{k+1}^j = -\Phi(x_{k+1}^j, x_k^*), \quad k = 0, \dots, N-1 \quad (10)$$

The stopping criterion of this method is based on the convergence of the *local* errors,

$$\left\| \Phi(x_{k+1}^j, x_k^*) \right\|_{\infty} \leq \epsilon, \quad k = 0, \dots, N-1. \quad (11)$$

One of the advantages of this approach is that it is not as memory demanding as a full-space Newton scheme. A limitation of the sequential scheme, however, is that it wastes computational time by tightening the error of the *local* nonlinear system at time step  $k$  when this might not be necessary for the next system  $k+1$ . In other words, the scheme lacks of a *global* view. This is particularly inefficient in simulation-based optimization where low-precision simulations are often needed. The parallel Newton scheme iterates simultaneously over the entire set of nonlinear equations and monitors the global error (8) as in the full-space Newton method. Consequently, significant amounts of computational time can be saved. In addition, the scheme can be run in a distributed memory system and thus can accommodate long time horizons and fine discretization resolutions.

### IV. CONVERGENCE ANALYSIS

A key observation that we make in this paper is that the parallel Newton scheme can be interpreted as a Newton

scheme under *parametric perturbations*. As we will see, this interpretation is key in assessing the convergence properties and computational limitations of the scheme.

To start the discussion we consider the local system at time step  $k + 1$ ,

$$\Phi(x_{k+1}, x_k) = 0. \quad (12)$$

The Newton scheme tries to find  $x_{k+1}^*$  by linearizing the above system around the current iterate  $x_{k+1}^j$  and treats  $x_k$  as an *exogenous* parameter with current value  $x_k^j$ . The Newton system is,

$$\Phi(x_{k+1}^j, x_k^j) + \Phi'(x_{k+1}^j, x_k^j)(x_{k+1}^{j+1} - x_{k+1}^j) = 0. \quad (13)$$

Here,  $x_k^j$  is the exogenous parameter sequence over the iteration sequence  $j = 0, \dots, J_k$  converging to a limit point  $x_k^*$ . For  $k = 0$  we have  $x_0^j = x_0$  which are the initial conditions.

Consider the following *perturbed* system,

$$\bar{\Phi}(x_{k+1}, x_k^*) = r. \quad (14)$$

where  $r$  is a residual or perturbation. The solution of this system is denoted as  $x_{k+1}(r)$  and satisfies  $x_{k+1}^* = x_{k+1}^*(x_k^*) = x_{k+1}(0)$ .

**Definition 1: (Strong Regularity.)** The system (14) is said to be strongly regular at  $x_{k+1}^*$  [18] if there exists  $L > 0$  such that,

$$\|x_{k+1}(r) - x_{k+1}^*\| \leq L\|r\|. \quad (15)$$

In [18], it is shown that a condition for the system (14) to be strongly regular is that the derivative matrix  $\Phi'(x_{k+1}^*, x_k^*)$  is non-singular.

The Newton system (13) at iteration  $j$  can be posed in the form of (14) by adding and subtracting  $\Phi(x_{k+1}^{j+1}, x_k^*)$ ,

$$\begin{aligned} \Phi(x_{k+1}^{j+1}, x_k^*) &= \Phi(x_{k+1}^{j+1}, x_k^*) - \Phi(x_{k+1}^j, x_k^j) \\ &\quad - \Phi'(x_{k+1}^j, x_k^j)(x_{k+1}^{j+1} - x_{k+1}^j). \end{aligned} \quad (16)$$

Under strong regularity we have that,

$$\|x_{k+1}^{j+1} - x_{k+1}^*\| \leq L\|r_{k+1}^j\|, \quad (17)$$

where,

$$\begin{aligned} r_{k+1}^j &= \Phi(x_{k+1}^{j+1}, x_k^*) - \Phi(x_{k+1}^j, x_k^j) \\ &\quad - \Phi'(x_{k+1}^j, x_k^j)(x_{k+1}^{j+1} - x_{k+1}^j). \end{aligned} \quad (18)$$

**Theorem 1: (Convergence of Perturbed Newton System.)** Assume that the system (14) is strongly regular at  $x_{k+1}^*, x_k^*$ . Assume also that the exogenous sequence  $x_k^j$  converges to the limit point  $x_k^*$ . If there exists  $\sigma \geq 0$  such that  $\|x_k^j - x_k^*\| \leq \sigma\|x_{k+1}^{j+1} - x_{k+1}^j\|$  then the Newton scheme converges superlinearly to  $x_{k+1}^*$ . If  $\|x_k^j - x_k^*\| \leq \sigma\|x_{k+1}^{j+1} - x_{k+1}^j\|^2$  and the mapping  $\Phi'$  is Lipschitz in both arguments then, convergence is quadratic.

**Proof:** We first note that the residual of the Newton system can be expanded as,

$$\begin{aligned} r_{k+1}^j &= \Phi(x_{k+1}^{j+1}, x_k^*) - \Phi(x_{k+1}^j, x_k^j) - \Phi'(x_{k+1}^j, x_k^j)(x_{k+1}^{j+1} - x_{k+1}^j) \\ &= \Phi(x_{k+1}^{j+1}, x_k^*) - \Phi(x_{k+1}^j, x_k^*) - \Phi'(x_{k+1}^j, x_k^*)(x_{k+1}^{j+1} - x_{k+1}^j) \\ &\quad + \Phi(x_{k+1}^{j+1}, x_k^j) - \Phi(x_{k+1}^j, x_k^j) - \Phi'(x_{k+1}^j, x_k^j)(x_{k+1}^{j+1} - x_{k+1}^j) \\ &\quad - \Phi(x_{k+1}^j, x_k^j) + \Phi(x_{k+1}^j, x_k^*) + \Phi'(x_{k+1}^j, x_k^*)(x_{k+1}^{j+1} - x_{k+1}^j) \\ &\quad + \Phi(x_{k+1}^{j+1}, x_k^*) - \Phi(x_{k+1}^{j+1}, x_k^j). \end{aligned} \quad (19)$$

Define the exogenous perturbation term,

$$\begin{aligned} w_{k+1}^j &= \Phi(x_{k+1}^{j+1}, x_k^j) - \Phi(x_{k+1}^j, x_k^j) - \Phi'(x_{k+1}^j, x_k^j)(x_{k+1}^{j+1} - x_{k+1}^j) \\ &\quad + \Phi(x_{k+1}^j, x_k^*) - \Phi(x_{k+1}^{j+1}, x_k^*) + \Phi'(x_{k+1}^j, x_k^*)(x_{k+1}^{j+1} - x_{k+1}^j) \\ &\quad + \Phi(x_{k+1}^{j+1}, x_k^*) - \Phi(x_{k+1}^{j+1}, x_k^j). \end{aligned} \quad (20)$$

so that,

$$\begin{aligned} r_{k+1}^j &= \Phi(x_{k+1}^{j+1}, x_k^*) - \Phi(x_{k+1}^j, x_k^*) \\ &\quad - \Phi'(x_{k+1}^j, x_k^*)(x_{k+1}^{j+1} - x_{k+1}^j) + w_{k+1}^j. \end{aligned} \quad (21)$$

Bounding and applying the mean value theorem we have,

$$\begin{aligned} \|r_{k+1}^j\| &\leq \sup_{\gamma \in [0,1]} \{ \Phi'(\gamma x_{k+1}^{j+1} + (1-\gamma)x_{k+1}^j, x_k^*) - \Phi'(x_{k+1}^j, x_k^*) \} \\ &\quad \cdot \|\Delta x_{k+1}^j\| + \|w_{k+1}^j\| \\ &= \kappa_1 \|\Delta x_{k+1}^j\| + \|w_{k+1}^j\|. \end{aligned} \quad (22)$$

The exogenous term is bounded as,

$$\begin{aligned} \|w_{k+1}^j\| &\leq \sup_{\gamma \in [0,1]} \{ \Phi'(\gamma x_{k+1}^{j+1} + (1-\gamma)x_{k+1}^j, x_k^j) - \Phi'(x_{k+1}^j, x_k^j) \} \\ &\quad \cdot \|\Delta x_{k+1}^j\| \\ &\quad + \sup_{\gamma \in [0,1]} \{ \Phi'(x_{k+1}^j, x_k^*) - \Phi'(\gamma x_{k+1}^{j+1} + (1-\gamma)x_{k+1}^j, x_k^*) \} \\ &\quad \cdot \|\Delta x_{k+1}^j\| \\ &\quad + L_\Phi \|x_k^j - x_k^*\| \end{aligned} \quad (23)$$

$$\leq \kappa_2 \|\Delta x_{k+1}^j\| + \kappa_3 \|x_k^j - x_k^*\|, \quad (24)$$

so that,

$$\|r_{k+1}^j\| \leq (\kappa_1 + \kappa_2) \|\Delta x_{k+1}^j\| + \kappa_3 \|x_k^j - x_k^*\|. \quad (25)$$

We first establish *superlinear* convergence from,

$$\begin{aligned} \|x_{k+1}^{j+1} - x_{k+1}^*\| &\leq L\|r_{k+1}^j\| \\ &\leq L(\kappa_1 + \kappa_2) \|\Delta x_{k+1}^j\| + L\kappa_3 \|x_k^j - x_k^*\| \\ &= L(\kappa_1 + \kappa_2) \|x_{k+1}^{j+1} - x_{k+1}^j\| + L\kappa_3 \|x_k^j - x_k^*\| \\ &= L(\kappa_1 + \kappa_2) \|x_{k+1}^{j+1} - x_{k+1}^* + x_{k+1}^* - x_{k+1}^j\| \\ &\quad + L\kappa_3 \|x_k^j - x_k^*\|. \end{aligned} \quad (26)$$

From the assumptions we have,

$$\begin{aligned} \|x_k^j - x_k^*\| &\leq \sigma \|\Delta x_{k+1}^j\| \\ &\leq \sigma \left( \|x_{k+1}^{j+1} - x_{k+1}^*\| + \|x_{k+1}^j - x_{k+1}^*\| \right), \end{aligned} \quad (27)$$

and

$$\begin{aligned} \|x_{k+1}^{j+1} - x_{k+1}^*\| &\leq L(\kappa_1 + \kappa_2 + \sigma\kappa_3) \cdot \\ &\cdot \left( \|x_{k+1}^{j+1} - x_{k+1}^*\| + \|x_{k+1}^j - x_{k+1}^*\| \right). \end{aligned} \quad (28)$$

Finally,

$$\|x_{k+1}^{j+1} - x_{k+1}^*\| \leq \frac{\alpha}{1-\alpha} \|x_{k+1}^j - x_{k+1}^*\|. \quad (29)$$

Here,  $\alpha = L(\kappa_1 + \kappa_2 + \sigma\kappa_3)$ . Consequently, the sequence converges superlinearly.

To establish quadratic convergence we notice that, if the derivative mapping is Lipschitz continuous, then

$$\|r_{k+1}^j\| \leq \kappa_1 \|\Delta x_{k+1}^j\|^2 + \|w_{k+1}^j\|. \quad (30)$$

Moreover, we have that,

$$\|w_{k+1}^j\| \leq \kappa_2 \|\Delta x_{k+1}^j\|^2 + \kappa_3 \|x_k^j - x_k^*\|. \quad (31)$$

From the assumptions we have that  $\|x_k^j - x_k^*\| \leq \sigma \|x_{k+1}^{j+1} - x_{k+1}^j\|^2$ . Consequently,

$$\begin{aligned} \|x_{k+1}^{j+1} - x_{k+1}^*\| &\leq L(\kappa_1 + \kappa_2) \|x_{k+1}^{j+1} - x_{k+1}^* + x_{k+1}^* - x_{k+1}^j\|^2 \\ &+ L\kappa_3 \|x_k^j - x_k^*\|^2. \end{aligned} \quad (32)$$

Expanding the squared term and dividing through by  $\|x_{k+1}^{j+1} - x_{k+1}^*\|$  we obtain,

$$\begin{aligned} &\frac{1}{\|x_{k+1}^{j+1} - x_{k+1}^*\| + 2 \cdot \|x_{k+1}^j - x_{k+1}^*\| + \frac{\|x_{k+1}^j - x_{k+1}^*\|^2}{\|x_{k+1}^{j+1} - x_{k+1}^*\|}} \\ &\leq L(\kappa_1 + \kappa_2 + \sigma\kappa_3). \end{aligned} \quad (33)$$

The sequence on the left-hand side is bounded only if there exists  $\gamma > 0$  such that,

$$\frac{\|x_{k+1}^j - x_{k+1}^*\|^2}{\|x_{k+1}^{j+1} - x_{k+1}^*\|} \geq \gamma. \quad (34)$$

Which implies,

$$\|x_{k+1}^{j+1} - x_{k+1}^*\| \leq \gamma \cdot \|x_{k+1}^j - x_{k+1}^*\|^2. \quad (35)$$

The proof is complete  $\square$ .

We note that, once  $x_k^j = x_k^*$ , then  $\|w_k^j\| = 0$  and  $\alpha = L\kappa_1$  so that the pure Newton convergence rate is recovered. The same holds for the sequential Newton scheme. The above result looks into the *local* equation system of a time step  $k$  and states that the Newton iteration *absorbs* the parametric perturbations induced by the incoming data  $x_k^j$  and converges to the local solution  $x_{k+1}^*$ . As can be seen, the convergence rate is dictated by the error of the previous time step. We now establish the convergence of the parallel Newton scheme for the full system of nonlinear equations (4).

**Theorem 2: (Convergence of Full System).** Assume that conditions of Theorem 1 hold for  $k = 0, \dots, N-1$ . Define the local errors  $\epsilon_k^j := \|x_k^j - x_k^*\|$ ,  $k = 0, \dots, N$  with initial values  $e_k^0 > 0$ ,  $k = 0, \dots, N$ . The iterations of the parallel Newton scheme (6) converge to the solution of the system (4)  $x_k^*$ ,  $k = 0, \dots, N$ . Furthermore, the minimum number of iterations is  $N$ .

**Proof:** We have the contraction condition,

$$\|x_{k+1}^{j+1} - x_{k+1}^*\| \leq \beta (\|x_{k+1}^j - x_{k+1}^*\|) + \omega \|x_k^j - x_k^*\|, \quad (36)$$

for  $k = 0, \dots, N-1$  where  $e_0^j = 0$  since  $x_0$  is fixed. Thus, we have the iteration sequence:

$$\begin{aligned} \epsilon_1^{j+1} &\leq \beta \epsilon_1^j \\ \epsilon_2^{j+1} &\leq \beta \epsilon_2^j + \omega \epsilon_1^j \\ &= \beta \epsilon_2^j + \omega \beta \epsilon_1^{j-1} \\ &\vdots \\ \epsilon_N^{j+1} &\leq \beta \epsilon_N^j + \omega \epsilon_{N-1}^j \\ &= \beta \epsilon_N^j + \beta \sum_{i=1}^{N-1} \omega^i \epsilon_{N-i}^{j-i}. \end{aligned} \quad (37)$$

We know that  $\epsilon_1^j \rightarrow 0$ . Applying a recursive argument we obtain  $\epsilon_k^j \rightarrow 0$ ,  $k = 0, \dots, N$  so that the scheme converges to  $x_k^*$ ,  $k = 0, \dots, N$ . From the above sequence we see that the local error for  $k = N$  and iteration  $j$  depends on the *delayed* error at  $k = N-1$  which in turn depends on the delayed error at  $k = N-2$  and so on. Since the error at each time step  $k$  takes at least one iteration to converge once the error at  $k-1$  converges, the perturbation term  $\sum_{i=1}^{N-1} \omega^i \epsilon_{N-i}^{j-i}$  requires at least  $N-1$  iterations to converge. Consequently, the local error  $e_N^j$  takes at least  $N$  iterations to converge and so does the global error.  $\square$

The previous convergence results shed some light into the convergence properties of the parallel Newton method. They do not provide much insight, however, on the performance gains compared to the sequential Newton method. To analyze this case, we interpret the parallel Newton method as a manifold-tracking algorithm [19]. In particular, we highlight that the method can be seen as a *warm-starting* technique that stays close to a moving manifold formed by the upcoming data  $x_k^j$  from the neighboring time step. In other words, the method stays close to the manifold given by the solution of the system

$$\Phi(x_{k+1}, x_k^j) = 0, \quad j = 0, \dots, J_k \quad (38)$$

which is denoted as  $x_{k+1}^*(x_k^j)$ . We will see that each iteration of the parallel Newton method  $x_{k+1}^j$  stays close and eventually converges to the manifold solution. Because of this, once  $x_k^j$  converges to  $x_k^*$ , a few (usually one) extra Newton iterations will be needed to converge to  $x_{k+1}^*(x_k^*)$ . This provides the savings compared to the sequential method which converges the system  $\Phi(x_{k+1}, x_k^*) = 0$  from the initial guess  $x_{k+1}^0$  once  $x_k^*$  is known. These savings can be significant. For instance, if the sequential method takes  $J_{seq}$  iterations per time step  $k$  and

the parallel Newton method takes one iteration per step then, the total iteration savings are  $(J_{seq} - 1) \cdot N$ . As can be seen, the savings scale with the length of the time horizon.

The following theorem establishes conditions under which the parallel Newton method converges to the moving manifold  $x_{k+1}^*(x_k^j)$ . To prove this, we compare the distances  $\|x_{k+1}^{j+1}(p_j) - x_{k+1}^*(p_j)\|$  and  $\|x_{k+1}^j(p_{j-1}) - x_{k+1}^*(p_{j-1})\|$  where  $p_j := x_k^j$  and  $p_{j-1} := x_k^{j-1}$ . Accordingly, the Newton system can be written as,

$$\Phi(x_{k+1}^j(p_{j-1}), p_j) + \Phi'(x_{k+1}^j(p_{j-1}), p_j)(x_{k+1}^{j+1} - x_{k+1}^j(p_{j-1})) = 0 \quad (39)$$

**Theorem 3: (Convergence to Moving Manifold).** Assume that the Jacobian  $\Phi'(x_{k+1}^j(p_{j-1}), p_j)$  is nonsingular. In addition, assume there exists  $\sigma \geq 0$  such that  $\|x_{k+1}^*(p_j) - x_{k+1}^*(p_{j-1})\| \leq \sigma \|x_{k+1}^j(p_{j-1}) - x_{k+1}^*(p_{j-1})\|$ . Then the iteration sequence  $x_{k+1}^{j+1}(p_j)$  given by the Newton iteration (39) converges to the manifold  $x_{k+1}^*(p_j)$  quadratically.

**Proof:** We have,

$$\begin{aligned} 0 &= \Phi(x_{k+1}^*(p_j), p_j) \\ &= \Phi(x_{k+1}^*(p_{j-1}), p_j) \\ &\quad + \int_0^1 \Phi'(x_{k+1}^*(p_{j-1}) + \tau(x_{k+1}^*(p_j) - x_{k+1}^*(p_{j-1})), p_j) \cdot \\ &\quad \cdot (x_{k+1}^*(p_j) - x_{k+1}^*(p_{j-1})) d\tau \end{aligned} \quad (40)$$

and,

$$\begin{aligned} \Phi(x_{k+1}^*(p_{j-1}), p_j) &= \Phi(x_{k+1}^j(p_{j-1}), p_j) \\ &\quad + \int_0^1 \Phi'(x_{k+1}^j(p_{j-1}) + \tau(x_{k+1}^*(p_{j-1}) - x_{k+1}^j(p_{j-1})), p_j) \cdot \\ &\quad \cdot (x_{k+1}^*(p_{j-1}) - x_{k+1}^j(p_{j-1})) d\tau. \end{aligned} \quad (41)$$

Combining (41) and (39) we have,

$$\begin{aligned} \Phi(x_{k+1}^*(p_{j-1}), p_j) &= -\Phi'(x_{k+1}^j(p_{j-1}), p_j)(x_{k+1}^{j+1} - x_{k+1}^j(p_{j-1})) \\ &\quad + \int_0^1 \Phi'(x_{k+1}^j(p_{j-1}) + \tau(x_{k+1}^*(p_j) - x_{k+1}^j(p_{j-1})), p_j) \cdot \\ &\quad \cdot (x_{k+1}^*(p_j) - x_{k+1}^j(p_{j-1})) d\tau. \end{aligned} \quad (42)$$

Combining (40) and (42) we have,

$$\begin{aligned} &\Phi'(x_{k+1}^j(p_{j-1}), p_j)(x_{k+1}^{j+1}(p_j) - x_{k+1}^j(p_{j-1})) \\ &= \int_0^1 \Phi'(x_{k+1}^*(p_{j-1}) + \tau(x_{k+1}^*(p_j) - x_{k+1}^*(p_{j-1})), p_j) \cdot \\ &\quad \cdot (x_{k+1}^*(p_j) - x_{k+1}^*(p_{j-1})) d\tau \\ &\quad + \int_0^1 \Phi'(x_{k+1}^j(p_{j-1}) + \tau(x_{k+1}^*(p_{j-1}) - x_{k+1}^j(p_{j-1})), p_j) \cdot \\ &\quad \cdot (x_{k+1}^*(p_j) - x_{k+1}^j(p_{j-1})) d\tau, \end{aligned} \quad (43)$$

Rearranging and bounding terms,

$$\begin{aligned} &\|\Phi'(x_{k+1}^j(p_{j-1}), p_j)\| \|x_{k+1}^{j+1}(p_j) - x_{k+1}^*(p_j)\| \\ &\leq \frac{1}{2} L \|x_{k+1}^j(p_{j-1}) - x_{k+1}^*(p_{j-1})\|^2 \\ &\quad + L \|x_{k+1}^*(p_j) - x_{k+1}^*(p_{j-1})\| \|x_{k+1}^j(p_{j-1}) - x_{k+1}^*(p_{j-1})\| \\ &\quad + \frac{1}{2} L \|x_{k+1}^*(p_j) - x_{k+1}^*(p_{j-1})\|^2. \end{aligned} \quad (44)$$

If  $\|x_{k+1}^*(p_j) - x_{k+1}^*(p_{j-1})\| \leq \sigma \|x_{k+1}^j(p_{j-1}) - x_{k+1}^*(p_{j-1})\|$  then,

$$\begin{aligned} &\|\Phi'(x_{k+1}^j(p_{j-1}), p_j)\| \|x_{k+1}^{j+1}(p_j) - x_{k+1}^*(p_j)\| \\ &\leq \frac{1}{2} (\sigma + 1)^2 \|x_{k+1}^j(p_{j-1}) - x_{k+1}^*(p_{j-1})\|^2. \end{aligned} \quad (45)$$

The proof is complete.  $\square$

**Corollary 1:** Assume conditions of Theorem 1 hold. Then, there exist iterates  $J_k$  such that  $p_{J_k} = x_k^*$  for all  $k = 0, \dots, N$  and  $\kappa \geq 0$  such that,

$$\begin{aligned} &\|x_{k+1}^{J_k+1}(x_{J_k}) - x_{k+1}^*(x_k^*)\| \\ &\leq \kappa \|x_{k+1}^{J_k}(x_k^{J_k-1}) - x_{k+1}^*(x_k^{J_k-1})\|^2, \end{aligned} \quad (46)$$

for  $k = 0, \dots, N - 1$ .

As can be seen, the parallel Newton scheme will converge rapidly to the solution  $x_{k+1}^*(x_k^*)$  once it is positioned at the moving manifold. In Figure 1 we illustrate this behavior.

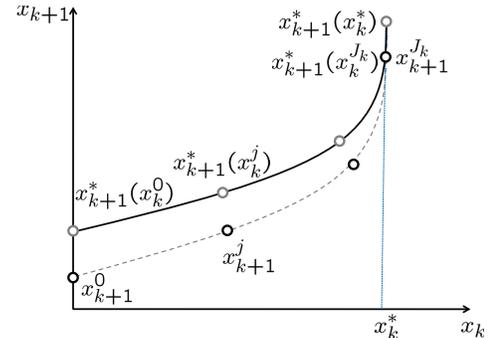


Fig. 1: Convergence of parallel Newton method to moving manifold.

## V. POWER SYSTEM MODEL

Figure 2 shows the circuit representation of the two-axis synchronous machine. This model was chosen since it accounts for transient responses and neglects the effects of subtransients [20], [21].

Consider a  $n$  bus network with  $m$  machines attached to it. The differential equations for each machine  $i = 1, 2, \dots, m$  are

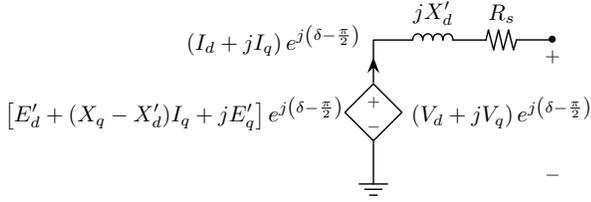


Fig. 2: Two-Axis Machine Model

given by

$$T'_{doi} \frac{dE'_{qi}}{dt} = -E'_{qi} - (X_{di} - X'_{di})I_{di} + E_{fdi} \quad (47)$$

$$T'_{qoi} \frac{dE'_{di}}{dt} = -E'_{di} + (X_{qi} - X'_{qi})I_{qi} \quad (48)$$

$$\frac{d\delta_i}{dt} = \omega_i - \omega_s \quad (49)$$

$$\frac{2H_i}{\omega_s} \frac{d\omega_i}{dt} = T_{Mi} - E'_{di}I_{di} - E'_{qi}I_{qi} - (X'_{qi} - X'_{di})I_{di}I_{qi} - T_{FWi}, \quad (50)$$

where Park's transformation has been used to switch the coordinate system. The exciter for the machines is an IEEE type 1 exciter given by

$$T_{Ei} \frac{dE_{fdi}}{dt} = -(K_{Ei} + S_{Ei}(E_{fdi}))E_{fdi} + V_{Ri} \quad (51)$$

$$T_{Fi} \frac{dR_{fi}}{dt} = -R_{fi} + \frac{K_{Fi}}{T_{Fi}}E_{fdi} \quad (52)$$

$$T_{Ai} \frac{dV_{Ri}}{dt} = -V_{Ri} + K_{Ai}R_{Fi} - \frac{K_{Ai}K_{Fi}}{T_{Fi}}E_{fdi} + K_{Ai}(V_i^{ref} - |V_i|), \quad (53)$$

with  $|V_i| = \sqrt{V_{di}^2 + V_{qi}^2}$ . For the remainder of this paper, it is assumed that every machine is driven by a steam turbine that is modeled as

$$T_{CHi} \frac{dT_{Mi}}{dt} = -T_{Mi} + P_{SVi} \quad (54)$$

$$T_{SVi} \frac{dP_{SVi}}{dt} = -P_{SVi} + P_{Ci} - \frac{1}{R_{Di}} \left( \frac{\omega_i}{\omega_s} - 1 \right), \quad (55)$$

where (54) is the behavior of a non-reheat steam turbine model and (55) is the speed governor for the system.

The power balance for equations of the network at each bus are given by

$$0 = \sum_{k=1}^n [V_i V_k Y_{ik}^* e^{j(\theta_i - \theta_k)} - (P_i + jQ_i)] - V_i e^{j\theta_i} (I_{di} - jI_{qi}) e^{-j(\delta_i - \pi/2)} \quad (56)$$

$$0 = \sum_{k=1}^n [V_j V_k Y_{jk}^* e^{j(\theta_j - \theta_k)} - (P_j + jQ_j)], \quad (57)$$

where (56) includes the complex power delivered to the generation buses  $i = 1, 2, \dots, m$  and (57) represents the load buses  $j = m + 1, \dots, n$ .

From Fig. 2, the coupling between the dynamic states and the network equations are

$$\begin{bmatrix} R_{si} & -X'_{qi} \\ X'_{di} & R_{si} \end{bmatrix} \begin{bmatrix} I_{di} \\ I_{qi} \end{bmatrix} = \begin{bmatrix} E'_{di} - V_i \sin(\delta_i - \theta_i) \\ E'_{qi} - V_i \cos(\delta_i - \theta_i) \end{bmatrix}, \quad (58)$$

where the terminal voltage  $V_i e^{j\theta_i}$  perceived by the system bus  $i$  is  $(V_d + jV_q) e^{j(\delta_i - \frac{\pi}{2})}$ .

Simplifying (47) - (58), the power system can be written as a differential algebraic equation of the form (2). The dynamic state variables are given by  $x_i = [E'_{qi}, E'_{di}, \delta_i, \omega_i, E_{fdi}, R_{fi}, V_{Ri}, T_{Mi}, P_{SVi}]^T \forall i = 1, 2, \dots, m$ . The algebraic variables are  $y_i = [I_{di}, I_{qi}, V_i, \theta_i]^T \forall i = 1, 2, \dots, m$  and  $y_j = [V_j, \theta_j]^T \forall j = m + 1, \dots, n$ . To solve the differential model, we apply a trapezoidal discretization rule. For a  $m$  machine,  $n$  bus system the number of states that need to be solved is  $11m + 2n$ .

## VI. CASE STUDIES

A case study was performed on the Western Electricity Coordinating Council's (WECC) 3 machine, 9 bus test system shown in Fig. 3 [20]. At  $t = 0.04$  sec a fault is created by severing the transmission lines that connect buses  $\{4, 5\}$  and  $\{8, 9\}$ . The fault is cleared after 0.15 sec and the transient response is computed with the traditional sequential approach and in parallel. The simulation has a duration of 10 sec and uses a step size of  $h = 0.01$  sec for the numerical integration. The simulations are currently implemented in MATLAB and the number of iterations required for the global residual to converge less than  $10^{-10}$  for each solution is recorded. The number of iterations is used as the metric to compare results between the two methods since the computational time cannot realistically be determined until the systems are tested in a parallel environment.

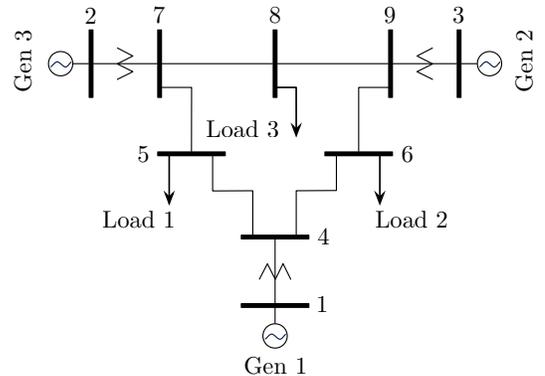


Fig. 3: WECC 3 Machine 9 Bus Network

Figures 4(a) and 4(b) are the frequency responses of the system using the sequential and parallel methods, respectively. In addition to the frequency response, the same results were achieved for the remaining 50 states.

Figure 5(a) shows the local residuals of the first 80 iterations for the sequential method. On average, it was observed that it

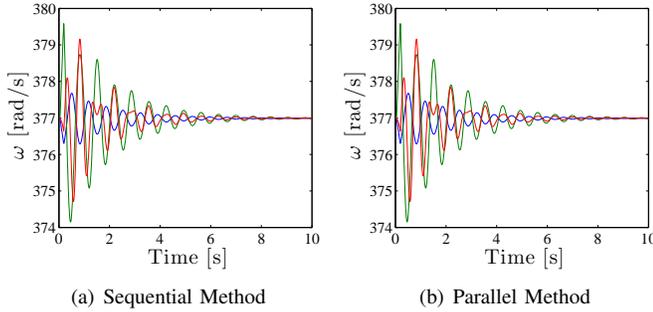


Fig. 4: Network Frequency Response

took approximately 4 iterations to solve the set of nonlinear equations at each time step. Notice that there are two prominent spikes at iterations 4 and 66. The first corresponds to the fault and the second is when the fault is cleared. Intuitively, the additional iterations are explained by the change in network structure and the initial conditions at time steps 4 and 19.

The global residuals are compared in Fig. 5(b). The sequential method required 4142 iterations to converge whereas the parallel method converged within the desired tolerances after 1017 iterations for the 1000 time step simulation which is close to the theoretical lower bound. The parallel method resulted in a 75% savings in the number of iterations over the sequential method. Additionally, we observed that as the length of the horizon is increased, the average number of iterations of the parallel method asymptotically converges to the theoretical lower bound.

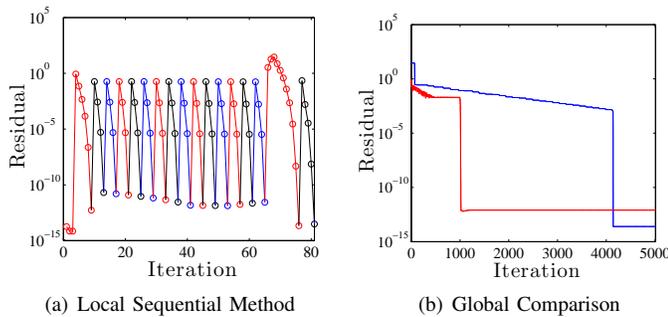


Fig. 5: Residuals

## VII. CONCLUSIONS AND FUTURE WORK

We provide a convergence analysis that explains why parallel block Newton methods can significantly reduce the number of iterations compared to sequential methods. In addition, we provide a theoretical lower bound on the number of iterations needed by the method. Numerical tests indicate that the method is scalable.

Future work will take advantage of the linear algebra structure of the problem to develop a multi-level parallelization scheme to reduce the time of each parallel Newton iteration. This will also enable us to group multiple time steps in a

single Newton system and thus achieve convergence properties that are closer to full-space Newton methods. Additionally, strategies with adaptive time steps will be implemented. Finally, an important research direction is the convergence of more general multi-splitting (e.g., parallel in space) methods.

## ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy, under Contract No. DE-AC02-06CH11357.

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