Bayesian uncertainty quantification for channelized subsurface characterization

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Abstract

In this paper, our goal is to characterize channelized permeability field given dynamic data, such as production data. Our prior models consist of log-Gaussian fields separated by interfaces. The interfaces are modeled by using a level set approach that allows reducing the parameterization of the interfaces to the parameterization of velocity fields that are smooth spatial fields. Smooth spatial fields are parameterized by using Karhunen-Loève expansion. We discuss regularity of the posterior and efficient sampling using approximate models. A numerical example is included.

1 Model problem and Bayesian sampling

The main objective of the paper is to use Bayesian uncertainty quantification methodology in subsurface characterization. Bayesian hierarchical models are used to integrate data from different sources. We consider channelized subsurface formations, and our goal is to quantify the uncertainty in the permeability field and, subsequently, in the predictions, given the dynamic water-cut or fractional flow data.

In this paper, for simplicity, we consider two-phase immiscible flow and transport under additional assumptions where we neglect the effects of gravity, compressibility, and capillary pressure. In this case, the forward model can be written as a coupled pressure and saturation equation:

$$\nabla \cdot (\lambda(S)\ k \nabla p) = Q_s,\ \frac{\partial S}{\partial t} + \ v \cdot \nabla f(S) = 0, \tag{1}$$

where λ is the total mobility, Q_s is a source term, f is the fractional flux of water, and $v = -\lambda(S)\ k \nabla p$ is the total velocity. The nonlinear functions $\lambda(S)$ and $f(S)$ depend on relative permeability fields. In our simulations, we are interested in conditioning permeability field to dynamic data such as the water-cut. The water cut is defined as the fraction of water produced in relation to the total production rate, denoted by F_t (denoted simply by F in further discussion). F for a two-phase water-oil flow is defined as the fraction of water in the produced fluid and is given by q_w/q_t, where $q_t = q_o + q_w$, with q_o and q_w the flow rates of oil and water at the production edge of the model, $F_t = \frac{\int_{\partial\Omega^{out}} v_n f(S) dl^F}{\int_{\partial\Omega^{out}} v_n dl^F}$, where $\partial\Omega^{out}$ is the outflow boundary and v_n is the normal velocity field.

The Bayesian model for the observed water-cut data and the unknown permeability field can be written as $z = F(k) + \epsilon$, where the unknown fine-scale permeability is denoted by k, z is the observed water-cut data, and ϵ is the model error. We assume $\epsilon \sim N(0, \sigma_k^2)$. The Bayesian model casts the inverse solution as the posterior distribution of k, given by

$$\pi(k) = P(k|z) \propto P(z|k)P(k), \tag{2}$$

where $P(z|k)$ is called the likelihood and $P(k)$ is called the prior.

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2 Parameterization of channelized fields

In parameterization, we would like to achieve a small-dimensional parameter space that can represent the permeabilities. We will assume that the permeability field consists of regions (called facies) where within each region the field can be described by a log-Gaussian field. Our goal is to represent boundaries and the permeability distribution within facies with fewer parameters. We use level set equations and parameterize the facies boundaries via velocity fields, and we use Karhunen-Loève (K-L) expansion to parameterize the permeability within each facies in an optimal way.

This decomposition allows us to write the permeability field in an hierarchical way:

$$k(x) = \sum_i k_i(x) I_{\Omega_i}(x), \tag{3}$$

where I_{Ω_i} is an indicator function of region Ω_i (i.e., $I(x) = 1$ if $x \in \Omega_i$ and $I(x) = 0$ otherwise) and $k_i(x)$ is the permeability within Ω_i.

In this paper, we seek the boundaries of the facies using adaptive representation. More precisely, level set functions τ representing the facies boundaries are defined such that $\tau = \tau_i$ for different interfaces. For the update of the facies, the level set equations (e.g., [5]) will be used. We assume $\frac{\partial \tau}{\partial s} + w \cdot \nabla \tau = 0$, where w is a vector field that is used to parameterize the velocity field (and subsequently, the interface) and s is a pseudo-time. $\tau(x)$ is the level set function such that a zero level set of τ represents the interfaces.

The permeability field within each facies is assumed to follow a log-Gaussian distribution with a known spatial covariance; that is, $Y(x,\omega) = \log[k(x,\omega)]$ follows a Gaussian distribution. For permeability fields given by a two-point correlation function $R(x,y) = E[Y(x,\omega)Y(y,\omega)]$ (where $E[\cdot]$ refers to the expectation and x, y are points in the spatial domain), the K-L expansion can be used to get an expression for the log permeability field $Y(x,\omega)$. The expansion is done by representing the permeability field in terms of an optimal L^2 basis given by $Y(x,\omega) = \sum_i \sqrt{\lambda_i}\theta_i(\omega)\Phi_i(x)$, where Φ_i and λ_i satisfy $\int_\Omega R(x,y)\Phi_i(y)dy = \lambda_i \Phi_i(x)$, $i = 1, 2, \ldots$, where $\lambda_i = E[Y_i^2] > 0$. By truncating the K-L expansion, we can represent the permeability field by a few (M) random parameters given by $Y_M = \sum_{i=1}^M \sqrt{\lambda_i}\theta_i\Phi_i$. In our numerical examples the covariance structure for the ith facies $R_i(x,y)$ is defined as $R_i(x,y) = \sigma_i^2 \exp\left(-\frac{|x_1-y_1|^2}{2l_{i1}^2} - \frac{|x_2-y_2|^2}{2l_{i2}^2}\right)$, $i = 1, 2, \ldots s$; l_{i1} and l_{2i} are the correlation lengths in each dimension, and $\sigma_i^2 = E(Y_i^2)$ is the variance. These lengths can be assumed to be random, and we have designed an efficient parameterization techniques for it [4]. Thus, the permeability field can be written as $k(\theta,\tau,l,\sigma^2) = \sum_{i=1}^s \exp(Y_{M_i}) I_{\Omega_i(\tau)}(x)$. We also use K-L to represent the velocity field w, in general and achieve a dimension reduction.

3 Likelihood setup and regularity

From the parameterization of interfaces with level sets we can say that the log permeability field Y is completely known given θ, τ, l and σ^2. The function τ can be represented by using the stochastic representation of w. By using the Bayes theorem, the posterior distribution of the parameters can be written as

$$\pi(\theta,\tau,l,\sigma^2) \propto P(z|\theta,\tau,l,\sigma^2)P(\theta,\tau,l,\sigma^2) = P(z|\theta,\tau,l,\sigma^2,\sigma_k^2)P(\theta)P(\tau)P(l)P(\sigma^2)P(\sigma_k^2).$$

We note that the hierarchical structure of the model is due to facies and a permeability distribution with each facies. The likelihood term $P(z|\theta,\tau,l,\sigma^2)$ is the probability density function of $MVN(F(\theta,\tau,l,\sigma^2),\sigma_k^2)$. In other words, $P(z|\theta,\tau,l,\sigma^2) \propto \exp\left(-\frac{\|z-F(\theta,\tau,l,\sigma^2)\|}{2\sigma_k^2}\right)$.

The prior distributions are taken as $\theta \sim MVN(0, \sigma_\theta^2)$, $\tau \sim MVN(\tau_o, \sigma_\tau^2)$ and $\sigma_i^2 \sim IG(a, b), i = 1, 2, \ldots, s$. The prior distributions for correlation lengths l_{ij} can be taken to be uniformly distributed over (l_{min}, l_{max}).

So the posterior can be written as

$$\pi(\theta, \tau, l, \sigma^2) \propto \exp\left(-\frac{\|z - F(\theta, \tau, l, \sigma^2)\|}{2\sigma_k^2}\right) \exp\left(-\frac{\theta'\theta}{2\sigma_\theta^2}\right) \exp\left(\frac{\|\tau - \tau_o\|}{2\sigma_\tau^2}\right) \exp(\frac{b}{\sigma^2})\sigma^{-a-1}.$$

We use a Bayesian framework. It can be shown that this Bayesian inverse problem is well-posed by proving that the posterior measure is continuous with respect to the data in total variation norm. In particular, we can estimate the error due to the truncation in K-L expansion of the velocity field that represent the interfaces and permeability fields within each facies. We show that the expectation of an arbitrary smooth function with respect to the full posterior and the reduced posterior is bounded by the error in K-L truncation. Moreover, the constant in this estimate is independent of the dimension of the space, an important feature for practical applications.

4 Efficient sampling

The posterior distribution of the parameters given the data is intractable, so we have to use Markov chain Monte Carlo methods to sample from the posterior. We denote the vector of all the parameters $(\theta, \tau, l, \sigma^2)$ as ν. At each MCMC iteration step, after proposing a new $\theta, l_1, l_2, \sigma^2$, we need to solve the eigenvalue problem for the K-L expansion to get the fine-scale permeability realizations. To speed computations, we can compute the eigenvalue problem (K-L expansion) for a certain number of pairs of l_1, l_2 beforehand (see [4] for details) and interpolate them to find the eigenvalues and eigenvectors at each step in the Metropolis-Hastings MCMC method.

The main disadvantage of the direct MCMC algorithm is the high computational cost of the forward problem. A large amount of CPU time is spent on simulating the rejected samples, making the direct (full) Metropolis Hastings MCMC simulations very expensive.

The direct Metropolis-Hastings MCMC method can be improved by adapting the proposal distribution $q(\nu|\nu_n)$ to the target distribution by using a coarse-scale model [3]. This can be achieved by a two-stage MCMC method [4]. First, we compare the fractional flow curves on the coarse-grid model. One can make the dimension of the parameter space to be random and use a two-stage reversible jump MCMC method [4]. If the proposal is accepted by the coarse-scale test, then a full fine-scale computation can be conducted and the proposal further tested as in the direct MCMC method. Otherwise, the proposal will be rejected by the coarse-scale test, and a new proposal will be generated from $q(\nu|\nu_n)$. The coarse-scale test filters the unacceptable bad proposals and avoids the expensive fine-scale tests for those proposals. The filtering process essentially modifies the proposal distribution $q(\nu|\nu_n)$ by incorporating the coarse-scale information of the problem. One of the key ingredients of two-stage MCMC approach is the choice of approximate models; we briefly discussed this next.

As for an approximate model, we will consider single-phase, flow-based multiscale simulation methods. This technique is similar to upscaling methods (e.g., [2]) except that, instead of computing effective properties, multiscale basis functions are calculated. These basis functions are coupled through a variational formulation of the problem. For multiphase flow and transport simulations, the conservative fine-scale velocity is often needed. For this reason, the mixed multiscale finite-element method (MsFEM) is used; see [2] for deatils on this method and its use in two-phase flow and transport. In our simulations, the multiscale basis functions are computed for the velocity once with $\lambda = 1$. These basis functions are used later without any update for solving two-phase flow equations. As a result, we obtain

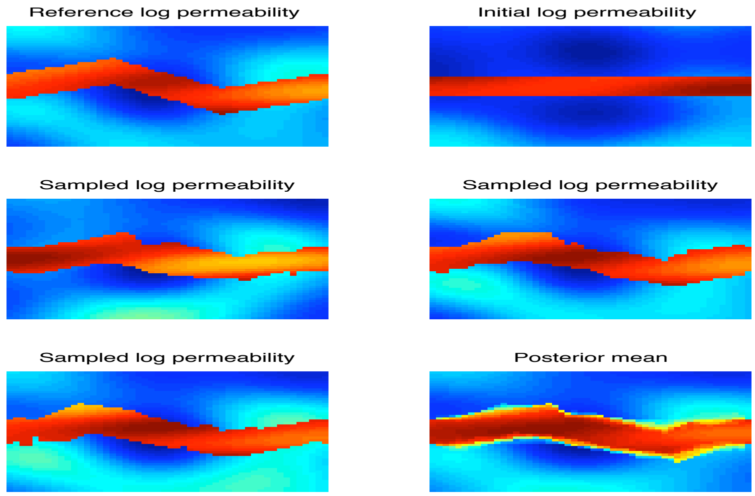

Figure 1: Top left: The true log permeability field. Top right: Initial log permeability field. Middle two and bottom left: Three accepted realizations of log permeability field. Bottom Right: The mean of the sampled log permeability field.

a coarse-scale velocity field that is used for solving the transport equation on the coarse grid. We also refer to [4], where various emulators have been developed that can be used as approximate models.

To describe our algorithm, we denote by F_ν^* the fractional flow computed by solving the problem on a coarse grid and introduce $\pi^*(\theta,\tau,l,\sigma^2) \propto \exp\left(-\frac{G(\|z-F^*(\nu)\|)}{2\sigma_c^2}\right)\exp\left(-\frac{\theta'\theta}{2\sigma_\theta^2}\right)\exp\left(\frac{\|\tau-\tau_o\|}{2\sigma_\tau^2}\right)\exp(\frac{b}{\sigma^2})\sigma^{-a-1}$, where the function G is estimated based on offline computations using independent samples from the prior. More precisely, by using independent samples from the prior distribution, the permeability fields are generated. Then both the coarse-scale and fine-scale simulations are performed and $\|z-F_\nu\|$ vs $\|z-F_\nu^*\|$ are plotted. This scatter plot data can be modeled by $\|z-F_\nu\| = G(\|z-F_\nu^*\|)+W$, where W is a random component representing the deviations of the true fine-scale error from the predicted error. Using the coarse-scale distribution $\pi^*(\nu)$ as a filter, we can describe the two-stage reversible jump MCMC as follows.

In the first stage, a coarse (approximate) model is run, and the acceptance probability is defined based on Metropolis-Hastings criteria. At this stage, one can use gradient information or simple random walk. If the proposal is accepted in the first stage, the fine-scale simulation is run, and the proposal is accepted or rejected (see [1] for details).

5 Numerical results

For our numerical example, we consider a 50×50 fine-scale permeability field on the unit square with only one high conductivity layer. Thus there are two interfaces, one for the upper interface and one for the lower interface. The permeability field is known at eightlocations along $x=0$ and $x=1$ boundaries. The ends of the interface are fixed at 0.4 and 0.6. One injection well is placed at $(0,0.5)$ and one production well at $(1,0.5)$. Two-phase flow model with quadratic relative permeabilities are considered. The log of the permeability field within the channel (middle facies) is assumed to be a Gaussian process with mean 3, where $l_1=0.3, l_2=.1$, and $\sigma^2=.4$. The log of the permeability field

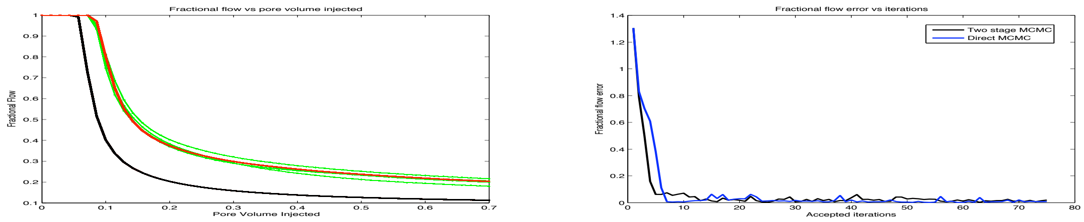

Figure 2: Left: The solid red line designates the fine-scale reference fractional flow, the black line designates the initial fractional flow and the green lines designate fractional flow corresponding to sampled permeability fields. Right: Fractional flow errors vs. accepted iterations.

outside the high conductivity is assumed to be Gaussian process with mean 0 and the same covariance function, where $l_1 = .2, l_2 = .2$ and $\sigma^2 = .4$. We retain the first 20 terms in the K-L. Initially τ's are taken to be equidistant points on the straight line joining the two ends of the interfaces. This approach is similar to taking w (velocity field in level set equation) to be independent at these points. We run the both the regular MCMC and two-stage MCMC taking a random walk sampler with $\sigma_\theta^2 = 0.16$ and $\sigma_\tau^2 = 0.04$. We use the mixed MsFEM described before for the upscaling method in the two-stage MCMC.

To assess the accuracy of the two-stage MCMC, we perform coarse-scale vs. fine-scale simulations for permeability samples from the prior. From the cross-plot between $E_k = \|z - F\|$ and $E_k^* = \|z - F^*\|$ the correlation coefficient between the two errors are found to be 0.93. The acceptance rate of the full MCMC is very low, approximately 0.002, using $\sigma_k^2 = .004$. The acceptance rate for the two-stage MCMC increases to 0.3 using $\sigma_k^2 = 0.004$ and $\sigma_c^2 = 0.01$

Figure 1 shows the reference log permeability field, the initial log permeability field, some of the sampled log permeability field and the mean of the sampled log permeability field for the two-stage MCMC. We can see that the sample mean is very close to the reference log permeability field. In Figure 2 (left plot), we plot the initial fractional flow and the fractional flow corresponding to some of the sampled permeability fields. We observe substantial improvement in fractional flow predictions. The convergence of the two-stage MCMC is plotted on the right in Figure 2. Clearly, both the two-stage and fine-scale MCMC methods have similar convergence properties; that is, they reach to the steady state within the same number of iterations.

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