1. Unconstrained Optimization

1.1. Taylor's Theorem

Theorem 2.1 (N&W)

1.1.1

A. Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and $p \in \mathbb{R}^n$. Then we have

$$f(x+p) = f(x) + \int_0^1 \nabla f(x+\tau p)^T p \, d\tau \overset{MVT}{=}$$

$$f(x) + \nabla f(x+tp) p \quad \text{for some } t \in (0,1)$$

B. Moreover, if f is twice continuously differentiable, we have that

$$\nabla f(x+p) = \nabla f(x) + \int_0^1 \nabla^2 f(x+\tau p) p \, d\tau$$

It works by Leibniz-Newton but MVT does not apply!

C. $$f(x+p) = f(x) + \nabla f(x)^T p + \frac{1}{2} p^T \nabla f(x+tp) p$$

for some $t \in (0,1)$

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1.1.2 Again by Leibniz-Newton and $\int$ by parts on the scalar function $f(x+tp)$.

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1. UNCONSTRAINED OPTIMIZATION

1.2 Necessary Conditions for optimality:

Theorem 2.3

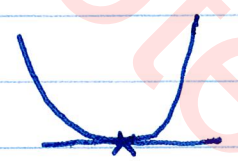
If x^* is a local minimizer of $f \in C^2$ in a neighborhood of x^*, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succeq 0$ is PSD

~~Sufficient Conditions~~

The conditions: $\begin{cases} \nabla f(x^*) = 0 \\ \nabla^2_{xx} f(x^*) \succeq 0 \end{cases}$

are called the necessary first and second-order optimality conditions.

"Proof"



Euler's principle;

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1. UNCONSTRAINED OPTIMISATION

1.3. ~~Necessary conditions~~

1.3.1 Sufficient conditions → Sometimes

~~Theorem 2.4~~

called second-order sufficient conditions even if there are no first-order sufficient conditions.: $\begin{cases} \nabla f(x^*) = 0 \\ \nabla^2 f(x^*) \succ 0 \end{cases}$ (SOSC)

<u>Theorem 2.4 (NfW)</u>

Suppose ~~$f \in C^2$~~ $\nabla^2_{xx} f$ is continuous in $\mathcal{N}(x^*)$; that $\nabla f(x^*) = 0$ and $\nabla^2_{xx} f(x^*)$ is positive definite. Then x^* is a local minimizer.

<u>Proof</u> Use Taylor Theorem C

Since $\nabla^2 f(x)$ is cts, $\exists$ r ~~such that~~ and $\varepsilon > 0$ such that

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$\nabla^2 f(x) \geq \varepsilon I \quad \forall\, x \in B(x^*, r).$

Define $p = x - x^*$.

By Taylor c, we have

$$f(x) = f(x^* + p) = f(x^*) + \underbrace{\nabla f(x^*)}_{=0} p + \frac{1}{2} p^T \nabla^2 f(x^* + tp) p$$

$$x^* + tp = x^* + t(x - x^*) = (1-t)x^* + tx \in B(x^*, r)$$

$$\Rightarrow f(x) = f(x^*) + \frac{1}{2} p^T \nabla f(x^* + tp) p \geq$$

$$f(x^*) + \frac{1}{2} \varepsilon \cdot \|p\|^2 =$$

$$f(x^*) + \frac{1}{2} \varepsilon \cdot \|x - x^*\|^2$$

↳ "Quadratic Growth Condition"

and $\Rightarrow f(x) > f(x^*) \quad \forall\, x \in B(x^*, r)$

$x \neq x^*$. QED ∎

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1.3.3 It is strict..., but is it isolated?

Theorem 2.4 A.

SOSC $\Rightarrow$ x^* is ILM.

Proof

Use Taylor Theorem B. Continue Proof

$$\nabla f(x) = \nabla f(x^*) + \int_0^1 \nabla^2 f(x+\tau p)\, p \, d\tau$$

$$\Rightarrow p^T(\nabla f(x) - \nabla f(x^*)) = \int_0^1 p^T \nabla^2 f(x+\tau p)\, p \, d\tau$$

$$\geq \varepsilon \cdot \|p\|^2$$

$$\Rightarrow (x-x^*)^T(\nabla f(x) - \nabla f(x^*)) \geq \varepsilon \|x-x^*\|^2$$

$\Rightarrow \nabla f(x)$ is "Monotone"

But $\nabla f(x^*) = 0 \overset{x \neq x^*}{\Rightarrow} \nabla f(x) \neq 0$ so

x cannot be a LM $\Rightarrow$ x^* is ILM. $\blacksquare$

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1. UNCONSTRAINED OPTIMIZATION

1.4. ~~Ex~~ How large is gap btwn. Nec Cond

1.4.1. and Suff Conditions?

Note: SOSC $\Rightarrow$ LM is both ILM and SLM so no gap B, C, D...

A. Say x^* is LM and satisfies NC

$\Rightarrow \tilde{f}(x) = f(x) + \varepsilon (x-x^*)^T (x-x^*)$ satisfies SC at x^* $\Rightarrow$ so it seems that any problem can be perturbed to have only LM that satisfy SOSC.

B. On the other hand; if (P) satisfies SOSC at any LM, so will any suff small perturbation.

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1.4.2

c) So if $\{f\} \subset (X, \mathcal{T})$

$\Rightarrow \{\{f\} \mid$ any LM has SOSC$\}$

~~seem~~ is open and dense.

Problem. w

$\Rightarrow$ So LM that are "degenerate" have measure 0.

This statement can be made precise using transversality arguments, but it indicates gap is negligible in this model.

So we will assume SOSC from here on at x^*

$\boxed{\text{J-J-T, Theorem 2.1}}$

<u>Exact Statement</u>: The set $\mathcal{J} = \{f \in C^3(\mathbb{R}^n, \mathbb{R}) \mid$ any NC point is not degenerate$\}$ is open and dense in the Whitney C^3_s topology

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1.5. Newton's method for optimization

1.5.1 ~~Newto~~ and nonlinear equation

Idea: Represent the optimality conditions as a nonlinear equation:

$$\nabla f(x) = 0 = F(x)$$

How do I solve it? Iteratively.

$$x^k \longrightarrow m_{F,x^k}(x) = F(x^k) + \nabla F(x^k)\cdot(x - x^k)$$

$$: \mathbb{R}^n \to \mathbb{R}^n$$

Therefore $\boxed{m_{F,x^k}(x^{k+1}) = 0}$

Defines an iterative process

$$F(x^k) + \nabla F(x^k)(x^{k+1} - x^k) = 0$$

$$\Rightarrow x^{k+1} = x^k - (\nabla F(x^k))^{-1} \cdot F(x^k)$$

Digression: **Notation** convention:

Sequence index: superscript. Component: subscript

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1.6. Rate of convergence

Digression: $G(x): \mathbb{R}^n \to \mathbb{R}^m$ is LC on $W(x^*)$ if
$$\|G(x^1) - G(x^2)\| \le L_{G,W} \|x^1 - x^2\| \quad \forall\, x^1, x^2 \in W$$

1.6.1 Suppose that:

1. $\nabla F(x)$ is Lipschitz continuous in
2. a neighborhood of x^*; s.t. $F(x^*) = 0$.
2. $\nabla F(x^*)$ is invertible

Then

α) If x^0 is sufficiently close to x^* then $x^k \to x^*$

β) Rate of convergence is quadratic

γ) $\|\nabla f(x^k)\| \to$ quadratically at 0.

Digression. Rate of convergence

$\{z^k\}_{k \ge 0}$ 1. $0 \le z^{k+1} \le c z^k \qquad c < 1$

(Q) Linearly convergent at 0 w.c

3. $0 \le z^{k+1} \le w^k z^k$, $w^k \to 0$ superlinear

2. $0 \le z^{k+1} \le c\{z^k\}^2$; $c z^0 < 1$

$\Rightarrow z^k$ is quadr convergent

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1.5.2 a)

$$x^{k+1} - x^* = x^k - x^* - \left[\nabla F(x^k)\right]^{-1} \cdot F(x^k)$$

$$= \underbrace{\left[\nabla F(x^k)\right]^{-1}}_{3} \cdot \Big(\underbrace{\nabla F(x^k) \cdot (x^k - x^*)}_{2} - \underbrace{\left(F(x^k) - F(x^*)\right)}_{1} \Big)$$

But

1. $$F(x^k) - F(x^*) = \int_0^1 \nabla F(x^k + t(x^* - x^k)) \cdot (x^k - x^*)\, dt$$

2. $$\nabla F(x^k) \cdot (x^k - x^*) = \int_0^1 \nabla F(x^k) \cdot (x^k - x^*)$$

$$\Rightarrow x^{k+1} - x^* = \int_0^1 \left[\nabla F(x^k) - \nabla F(x^k + t(x^* - x^k))\right] (x^k - x^*)\, dt$$

$$\|x^{k+1} - x^*\| \le \int_0^1 \|\nabla F(x^k) - \nabla F(x^k + t(x^* - x^k))\| \, \|x^k - x^*\|\, dt$$

$$\le \left(\int_0^1 Lt\right) \cdot \|x^k - x^*\|^2 = \frac{1}{2} L \|x^k - x^*\|^2$$

3. $B(x^*, r)$; s.t $\|\nabla F(x^k)^{-1}\| \le 2 \cdot \|\nabla F(x^*)^{-1}\|$

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1.6.3. $\Rightarrow \|x^{k+1} - x^*\| \leq L \cdot \|\nabla F(x^*)\| \cdot \|x^k - x^*\|^2$

So

$x^k \in B(x^*, r)$ s.t. $\begin{cases} L \text{ for all of } \nabla F(x) \\ \nabla f(x)^{-1} \leq 2 \nabla f(x) \end{cases}$

the above holds

Then

1) x^0 is such that $\boxed{\text{induction}}$

$\|x^0\| < r$; $\|x^0 - x^*\| \leq \dfrac{1}{2L\|\nabla F(x^*)\|}$ $\Rightarrow$ $\|x^k\| < r$, $\|x^k - x^*\| \leq \dfrac{1}{2L\|\nabla F(x^*)\|}$

and

$$\|x^{k+1} - x^*\| < \frac{1}{2}\|x^k - x^*\|$$

So α) $\|x^k - x^*\| \to 0$; and

β) quad convergent

QED

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1.6.4

How does this apply to optimization?

$\nabla f(x^*) = 0$; $\nabla f(x^*)$ is PD

$\Rightarrow F(x) = \nabla f(x)$ satisfies conditions.

$\boxed{\text{Goal 2: Have my iterative algorithms achieve quadratic convergence}}$

(or superlinear convergence) since linear convergence can be very slow (see steepest descent).

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1.7.

What is the problem with Newton's Method?

We have proved we achieve goal 2, but not a word about goal 1 !! So if x^0 is not in the right neighborhood of x^* this iteration may truly go anywhere!! Divergence.

Example of nonconvergence.

How do I achive both goals? Answer globalization (global convergence to a stationary point). But make sure I do not destroy second goal!!

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