3. Trust-Region Methods

3.1

Rationale and idea: the subproblem.

- The problem with line-search methods. Very solid, (such as Armijo w. modified factorization), but one flaw. There is no way to distinguish a simple stationary point from SONC. It is unlikely, but...

- Alternative: Use second-order model in a region "of trust" and TAKE IT SERIOUSLY. that is, DO NOT MODIFY the matrix. If fail: reduce; if big success, enlarge.

- NOTATIONS:

  $f^k = f(x^k)$; $\nabla f^k = \nabla f(x^k)$;

3.1.2 Quadratic model:

$$m_k(p) = f^k + p^T g^k + \frac{1}{2} p^T B^k p$$

Taylor Series:

$$f(x^k + p) = f^k + p^T g^k + \frac{1}{2} p^T \nabla^2 f(x^k + tp) p$$

Difference: $m_k(p) - f(x^k + p) = O(\|p\|^2)$

[if $B^k = \nabla^2 f(x^k) \Rightarrow O(\|p\|^3)$]

Trust-Region Subproblem:

$$\boxed{\min_{p \in \mathbb{R}^n} m_k(p) \quad \|p\| \le \Delta^k}$$

$\rightsquigarrow$ called the trust region

Discussion: If $B^k > 0$; and $\|(B^k)^{-1} g^k\| \le \Delta$

Then that is the solution. But the interesting case is elsewhere.

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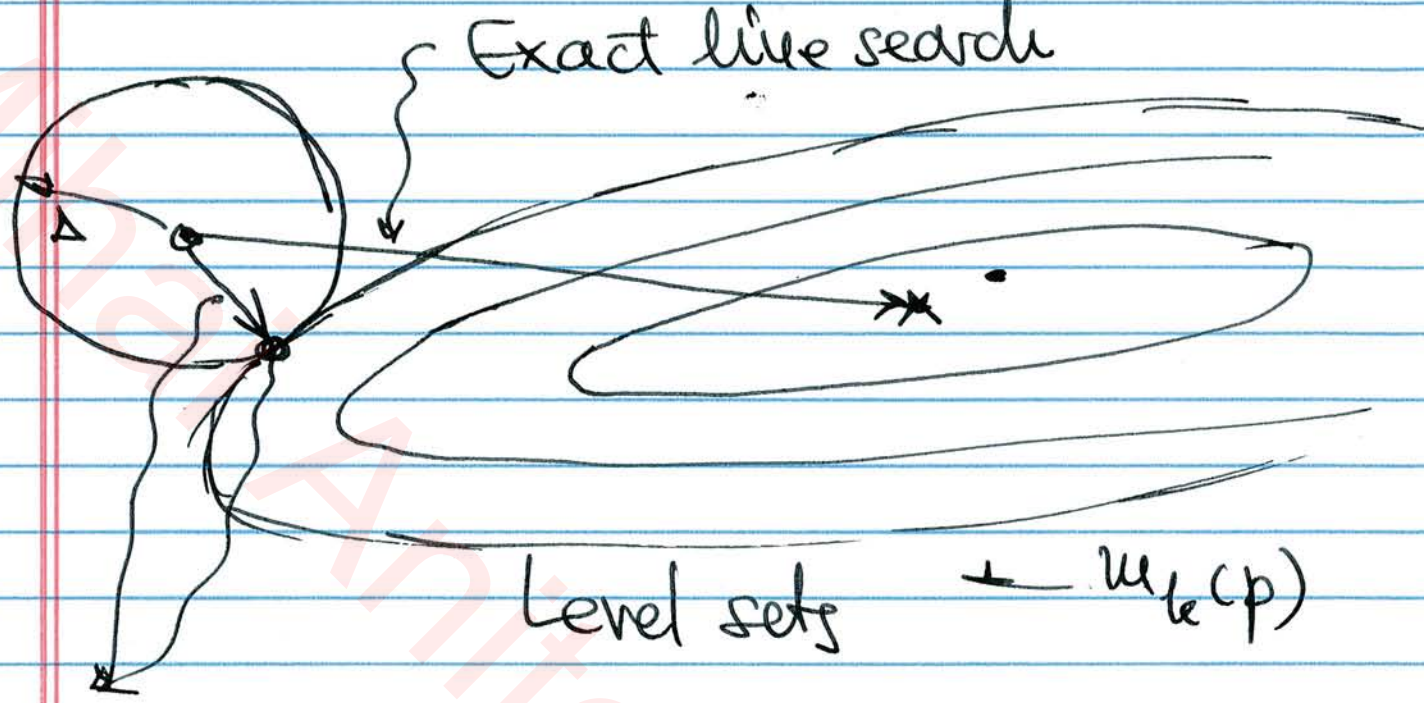
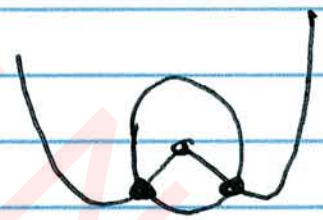
3.1.3 ⊛ Intuition



Trust-region solution.

⊛ Note :- Example



$\cancel{((x^2-1)^2+1)}$

$(x^2-1)^2$ at 0. LS. is cooked

$2x(x^2-1)$

$4x^2-2$ ⇒ I escape saddle-point!

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3.2 Trust-Region Management

(*) How do we update TR size?

How well are we doing?

$$\rho^k = \frac{f(x^k) - f(x^k + p^k)}{m_k(0) - m_k(p^k)}$$

← actual reduction

← predicted reduction

Both are > 0

Define $\eta \in [0, \frac{1}{4})$; $\hat{\Delta} > 0$

$\Delta_0 \in (0, \hat{\Delta})$

Obtain p^k by TR

Compute ρ^k

Δ^k manage

if $\rho^k < \frac{1}{4}$ → "too optimistic"

$$\Delta^{k+1} = \frac{1}{4}\Delta^k$$

else

if $\rho^k > 3/4$ and $\|p^k\| = \Delta^k$ → "too pessimistic"

$$\Delta^{k+1} = \min(2\Delta^k, \hat{\Delta})$$

else $\Delta^{k+1} = \Delta^k$

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3.2.2

If $\rho^k \geq \eta$

$$x^{k+1} = x^k + p^k \quad \leftarrow \text{accept step}$$

else

$$x^{k+1} = x^k \quad \leftarrow \text{reject step}$$

How do we solve the TR subproblem

1) B^k is positive definite $\rightarrow$ LS, $\rightarrow$ NLE

"dogleg method"

2) B^k is not P.S.D., we have to mess around w. eigenvalues

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3. Trust-region

3.3. Cauchy Point

① Why: We need a convergence framework that allows us to play around with B^k.

⊛ Idea: $l_k(s) = f^k + g^{k^T} p$: linear model

Find: $$\boxed{p^{k,s} = \arg\min_{\substack{p \in \mathbb{R}^n \\ \|p\| \le \Delta^k}} f^k + g^{k^T} p}$$

Use $m_k(p)$ to minimize along $p^{k,s}$

$$\tau^k = \boxed{\arg\min_{\substack{\tau \ge 0 \\ \|\tau p^{k,s}\| \le \Delta^k}} m_k(\tau p_k^{k,s})}$$

Define $p^{k,c} = \tau^k p^{k,s}$

Note: Unless $g^k = 0$; the solution $p^{k,s} \neq 0$ and $\|p^{k,s}\| = \Delta^k$ $\Rightarrow$ $\boxed{p^{k,s} = \frac{-\Delta^k}{\|g^k\|} \cdot g^k}$

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3.3.2 Solution

$\alpha)$ $g^{k,T} B^k g^k \leq 0 \Rightarrow \tau^k = 1$

so $p^{k,s} = p^{k,c}$

$\beta)$ $g^{k,T} B^k g^k > 0 \Rightarrow$ quadratic equation

$$\tau^k = \begin{cases} 1 & g^{k,T} B^k g^k \leq 0 \\ \min\left(\|g^k\|^3 / (\Delta^k g^{k,T} B^k g^k), 1 \right) & \text{ow} \end{cases}$$

Idea: We converge if we use only Cauchy point!! Therefore the Cauchy point decrease becomes the yardstick by which we accept even incomplete TR solutions (cause they are hard).

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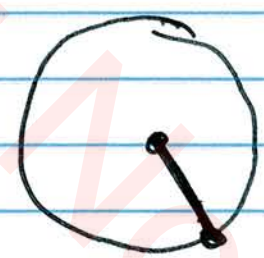
3. Trust-region approach; circumventing TR :-)

3.4 Dogleg Method

Q: How do we enhance upon the Cauchy point, and, hopefully, obtain superlinear convergence?

A: Go to Cauchy point, then move as far as you can on Newton direction. (makes sense really only for $B_k \succeq 0$ such as NE and LS)

The dogleg method.



If [redacted] Cauchy point is interior (otherwise just stop at it)

$$\Rightarrow g^{k,T} B^k g^k > 0 \quad \text{and}$$

$$\boxed{\tau^k = \frac{\|g_k\|^3}{\Delta^k g^{k,T} B^k g^k}} \Rightarrow p^u = \tau^k p^{k,S} = \frac{\|g_k\|^{\cancel{3}}}{\cancel{\|\Delta^k\|} g^{k,T} B^k g^k} \cdot \frac{-\Delta^k g^k}{\cancel{\|g^k\|}}$$

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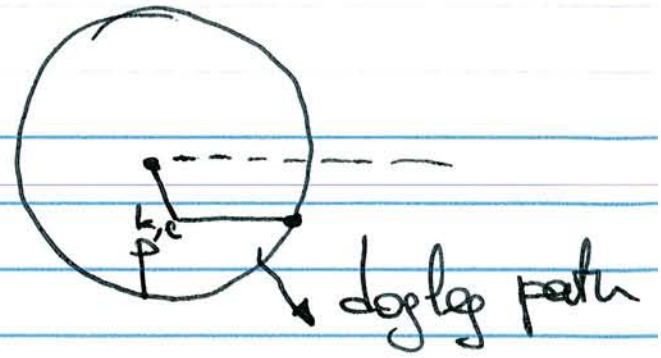
3.4.2 $$\boxed{p^u = -\frac{\|g^k\|^2}{g^{k,T} B^k g^k} \cdot g^k}$$



$$\boxed{p^B = -B_k^{-1} g^k}$$

dogleg path: $$\tilde{p}(\tau) = \begin{cases} \tau p^u & \tau \in (0,1) \\ p^u + (\tau - 1)(p^B - p^u) & 1 \le \tau \le 2 \end{cases}$$

If sufficiently close to solution, it will allow me to choose $\tau = 2$; the full Newton step and allow for SLC.

Lemma 4.2

Let B be positive definite. Then

i) $\|\tilde{p}(\tau)\|$ is an increasing function of τ

"moves away" from center

ii) $m_k(\tilde{p}(\tau))$ is a decreasing fcn. of τ

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3.4.3 <u>Proof</u>. For $\tau \in [0,1]$ the statement holds $\Rightarrow$ from Cauchy point properties

$$\boxed{\begin{aligned} h(\alpha) &= \frac{1}{2}\|\tilde{p}(1+\alpha)\|^2 = \\ &\frac{1}{2}\|p^u + \alpha(p^B - p^u)\|^2 = \frac{1}{2}\|p^u\|^2 + \\ &+ \alpha(p^u)^T(p^B - p^u) + \frac{1}{2}\alpha^2\|p^B - p^u\|^2 \end{aligned}}$$

$\Rightarrow$

$$h'(\alpha) = \alpha\|p^B - p^u\| + (p^u)^T(p^B - p^u)$$

$$\geq -(p^u)^T(p^u - p^B) = -\frac{(g^Tg)^3}{(g^TBg)^2} + \frac{g^TB^{-1}g \cdot g^Tg}{g^TBg}$$

$$= \frac{g^Tg}{(g^TBg)^2}\left[(g^TB^{-1}g)(g^TBg) - (g^Tg)^2\right] \geq 0$$

By Cauchy-Schwarz

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3.4.4 $\hat{h}(\alpha) = u(\tilde{p}(1+\alpha)) =$

$\cancel{f^k + \alpha g^{k,T} f}$

$$f + g^T(p^u + \alpha(p^B - p^u)) +$$

$$\frac{1}{2}(p^u + \alpha(p^B - p^u))^T B(p^u + \alpha(p^B - p^u))$$

$$\hat{h}' = g^T(p^B - p^u) + \cancel{\alpha(p^B - p^u)^T B(p^u + \alpha}$$

$$(p^B - p^u)^T B(p^u + \alpha(p^B - p^u)) \overset{\boxed{\alpha = 1}}{\leq}$$

$$(p^B - p^u)^T \cdot (\underbrace{g + Bp^B}_{=-g}) = 0.$$

QED

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3. TR methods

3.5 Convergence theory

⊛ Minimal decrease of Cauchy point

Lemma 4.3. $p^{k,c}$ satisfies

$$m_k(0) - m_k(p_k^c) \geq \frac{1}{2}\|g^k\| \min\left(\Delta^k, \frac{\|g^k\|}{\|B^k\|}\right)$$

<u>Proof</u>: Only one case: we prove:

$$g^T B g > 0 \quad \text{and} \quad \frac{\|g\|^3}{\Delta g^T B g} \leq 1 \quad (\zeta < 1)$$

$$\zeta = \frac{\|g\|^3}{\Delta g^T B g}$$

$$\Rightarrow \boxed{\cancel{m_k(p^c) - m_k(p^{k,c})}} \quad p^c = -\frac{\Delta g}{\|g\|}$$

$$m(p^c) - m(0) = g^T p^c + \frac{1}{2} p^{c,T} B p^c =$$

$$-\frac{\|g\|^4}{g^T B g} + \frac{1}{2}\cancel{\Delta^2} \cdot \frac{1}{\cancel{\|g\|^2}} \cancel{g^T B g} \cdot \frac{\|g\|^6}{\cancel{\Delta^2}(g^T B g)^{\cancel{2}}}$$

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3.5.B Theorem 4.5

Assume: 1. $\|B^k\| \leq \beta$; $\forall k$

DEFINE

2. $S = \{ x \mid f(x) \leq f(x^0) \}$

$$S(R_0) = \bigcup_{x \in S} \mathbb{B}(x \mid R_0)$$

f is B.B. on $S(R_0)$; for some $R_0 > 0$

3. ∇f is Lip on $S(R_0)$

4. We solve the TR subproblem $\boxed{\text{aproximately}}$ while satisfying:

$$(4.20): \; m_k(0) - m_k(p^k) \geq c_1 \cdot \|g^k\| \cdot \min\left(\Delta^k, \frac{\|g^k\|}{\|B^k\|}\right)$$

$$(4.25): \; \|p^k\| \leq \gamma \cdot \Delta^k \text{ for some } \gamma \leq 1$$

Then, $\liminf_{k \to \infty} \|g^k\| = 0$

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Intuition:

Assume:

1. $m_k(0) - m_k(p_k^S) \leq f(x^k) - f(x^{k+1})$
   + CP property
2. Δ_k is bounded below by $\bar{\Delta}$

$$\Rightarrow \sum \|g^k\| \cdot \min\left(\Delta^k, \frac{\|g^k\|}{\|B^k\|}\right) < \infty$$

$$\Rightarrow g^k \to 0$$

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3. Trust Region Methods

3.6. Iterative resolution of the subproblem

Theorem 4.1 : $\boxed{\text{Characterization of TR solution}}$

The vector p^* is a global solution of TR problem

$$\min_{p \in \mathbb{R}^n} m(p) = f + g^T p + \frac{1}{2} p^T B p \quad \text{st} \quad \|p\| \leq \Delta$$

$\Leftrightarrow$ p^* is feasible and $\exists\, \lambda \geq 0$ such that

$$(B + \lambda I)\, p^* = -g$$

$$\lambda(\Delta - \|p^*\|) = 0$$

$$(B + \lambda I) \text{ is P.S.D}$$

$\boxed{\text{Proof}}$

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3.6.1 Lemma 4.7. W.O Proof

Let $m(p) = g^T p + \frac{1}{2} p^T B p$; B symmetric

i) m attains minimum $\Leftrightarrow$ B is psd and g is in range of B

ii) m has a unique minimizer $\Leftrightarrow$ B is p.d.

<u>Proof, theorem</u>

a) $\exists\ \lambda \geq 0$ such that conditions are satisfied. $\overset{L.4.7}{\Rightarrow}$ p^* is a global min of

$$\hat{m}(p) = g^T p + \frac{1}{2} p^T (B + \lambda I) p =$$

$$m(p) + \frac{\lambda}{2} p^T p$$

Since $\hat{m}(p) \geq \hat{m}(p^*)$ $\Rightarrow$

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3.6.2

$$m(p) \geq m(p^*) + \frac{\lambda}{2}\left(p^{*T}p^* - p^T p\right)$$

$$\lambda \cdot \left(\|\Delta u - p^*\|\right) = 0 \quad \Rightarrow$$

$$m(p) \geq m(p^*) + \frac{\lambda}{2}\left(\Delta^2 - p^T p\right) \geq m(p^*)$$

↑ TR constraint

QED

$\Rightarrow$

Assume p^* is global solution

α) $\|p^*\| < \Delta$; verify yourselves by unconstrained minimization

$\lambda = 0$

β) $\|p^*\| = \Delta \quad \Leftrightarrow \quad p^T p = \Delta^2$

$\mathcal{L}(p, \lambda) = m(p) + \frac{\lambda}{2}\left(p^T p - \Delta^2\right)$ has a stationary point at p^*

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3.6.3 $\mathbb{B}p^* + g + \lambda p^* = 0 \Rightarrow (B + \lambda I)p^* = g$ (1)

(2) Since $m(p) \geq m(p^*)\ \forall p$ with $p^T p = \Delta^2$

$$m(p) \geq m(p^*) + \frac{\lambda}{2}\left((p^*)^T p^* - p^T p\right)$$

$$\cancel{g^T p + \frac{1}{2} p^T B p \geq g^T p^* +}$$

$$g^T p + \frac{1}{2} p^T B p \geq g^T p^* + \frac{1}{2} p^{*T} B p$$

$$+\lambda I \qquad\qquad +\lambda I$$

$$\boxed{g^T(p - p^*) =} -p^{*T}(B + \lambda I)(p - p^*) +$$

$$+ \frac{1}{2} p^T (B + \lambda I) p - \frac{1}{2} p^{*T}(B + \lambda I) p^* \geq 0 \quad \forall p$$

$$= \frac{1}{2} p^T (B + \lambda I) p + \frac{1}{2} p^{*T} (B + \lambda I) p^*$$

$$- \frac{1}{2} p^{*T} (B + \lambda I) p - \frac{1}{2} p^T (B + \lambda I) p^* =$$

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3.6.4 $\frac{1}{2}(p-p^*)^T(B+\lambda I)(p-p^*) \geq 0$

$\forall p; \|p\| \leq \Delta$

<u>Prove</u> Set of directions

$$\left\{ w : w = \pm \frac{p-p^*}{\|p-p^*\|} \text{ ; for some } p; \|p\| = \Delta \right\}$$

is dense on unit sphere $\Rightarrow$ P.S.D

(3) $\lambda \geq 0$

Because $\begin{cases} B+\lambda I \text{ is p.S.d.; and} \\ (B+\lambda I)p^* = -g \end{cases}$

$\Leftarrow$

<u>Lemma</u>

α) p^* is unconstrained min of

$$\hat{m}(p) = g^T p + \frac{1}{2} p^T (B+\lambda I) p$$

β) Point p^* in unc. min of $m(p)$

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3.6J From a):

$\hat{m}(p) \geq \hat{m}(p^*) \Rightarrow$ $\boxed{\forall p}$

$m(p) \geq m(p^*) + \frac{1}{2}\lambda(p^{*T}p^* - p^T p)$

Assume $\lambda \overset{\text{only}}{<} 0$ satisfies it.

$\Rightarrow \forall p;\ \|p\| \geq \Delta \Rightarrow$

$m(p) \geq m(p^*) + \frac{1}{2}\lambda(p^{*T}p^* - p^T p) \geq m(p^*)$

$\Rightarrow p^*$ is a global min. $\Rightarrow$

$Bp = -g$; B is p.s.d $\Rightarrow$ $\lambda = 0$ works

(Lemma).

$\boxed{\Sigma}$

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3.6.6 Solving the TR problem.

1) $\|p\| \le \Delta$; $\lambda = 0$; or

$$\boxed{2)\ p(\lambda) = -(B + \lambda I)^{-1} g}$$

$B + \lambda I$ is P.D.

& $\|p(\lambda)\| = \Delta$.

Since B is symmetric ; $\exists$ Q orthogonal

$Q = [q^1, q^2, \dots q^n]$; such that

$$B q^i = \lambda^i q^i \Rightarrow BQ = Q\Lambda$$

$$\lambda^1 \le \lambda^2 \le \dots \lambda^n$$

$$\Rightarrow B + \lambda I = Q(\Lambda + \lambda I)Q^T$$

$$p(\lambda) = -(B + \lambda I)^{-1} \cdot g =$$

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3.6.7

$$= -Q(\Lambda+\lambda I)Q^T g =$$

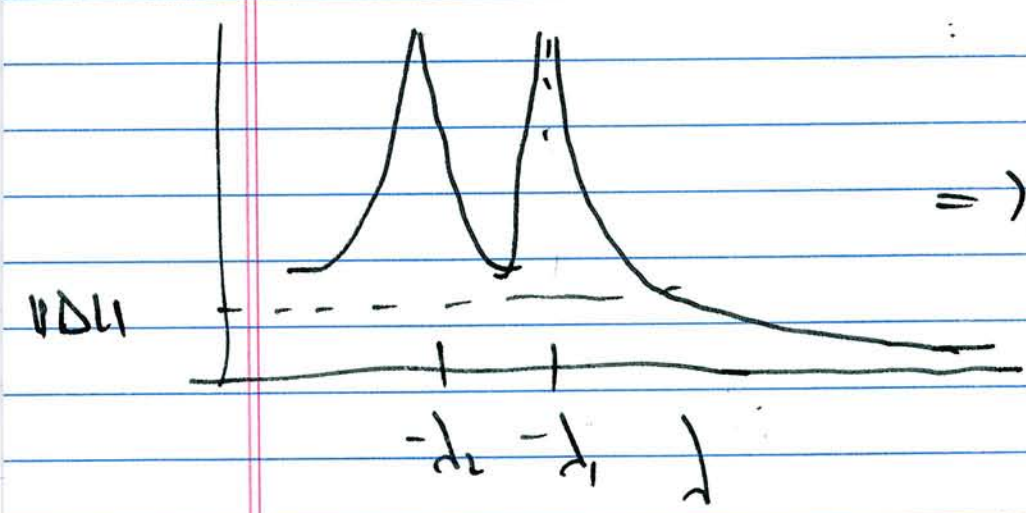
$$= -\sum_{j=1}^{n} \frac{q^{jT} g}{(\lambda+\lambda^j)} q^j$$

$$\Rightarrow \|p(\lambda)\|^2 = \sum_{j=1}^{n} \frac{(q^{jT} g)^2}{(\lambda+\lambda^j)^2} = \Phi(\lambda)$$

Observations

1) $\lambda > -\lambda^1$; $\Phi(\lambda)$ is decreasing

2) $q^{1T} g \neq 0 \Rightarrow \lim_{\lambda \to \rho} \|p(\lambda)\| = +\rho$



$\Rightarrow$ $\boxed{\text{MUST CROSS } \Delta.}$

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3.6.8 $\boxed{\text{Find } \phi_1(\lambda)=0 \quad \lambda \in [-\lambda_1, \infty).}$

For stability Purposes; Solve as

$$\phi_2(\lambda) = \frac{1}{\Delta} - \frac{1}{\|p(\lambda)\|}$$ (Note; diff!! at $\lambda > -\lambda_1$)

Use newton

$$\lambda^{\ell+1} = \lambda^{\ell} - \frac{\phi_2(\lambda^{(\ell)})}{\phi_2'(\lambda^{\ell})}$$

$\Rightarrow$ $\boxed{\text{Algorithm 4.3 (using Cholesky)}}$

Factor $B + \lambda^{(\ell)} I = R^T R$

Solve $R^T R\, p^{\ell} = -g$; $R^T q^{\ell} = p^{\ell}$

Set $\lambda^{(\ell+1)} = \lambda^{(\ell)} + \left(\frac{\|p^{\ell}\|}{\|q^{\ell}\|}\right)^2 \left(\frac{\|p^{\ell}\| - \Delta}{\Delta}\right)$.

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3.6.9 Problems: Need safeguards. Need estimate of λ_1. See More-Sorensen

Hard Case

$q_1^T g = 0 \quad ; \quad \lambda_1 < \lambda_2 \leq \ldots$

$$\Rightarrow p(\lambda) = -\sum \frac{q^{j^T} g}{\lambda + \lambda^j} q^j$$

$\Rightarrow$ ~~We cannot~~ What if $\|p(-\lambda_1)\| < \|\Delta\|$

$\Rightarrow B + \lambda I$ is invertible for $\lambda \in (-\lambda_1, \infty)$

$\Rightarrow \|p(\lambda)\| < \|\Delta\| \Rightarrow$ no solution

$\Rightarrow$ $\boxed{\text{WE MUST HAVE } \lambda = -\lambda_1}$

$(B + \lambda_1) p = -g \quad \Rightarrow \quad p = p + \tau q^1$

is also a solution

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3.6.10. $\hat{p}(\lambda) = p(\lambda_1) + z q^1 \Rightarrow$ Choose z

such that $\|\hat{p}(\lambda)\| = 0$

(TEST $(B + \lambda_1 I) \cdot \hat{p}(\lambda) = q^1$.)

Because $p(\lambda_1) \perp q^1 \Rightarrow$

$\|\hat{p}(\lambda)\|^2 = \|p(\lambda_1)\|^2 + z^2 \Rightarrow$ Two Choices

$\boxed{\text{Extend to multiple eigs...}}$

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3. Trust-region methods

3.8. Convergence of algorithms

Observation: Need not solve TR exactly !!

Approximate resolution.

$$m(0) - m(p) \geq c_1 [m(0) - m^*(p^*)]$$

$$\|p\| \leq \gamma \Delta. \qquad \gamma > 1$$

Theorem 4.8.

Assume that: 1) $\eta \in (0, 1/4)$

2) $\|B_k\| \leq \beta$ for some β

3) f is bdd below on S

4) f is Lipsch cts diff on $S(R_0)$

5) AR holds

6) $B_k = \nabla^2 f(x^k)$

Then

$$\boxed{\lim_{k \to \infty} \|g^k\| = 0}$$

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7)

3.8.2 If, in addition, level set is compact

then either α) x^k ~~$\to x^*$~~ terminates at x^*, SONC

β) $\{x^k\}$ has a limit point x^* SONC.

~~Theorem 4.9~~ :

Theorem 4.9 : "In addition, superlinearly convergent".

{ Idea: If Δ^k not too small; $B^k > 0$

$\Rightarrow$ choose Newton step

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3. Trust Region

3.9 Enhancements

Scaling:

$$\min_{p \in \mathbb{R}^n} \quad m_k(p) = f^k + g^{k^T} p + \frac{1}{2} p^T B^k p$$

$$\text{s.t.} \quad \|Dp\| \le \Delta^k$$

$$\Leftrightarrow \quad Dp = r$$

$$\boxed{\begin{aligned} \min_{r \in \mathbb{R}^n} \quad & m_k(r) = f^k + g^{k^T} D^{-1} r + \\ & + \frac{1}{2} r^T D^{-1} B^k D^{-1} r \end{aligned}}$$

$$\|r\| \le \Delta^k$$

a good choice:

$$\boxed{D^{-1} B D^{-1} \sim I}$$

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3. TRUST REGION

3.10 LS v. TR.

	LS	TR
SoNC	Cannot hope to achieve SoNC $\nabla f(x^*) = 0$	Yes, sometimes $\nabla f(x^*) = 0$

	LS	TR
SoNC	N	Y, under some assumption
FoNC	Y (Z)	Y : Thm. 4.5, 4.6, 8
SLC	Y if Newton	Y if Newton
SUBPROBLEM	LE, but for which matrix.	TR, iterative, harder

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