THE UNIVERSITY OF CHICAGO

Lecture 7

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Nonlinear Equations

- Formulation: $r : \mathbb{R}^n \to \mathbb{R}^n \qquad r(x) = 0,$
- Newton's Method … which may not converge.

$$r(x) = \begin{bmatrix} r_1(x) \\ r_2(x) \\ \vdots \\ r_n(x) \end{bmatrix}.$$

Algorithm 11.1 (Newton's Method for Nonlinear Equations).

Choose x_0;
for $k = 0, 1, 2, \ldots$
  Calculate a solution p_k to the Newton equations

$$J(x_k)p_k = -r(x_k);$$

  $x_{k+1} \leftarrow x_k + p_k$;
end (for)

Trust-region for nonlinear equations

- Model:

$$m_k(p) = \frac{1}{2}\|r_k + J_k p\|_2^2 = f_k + p^T J_k^T r_k + \frac{1}{2} p^T J_k^T J_k p_k.$$

- Trust-region problem:

$$\min_p m_k(p), \qquad \text{subject to } \|p\| \le \Delta_k,$$

- Reduction ratio:

$$\rho_k = \frac{\|r(x_k)\|^2 - \|r(x_k + p_k)\|^2}{\|r(x_k)\|^2 - \|r(x_k) + J(x_k)p_k\|^2}.$$

- Can use dogleg ! (with positive definite B)

TR algorithm for nonlinear equtions

Algorithm 11.5 (Trust-Region Method for Nonlinear Equations).

Given $\bar{\Delta} > 0$, $\Delta_0 \in (0, \bar{\Delta})$, and $\eta \in \left[0, \frac{1}{4}\right)$:

for $k = 0, 1, 2, \ldots$

Calculate p_k as an (approximate) solution of (11.46);

Evaluate ρ_k from (11.47);

if $\rho_k < \frac{1}{4}$

$\Delta_{k+1} = \frac{1}{4}\|p_k\|$;

else

if $\rho_k > \frac{3}{4}$ and $\|p_k\| = \Delta_k$

$\Delta_{k+1} = \min(2\Delta_k, \bar{\Delta})$;

else

$\Delta_{k+1} = \Delta_k$;

end (if)

end (if)

if $\rho_k > \eta$

$x_{k+1} = x_k + p_k$;

else

$x_{k+1} = x_k$;

end (if)

end (for).

Section 8: Constrained Optimization

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Reference: Section 12 Nocedal and Wright

8.1 INTRODUCTION IN CONSTRAINED OPTIMIZATION

Notations

- Problem Formulation

$$\min_{x\in\mathbb{R}^n} f(x) \text{ subject to } \begin{cases} c_i(x)=0 & i\in\mathcal{E} \\ c_i(x)\geq 0 & i\in\mathcal{I} \end{cases}$$

- Feasible set

$$\Omega=\{x \mid c_i(x)=0, i\in\mathcal{E};\quad c_i(x)\geq 0, i\in\mathcal{I}\}$$

- Compact formulation

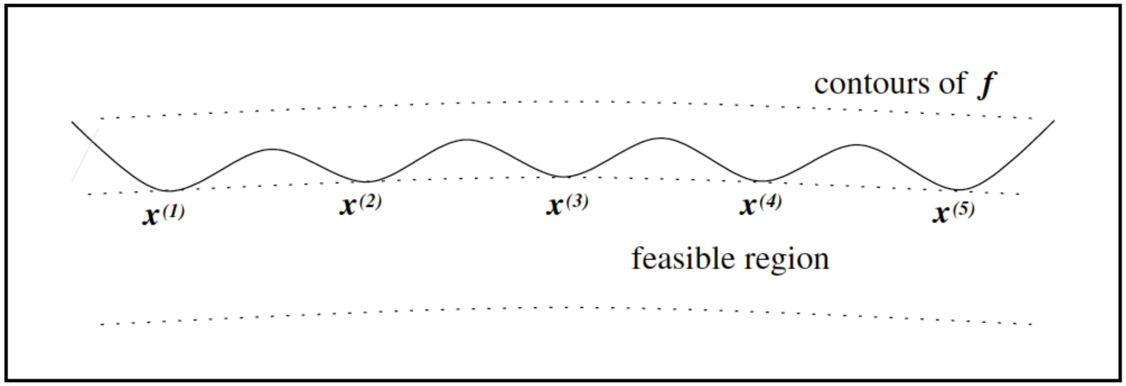
$$\min_{x\in\Omega} f(x)$$

Local and Global Solutions

- Constraints make make the problem simpler since the search space is smaller.
- But it can also make things more complicated.

$$\min (x_2 + 100)^2 + 0.01 x_1^2 \text{ subject to } x_2 - \cos x_1 \geq 0$$

- Unconstrained problem has one minimum, constrained problem has MANY minima.



Types of Solutions

- Similar as the unconstrained case, except that we now restrict it to a neighborhood of the solution.
- Recall, we aim only for local solutions.

A vector x^* is a *local solution* of the problem (12.3) if $x^* \in \Omega$ and there is a neighborhood $\mathcal{N}$ of x^* such that $f(x) \geq f(x^*)$ for $x \in \mathcal{N} \cap \Omega$.

A point x^* is an *isolated local solution* if $x^* \in \Omega$ and there is a neighborhood $\mathcal{N}$ of x^* such that x^* is the only local solution in $\mathcal{N} \cap \Omega$.

A vector x^* is a *strict local solution* (also called a *strong local solution*) if $x^* \in \Omega$ and there is a neighborhood $\mathcal{N}$ of x^* such that $f(x) > f(x^*)$ for all $x \in \mathcal{N} \cap \Omega$ with $x \neq x^*$.

Smoothness

- It is ESSENTIAL that the problem be formulated with smooth constraints and objective function (since we will take derivatives).
- Sometimes, the problem is just badly phrased. For example, when it is done in terms of max function. Sometimes the problem can be rephrased as a constrained problem with SMOOTH constrained functions.

$$\max\left\{f_1(x), f_2(x)\right\} \le a \Leftrightarrow \begin{cases} f_1(x) \le a \\ f_2(x) \le a \end{cases}$$

Examples of max nonsmoothness removal

- In Constraints:

$$\|x\|_1 = |x_1| + |x_2| \le 1 \Leftrightarrow \max\{-x_1, x_1\} + \max\{-x_2, x_2\} \le 1 \Leftrightarrow$$
$$-x_1 - x_2 \le 1, \quad x_1 - x_2 \le 1, \quad -x_1 + x_2 \le 1, \quad x_1 + x_2 \le 1$$

- In Optimization:

$$\min f(x); \quad f(x) = \max\{x^2, x\}; \quad \Leftrightarrow \begin{cases} \min & t \\ \text{subject to} & \max\{x^2, x\} \le t \end{cases}$$

$$\Leftrightarrow \begin{cases} \min & t \\ \text{subject to} & x^2 \le t, x \le t \end{cases}$$

8.2 EXAMPLES

Examples

- Single equality constraint (put in KKT form)

$$\min x_1 + x_2 \quad \text{subject to} \quad x_1^2 + x_2^2 - 2 = 0$$

- Single inequality constraint (put in KKT form, point out complementarity relationship)

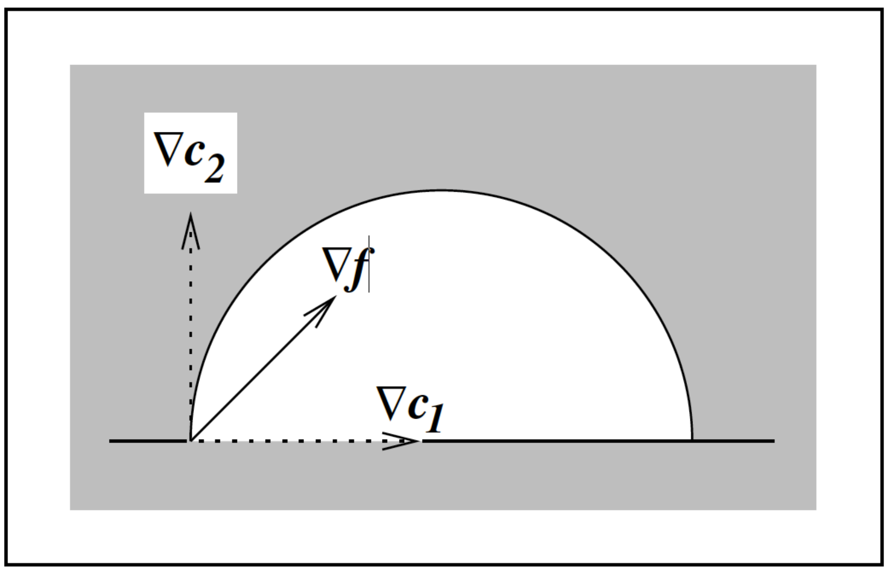
$$\min x_1 + x_2 \quad \text{subject to} \quad -\left(x_1^2 + x_2^2 - 2\right) \geq 0$$

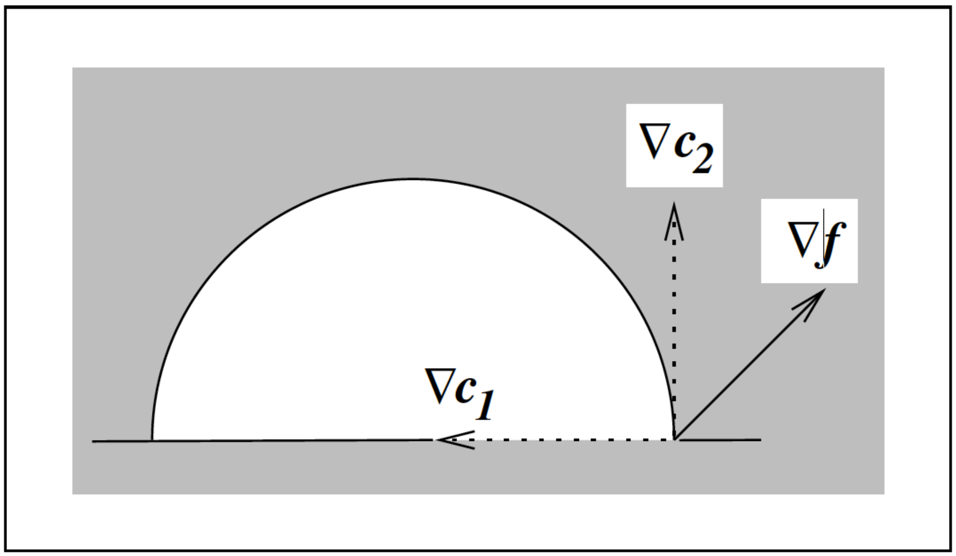
- Two inequality constraints (KKT, complementarity relationship, sign of the multiplier)

$$\min x_1 + x_2 \quad \text{subject to} \quad -\left(x_1^2 + x_2^2 - 2\right) \geq 0, x_1 \geq 0$$

Multiplier Sign Example

- There are two solutions for the Lagrangian equation, but only one is the right.





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8.3 IMPLICIT FUNCTION THEOREM REVIEW

Refresher (Marsden and Tromba)

3.5 The Implicit Function Theorem

Key Points in this Section.

1. **One-Variable Version.** If $f : (a, b) \to \mathbb{R}$ is C^1 and if $f'(x_0) \neq 0$, then locally near x_0, f has a C^1 inverse function $x = f^{-1}(y)$. If $f'(x) > 0$ on all of (a, b) and is continuous on $[a, b]$, then f has an inverse defined on $[f(a), f(b)]$. This result is used in one-variable calculus to define, for example, the log function as the inverse of $f(x) = e^x$ and $\sin^{-1}$ as the inverse of $f(x) = \sin x$.

2. **Special n-variable Version.** If $F : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^1 and at a point $(\mathbf{x}_0, z) \in \mathbb{R}^{n+1}$, $F(\mathbf{x}_0, z) = 0$ and $\frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0$, then locally near $(\mathbf{x}_0, z_0)$ there is a unique solution $z = g(\mathbf{x})$ of the equation $F(\mathbf{x}, z) = 0$. We say that $F(\mathbf{x}, z) = 0$ ***implicitly defines*** z as a function of $\mathbf{x} = (x_1, \ldots, x_n)$.

3. The partial derivatives are computed by ***implicit differentiation***:

$$\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x_i} = 0,$$

so

$$\frac{\partial z}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z}$$

4. The special implicit function theorem guarantees that if $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$, then the level set $g = c$ is a smooth surface near $\mathbf{x}_0$, a fact needed in the proof of the Lagrange multiplier theorem.

5. The general implicit function theorem deals with solving m equations

$$\begin{aligned} F_1(x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \\ \vdots \quad &\quad \vdots \\ F_m(x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \end{aligned}$$

for m unknowns $\mathbf{z} = (z_1, \ldots, z_m)$. If

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial z_1} & \cdots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial z_1} & \cdots & \dfrac{\partial F_m}{\partial z_m} \end{vmatrix} \neq 0$$

at $(\mathbf{x}_0, \mathbf{z}_0)$, then these equations define $(z_1, \ldots, z_m)$ as functions of $(x_1, \ldots, x_n)$. The partial derivatives $\partial z_i / \partial x_j$ may again be computed by using implicit differentiation.

8.4 FIRST-ORDER OPTIMALITY CONDITIONS FOR NONLINEAR PROGRAMMING

Inequality Constraints: Active Set

$$\min_{x\in\mathbb{R}^n} f(x) \text{ subject to } \begin{cases} c_i(x)=0 & i\in\mathcal{E} \\ c_i(x)\geq 0 & i\in\mathcal{I} \end{cases}$$

- One of the key differences with equality constraints.
- Definition at a feasible point x.

$$x\in\Omega(x) \quad \mathcal{A}(x)=\mathcal{E}\cup\left\{i\in\mathcal{I};\, c_i(x)=0\right\}$$

"Constraint Qualifications" for inequality constraints

- We need the equivalent of the "Jacobian has full rank" condition for the case with equality-only.
- This is called "the constraint qualification".
- Intuition: "geometry of feasible set"="algebra of feasible set"

Tangent and linearized cone

- Tangent Cone at x (can prove it is a cone)

$$T_\Omega(x) = \left\{ d \,\middle|\, \exists\{z_k\} \in \Omega, z_k \to x, \exists\{t_k\} \in \mathbb{R}_+, t_k \to 0, \lim_{k\to\infty} \frac{z_k - x}{t_k} = d \right\}$$

- Linearized feasible direction set (EXPAND)

$$\mathcal{F}(x) = \left\{ d \,\middle|\, d^T \nabla c_i(x) = 0, i \in \mathcal{E}; d^T \nabla c_i(x) \geq 0, i \in \mathcal{A}(x) \cap \mathcal{I} \right\} \Rightarrow T_\Omega(x) \subset \mathcal{F}(x)$$

- Essence of constraint qualification at a point x ("geometry=algebra"):

$$T_\Omega(x) = \mathcal{F}(x)$$

What are sufficient conditions for constraint qualification?

- The most common (and only one we will discuss in the class): the linear independence constraint qualification (LICQ).
- We say that LICQ holds at a point $x \in \Omega$ if $\nabla c_{A(x)}$ has full row rank.
- How do we prove equality of the cones ? If LICQ holds, then, from IFT

$$d \in \mathcal{F}(x) \Rightarrow c_{\mathcal{A}(x)}\left(\tilde{x}(t)\right) = t\nabla c_{\mathcal{A}(x)} d \Rightarrow \exists \tau > 0, \forall 0 < t < \tau;$$

$$c_{\bar{\mathcal{A}}(x)}\left(\tilde{x}(t)\right) > 0; c_{\mathcal{A}(x)\cap\mathcal{I}}\left(\tilde{x}(t)\right) \geq 0; c_{\mathcal{E}}\left(\tilde{x}(t)\right) = 0 \Rightarrow \tilde{x}(t) \in \Omega \Rightarrow d \in T_{\Omega}(x)$$

8.4.1 OPTIMALITY CONDITIONS FOR EQUALITY CONSTRAINTS

Constrained optimality

- If problem is constrained, only *feasible* directions are relevant
- For equality-constrained problem

$$\min f(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{g}(\boldsymbol{x}) = \mathbf{0}$$

where $f: \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{g}: \mathbb{R}^n \to \mathbb{R}^m$, with $m \leq n$, necessary condition for feasible point x^* to be solution is that negative gradient of f lie in space spanned by constraint normals,

$$-\nabla f(\boldsymbol{x}^*) = \boldsymbol{J}_g^T(\boldsymbol{x}^*)\boldsymbol{\lambda}$$

where $\boldsymbol{J}_g$ is Jacobian matrix of g, and $\boldsymbol{\lambda}$ is vector of *Lagrange multipliers*

- This condition says we cannot reduce objective function without violating constraints

Expand, use implicit function theorem. Jacobian full rank

Constrained optimality

- *Lagrangian function* $\mathcal{L}\colon \mathbb{R}^{n+m} \to \mathbb{R}$, is defined by

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \boldsymbol{\lambda}^T \boldsymbol{g}(\boldsymbol{x})$$

- Its gradient is given by

$$\nabla \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \begin{bmatrix} \nabla f(\boldsymbol{x}) + \boldsymbol{J}_g^T(\boldsymbol{x})\boldsymbol{\lambda} \\ \boldsymbol{g}(\boldsymbol{x}) \end{bmatrix}$$

- Its Hessian is given by

$$\boldsymbol{H}_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{\lambda}) = \begin{bmatrix} \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{\lambda}) & \boldsymbol{J}_g^T(\boldsymbol{x}) \\ \boldsymbol{J}_g(\boldsymbol{x}) & \boldsymbol{O} \end{bmatrix}$$

where

$$\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{H}_f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i \boldsymbol{H}_{g_i}(\boldsymbol{x})$$

Constrained optimality

- Together, necessary condition and feasibility imply critical point of Lagrangian function,

$$\nabla \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \begin{bmatrix} \nabla f(\boldsymbol{x}) + \boldsymbol{J}_g^T(\boldsymbol{x})\boldsymbol{\lambda} \\ \boldsymbol{g}(\boldsymbol{x}) \end{bmatrix} = \mathbf{0}$$

- Hessian of Lagrangian is symmetric, but not positive definite, so critical point of $\mathcal{L}$ is saddle point rather than minimum or maximum

- Critical point $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ of $\mathcal{L}$ is constrained minimum of f if $\boldsymbol{B}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ is positive definite on *null space* of $\boldsymbol{J}_g(\boldsymbol{x}^*)$

- If columns of $\boldsymbol{Z}$ form basis for null space, then test *projected* Hessian $\boldsymbol{Z}^T\boldsymbol{B}\boldsymbol{Z}$ for positive definiteness

Expand: Implicit Functions Theorem

Summary: Necessary Optimality Conditions

- First order:

$$\nabla_x \mathcal{L}\left(x^*, \lambda^*\right) = 0$$

- Second order necessary conditions.

$$\nabla_x c\left(x^*\right) w = 0 \Rightarrow w^T \nabla_{xx}^2 \mathcal{L}\left(x^*, \lambda^*\right) w \geq 0$$

Sufficient Optimality Conditions

- The point is a local minimum if LICQ and the following holds:

$$(1)\nabla_x \mathcal{L}\left(x^*,\lambda^*\right)=0;\,(2)\nabla_x c\left(x^*\right)w=0 \Rightarrow \exists\sigma>0\; w^T\nabla_{xx}^2\mathcal{L}\left(x^*,\lambda^*\right)w \geq \sigma\|w\|^2$$

- Proof: By IFT, there is a change of variables such that

$$u\in\mathcal{N}(0)\subset\mathbb{R}^{n-n_c}u\leftrightarrow x(u);\,\tilde{x}\in\mathcal{N}\left(x^*\right),c(\tilde{x})=0\Leftrightarrow\exists\tilde{u}\in\mathcal{N}(0);\,\tilde{x}=x(\tilde{u})$$

$$\nabla_x c\left(x^*\right)\nabla_u x(\tilde{u})\Big|_{\tilde{u}=0}=0;\quad Z=\nabla_u x(\tilde{u})$$

- The original problem can be phrased as

$$\min_u f\left(x(u)\right)$$

Sufficient Optimality Conditions

- We can now piggy back on theory of unconstrained optimization, noting that.

$$\nabla_u f\left(x(u)\right)\Big|_{u=0} = \nabla_x \mathcal{L}\left(x^*, \lambda^*\right) = 0;$$

$$\nabla^2_{uu} f\left(x(u)\right)\Big|_{u=0} = Z^T \nabla^2_{xx} \mathcal{L}\left(x^*, \lambda^*\right) Z \succ 0;\ Z = \nabla_u x(u)$$

- Then from theory of unconstrained optimization

we have a local isolated minimum at 0 and thus the original problem at . (following the local isomorphism above)x^*

Another Essential Consequence

- If LICQ+ second-order conditions hold at the solution x^* , then the following matrix must be nonsingular (EXPAND).

$$\begin{bmatrix} \nabla_{xx}^2 \mathcal{L}\left(x^*,\lambda^*\right) & \nabla_x c\left(x^*\right) \\ \nabla_x^T c\left(x^*\right) & 0 \end{bmatrix}$$

- The system of nonlinear equations has an invertible Jacobian,

$$\begin{bmatrix} \nabla_x \mathcal{L}\left(x^*,\lambda^*\right) \\ c\left(x^*\right) \end{bmatrix} = 0$$

8.4.2 FIRST-ORDER OPTIMALITY CONDITIONS FOR MIXED EQ AND INEQ CONSTRAINTS

The Lagrangian

- Even in the general case, it has the same expression

$$\mathcal{L}(x) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{A}} \lambda_i c_i(x)$$

First-Order Optimality Condition Theorem

Suppose that x^* is a local solution of (12.1), that the functions f and c_i in (12.1) are continuously differentiable, and that the LICQ holds at x^*. Then there is a Lagrange multiplier vector λ^*, with components λ_i^*, $i \in \mathcal{E} \cup \mathcal{I}$, such that the following conditions are satisfied at (x^*, λ^*)

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0, \tag{12.34a}$$

$$c_i(x^*) = 0, \quad \text{for all } i \in \mathcal{E}, \tag{12.34b}$$

$$c_i(x^*) \geq 0, \quad \text{for all } i \in \mathcal{I}, \tag{12.34c}$$

$$\lambda_i^* \geq 0, \quad \text{for all } i \in \mathcal{I}, \tag{12.34d}$$

$$\lambda_i^* c_i(x^*) = 0, \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I}. \tag{12.34e}$$

Equivalent Form:

$$\nabla f\left(x^*\right) - \lambda^T_{\mathcal{A}\left(x^*\right)} \nabla c_{\mathcal{A}\left(x^*\right)}\left(x^*\right) = 0 \Rightarrow \text{Multipliers are unique !!}$$

Sketch of the Proof

- If x^* is a solution of the original problem, it is also a solution of the problem.

$$\min f(x) \text{ subject to } c_{\mathcal{A}(x^*)}(x) = 0$$

- From the optimality conditions of the problem with equality constraints, we must have (since LICQ holds)

$$\exists \{\lambda_i\}_{i \in \mathcal{A}(x^*)} \quad \text{such that} \quad \nabla f(x^*) - \sum_{i \in \mathcal{A}(x^*)} \lambda_i \nabla c_i(x^*) = 0$$

- But I cannot yet tell by this argument $\lambda_i \geq 0$

Sketch of the Proof: The sign of the multiplier

- Assume now one multiplier has the "wrong" sign. That is

$$j \in \mathcal{A}\left(x^*\right) \cap \mathcal{I}, \quad \lambda_j < 0$$

- Since LICQ holds, we can construct a feasible path that "takes off" from that constraint (inactive constraints do not matter locally)

- $$c_{\mathcal{A}\left(x^*\right)}\left(\tilde{x}(t)\right) = te_j \Rightarrow \tilde{x}(t) \in \Omega \quad \text{Define } b = \frac{d}{dt}\tilde{x}(t)_{t=0} \Rightarrow \nabla c_{\mathcal{A}(x)} b = e_j$$

$$\frac{d}{dt} f\left(\tilde{x}(t)\right)_{t=0} = \nabla f\left(x^*\right)^T b = \lambda_{c_{\mathcal{A}(x)}}^T \nabla c_{\mathcal{A}(x)} b = \lambda_j < 0 \quad \Rightarrow$$

$$\exists t_1 > 0, \quad f\left(\tilde{x}(t_1)\right) < f\left(\tilde{x}(0)\right) = f\left(x^*\right), \quad \text{CONTRADICTION!!}$$

Strict Complementarity

- It is a notion that makes the problem look "almost" like an equality.

Definition 12.5 (Strict Complementarity).

Given a local solution x^* of (12.1) and a vector λ^* satisfying (12.34), we say that the strict complementarity condition *holds if* exactly one *of λ_i^* and $c_i(x^*)$ is zero for each index $i \in \mathcal{I}$. In other words, we have that $\lambda_i^* > 0$ for each $i \in \mathcal{I} \cap \mathcal{A}(x^*)$.*

8.5 SECOND-ORDER CONDITIONS

Critical Cone

- The subset of the tangent space, where the objective function does not vary to first-order.
- The book definition.

$$\mathcal{C}(x^*, \lambda^*) = \{w \in \mathcal{F}(x^*) \mid \nabla c_i(x^*)^T w = 0, \text{ all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0\}.$$

- An even simpler equivalent definition.

$$\mathcal{C}\left(x^*, \lambda^*\right) = \left\{ w \in T_\Omega\left(x^*\right) \middle| \nabla f\left(x^*\right)^T w = 0 \right\}$$

Rephrasing of the Critical Cone

- By investigating the definition

$$w \in \mathcal{C}\left(x^*, \lambda^*\right) \Leftrightarrow \begin{cases} \nabla c_i\left(x^*\right)^T w = 0 & i \in \mathcal{E} \\ \nabla c_i\left(x^*\right)^T w = 0 & i \in \mathcal{A}\left(x^*\right) \cap \mathcal{I} \quad \lambda_i^* > 0 \\ \nabla c_i\left(x^*\right)^T w \geq 0 & i \in \mathcal{A}\left(x^*\right) \cap \mathcal{I} \quad \lambda_i^* = 0 \end{cases}$$

- In the case where strict complementarity holds, the cones has a MUCH simplex expression.

$$w \in \mathcal{C}\left(x^*, \lambda^*\right) \Leftrightarrow \nabla c_i\left(x^*\right) w = 0 \ \forall\, i \in \mathcal{A}\left(x^*\right)$$

Statement of the Second-Order Conditions

Theorem 12.5 (Second-Order Necessary Conditions).

Suppose that x^* is a local solution of (12.1) and that the LICQ condition is satisfied. Let λ^* be the Lagrange multiplier vector for which the KKT conditions (12.34) are satisfied. Then

$$w^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w \geq 0, \quad \text{for all } w \in \mathcal{C}(x^*, \lambda^*). \tag{12.57}$$

- How to prove this? In the case of Strict Complementarity the critical cone is the same as the problem constrained with equalities on active index.
- Result follows from equality-only case.

Statement of second-order sufficient conditions

Theorem 12.6 (Second-Order Sufficient Conditions).

Suppose that for some feasible point $x^* \in \mathbb{R}^n$ there is a Lagrange multiplier vector λ^* such that the KKT conditions (12.34) are satisfied. Suppose also that

$$w^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w > 0, \quad \text{for all } w \in \mathcal{C}(x^*, \lambda^*), w \neq 0. \tag{12.65}$$

Then x^* is a strict local solution for (12.1).

- How do we prove this? In the case of strict complementarity again from reduction to the equality case.

$$x^* = \arg\min_x f(x) \text{ subject to } c_{\mathcal{A}}(x) = 0$$

How to derive those conditions in the other case?

- Use the slacks to reduce the problem to one with equality constraints.

$$
\begin{array}{llll}
\min_{x\in\mathbb{R}^n, z\in\mathbb{R}^{n_I},} & f(x) & & \\
s.t. & c_E(x) & =0 & \\
 & \left[c_I(x)\right]_j - z_j^2 & =0 & j=1,2,\ldots n_I
\end{array}
$$

- Then, apply the conditions for equality constraints.
- I will assign it as homework.

Summary: Why should I care about Lagrange Multipliers?

- Because it makes the optimization problem in principle equivalent to a nonlinear equation.

$$\begin{bmatrix} \nabla_x \mathcal{L}\left(x^*,\lambda^*\right) \\ c_{\mathcal{A}}\left(x^*\right) \end{bmatrix} = 0; \ \det\begin{bmatrix} \nabla^2_{xx}\mathcal{L}\left(x^*,\lambda^*\right) & \nabla_x c_{\mathcal{A}}\left(x^*\right) \\ \nabla^T_x c_{\mathcal{A}}\left(x^*\right) & 0 \end{bmatrix} \neq 0$$

- I can use concepts from nonlinear equations such as Newton's for the algorithmics.