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# Stochastic Finite-Element Approach in Nuclear Reactor Uncertainty Quantification

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## INTRODUCTION

Polynomial approximations are arguably the most common representation of uncertainty [2]. Such approximations are computed by either local Taylor expansions, e.g., first-order derivatives, or by fitting the approximation at well-chosen sample points, e.g., collocation or surface response methods [4,6]. In this work we present a *hybrid* approach, where the polynomial approximation is computed based on function and derivative information at sample points in the uncertainty region. This approach is shown to efficiently approximate both the distribution and the major global sensitivity effects of the target functional.

## UNCERTAINTY QUANTIFICATION

Our generic model with uncertainty is a system of algebraic-differential equations

$$F(T,R)=0 \tag{1}$$

$$R=R(T)\cdot(1+\Delta R(T,\alpha)) \tag{2}$$

For example, the temperature field $T$ in a reactor core is described by neutron transport, coolant flow, and heat conduction equations that include various parameters $R$ such as thermo-physical parameters and cross sections.

The variables $T=(T_1,T_2,...,T_n)$ characterize the model state. The expressions for the dependence of intermediate physical parameters $R=(R_1,R_2,...,R_N)$ on the state of the model include experimental errors $\Delta R=(\Delta R_1,\Delta R_2,...,\Delta R_N)$.

The dependency of the experimental errors on the state of the model is described by a set of stochastic parameters, or uncertainty quantifiers $\alpha=(\alpha_1,\alpha_2,...,\alpha_m)$. as described in Equation (8). For a given merit function $J=J(T)$ the task is to find the influence of uncertainties in the parameters on the uncertainty of the output. We express an output as the function of parameters $\alpha$

$$J\approx\hat{J}(T(R(T)+\Delta R(T,a)))=\hat{J}(\alpha)\,, \tag{3}$$

and use a polynomial approximation $\hat{J}$ as a *surrogate* (an approximation) for the full model.

### Stochastic Finite-Element Method

The stochastic finite-element method (SFEM) [2] is a generalization of the deterministic finite-element method incorporating stochastic variables into deterministic relationships. A basis of multivariable polynomials $\{\phi_i\}$ is used to approximate the merit function. The coefficients $x_i$ in the expansion $\hat{J}=\sum_i x_i\phi_i(\alpha)$ are found by matching ("collocating [6]") both function and derivative information

$$J(\alpha^{(k)})=\sum_i x_i\phi_i(\alpha^{(k)}) \tag{4}$$

$$\frac{\partial J(\alpha^{(k)})}{\partial\alpha_j}=\sum_i x_i\frac{d\phi_i(\alpha^{(k)})}{d\alpha_j} \tag{5}$$

at the sample points $\alpha^{(k)}$. The first-order derivative information may be obtained with an eventually negligible computational overhead [3]. The number of sample points required to find the same coefficients is decreased by a factor of *m+1*, compared with the approaches that do not match derivative information [2,4].

### Sensitivity Information and Basis Truncation

The relative importance of each component of $\alpha$ in the model is measured by

$$I(\alpha_i)=\frac{\mathrm{var}[E[J_i]]}{\mathrm{var}[J]}\ . \tag{6}$$

Here $E[J_i]$ is the marginal expected value, defined as

$$E[J_i]=\frac{1}{|\Omega(\alpha_i)|}\int_{\Omega(\alpha_i)}J(\alpha_1,...,\alpha_m)d\alpha_1...d\alpha_{i-1}d\alpha_{i+1}...d\alpha_m \tag{7}$$

where $\Omega(\alpha_i)$ is the set of all possible values of uncertainty, with the value $\alpha_i$ fixed.

To reduce the number of polynomials in the basis $\{\phi_i\}$, the variables of high importance are allowed higher polynomial degrees, within a given number of total polynomials, resulting in a *truncated basis*.

## APPLICATION OF THE METHOD

We apply the method to a simplified steady-state model for a coupled neutronics and thermo-hydraulics problem in a seven pin configuration [5], with uncertainty in the material properties. The thermo-hydraulic model is three-dimensional heat conduction and convection equation, and the neutronics model is one-dimensional, 4-group diffusion equation. The intermediate parameters $R$ are: heat conductivity in fuel and coolant, specific coolant heat, heat transfer coefficient, and 4-group microscopic cross-sections. All of these have temperature-dependent uncertainty, approximated by an expansion

$$R_i = R_i(T)\cdot(1+\alpha_{i_0}C^{(0)}(T)+ \\ +\alpha_{i_1}C^{(1)}(T)+\alpha_{i_2}C^{(2)}(T)+...) \quad (8)$$

in terms of Chebyshev polynomials $\{C(T)\}$. Depending on the maximal experimental error, the expansion (8) includes 1-3 terms, resulting in 38 uncertainty quantifiers.

We use uncertainty range information from [1] to create a probabilistic model for the uncertainty coefficients $\alpha$ with uniform distribution *that satisfies all the constraints in* [1]. A surrogate model $\hat{J}(\alpha)$ for the functional $J(\alpha)$ is obtained from (4), (5). We used an SFEM basis of total degree at most 2, truncated by the above procedure. As a functional $J(\alpha)$ we choose the maximum centerline temperature.

## RESULTS

Table I presents the maximum fuel centerline temperature distributions determined with three sampling methods: Monte Carlo, linear approximation and SFEM. The calculation of the uncertainty using Monte Carlo took 100 samples. The SFEM model and the linear approximation required an effort equivalent to 10 and 1 samples correspondingly. Compared to the Monte Carlo results, the error range in 1-norm is 10 degrees K for linear approach and less than 0.5 K for SFEM. Sensitivity calculations presented in Table II for the fuel heat conductivity show that SFEM model faithfully reproduces the sensitivity (6) of the full model with respect to the most important parameters.

## CONCLUSIONS

We conclude that SFEM approaches with collocation using first-derivative information can improve (by a factor of 10) the uncertainty prediction of linear approach.

In addition, the sensitivities (6) are also accurately captured. This leads to the ability of far more accurate uncertainty calculation *without the need for higher-order sensitivity implementations*.

Table I. Performance of SFEM model: maximal centerline temperature, K.

| | Random sampling | Linear approximation | SFEM |
|---|---|---|---|
| Sample size | 100 | 1 | 10 |
| Range | 2237 – 2460 | 2227 – 2450 | 2237 – 2459 |
| St. deviation | 59.05 | 59.12 | 58.96 |

Table II. Sensitivities of the model, (6).
**Note:** *evaluations of the marginal expected values required 8550 full solutions of (1), (2) for the full model, and 15200 evaluations of the surrogate (3); the latter needed only 10 full solutions of (1),(2).*

| | Full model | SFEM model |
|---|---|---|
| $K_{fuel}$ | $\alpha_1 = 0.3876$ | $\alpha_1 = 0.4090$ |
| | $\alpha_2 = 0.0005$ | $\alpha_2 = 0.0017$ |
| | $\alpha_3 = 0.6097$ | $\alpha_3 = 0.5893$ |

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