Using Linear Complementarity Techniques to Model and Simulate Multi-Rigid-Body Dynamics with Contact and Friction

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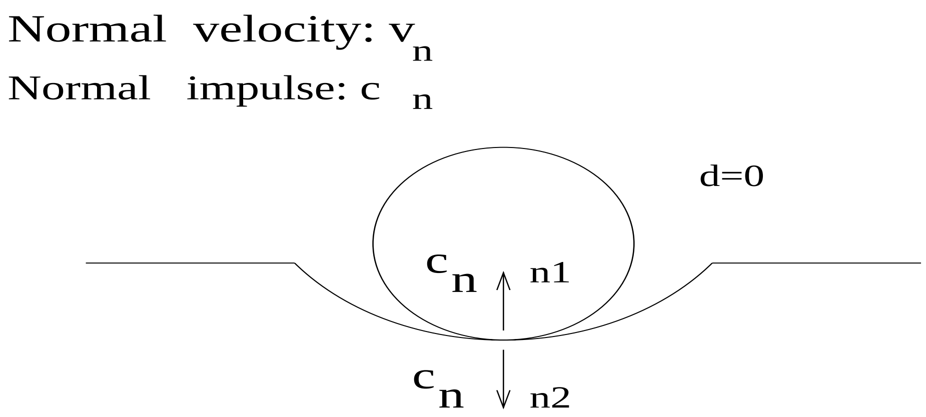
Friction: an essential component of MBD

- Robotics: Prehensile manipulation is not possible without friction. In some devices friction is used as an active element (for example cheap, nonprehensile manipulators).
- Virtual Reality: The lack of it would substantially reduce the believability of a scene.
- The Coulomb Friction Model is the widely used model for static and dynamic friction.
- Unfortunately, Friction creates major difficulties in setting up a consistent model.

Model Requirements and Notations

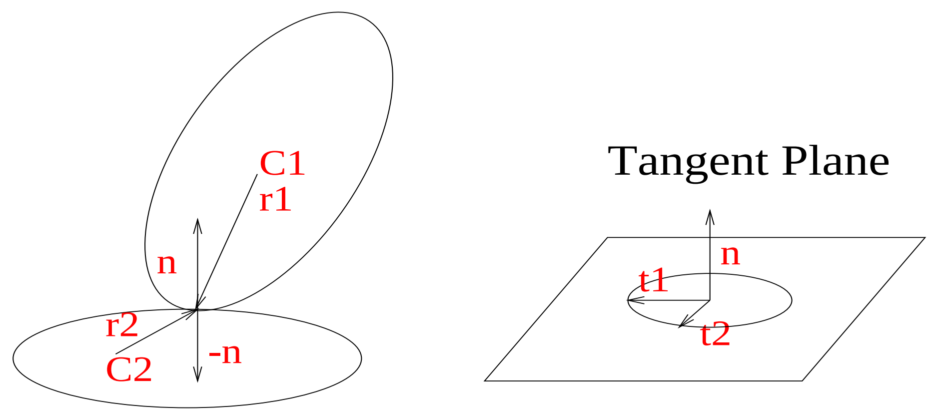
- MBD system : generalized positions q and velocities v.
- No interpenetration $\Phi^{(j)}(q) \geq 0,\ 1 \leq j \leq n_{total}$.
- Compressive contact forces at a contact.
- Joint constraints $\Theta^{(i)}(q) = 0,\ 1 \leq i \leq m$.
- Coulomb friction, for friction coefficients $\mu^{(j)}$.
- Satisfaction of acceleration based Newton laws.
- Dynamic parameters: mass $M(q)$, external force $k(t, q, v)$.
- Impact resolution.



Contact Model

- Contact configuration described by the (generalized) distance function $d = \Phi(q)$, which is defined for some values of the interpenetration. Feasible set: $\Phi(q) \geq 0$.
- Contact forces are compressive, $c_n \geq 0$.
- Contact forces act only when the contact constraint is exactly satisfied, or

$\Phi(q)$ is complementary to c_n or $\Phi(q)c_n = 0$, or $\Phi(q) \perp c_n$.



c_n is the normal impulse and $\beta = (\beta_1, \beta_2)^T$ is the tangential impulse;
In generalized coordinates, q (Newton-Euler world coordinates):

$$n(q) = \begin{pmatrix} n \\ r_1 \times n \\ -n \\ r_2 \times (-n) \end{pmatrix} \quad d_i(q) = \begin{pmatrix} t_i \\ r_1 \times t_i \\ -t_i \\ r_2 \times (-t_i) \end{pmatrix}, \quad i = 1, 2.$$

Here F_c is the total contact force, $F_c = c_n n(q) + \widehat{D}(q)\beta$.
$\widehat{D}(q)$ are the tangential directions, $\widehat{D}(q) = [d_1(q), d_2(q)]$.

Coulomb Friction Model

- The contact force lies in a (circular) cone in 3D, or $||\beta|| \leq \mu c_n$, where μ is the friction coefficient.
- When sliding exists at a contact, the tangential force is opposed to the sliding velocity, or

$$\beta = \text{argmin}_{\widehat{\beta}} v^T \widehat{D}(q)\widehat{\beta} \quad \text{subject to} \quad \left|\left|\widehat{\beta}\right|\right| \leq \mu c_n.$$

- We have that the tangential velocity at the contact is v_T such that

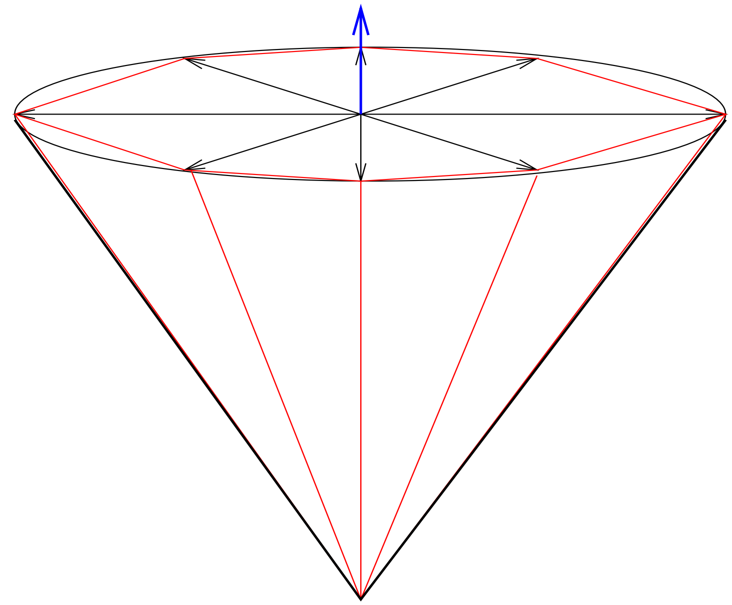
$$|v_T| = \lambda = -v^T \widehat{D}(q)\frac{\beta}{||\beta||}$$

For given c_n and v, the frictional impulse maximize dissipation over all feasible frictional contact impulses.

Discretized Friction Model

- d_i(GC) is the column corresponding to $t(\alpha_i)$, $\alpha_i \in [0, \pi]$, $i = 1, 2, \ldots, l$, $D(q) = [d_1, d_2, ...d_l]$.
- To each tangential direction we attach a force $\beta_i \geq 0$, $i = 1, 2, \ldots, l$. We denote by $\beta = (\beta_1, \beta_2, \ldots, \beta_l)$.
- The frictional constraints become

Polygonal cone approximation to the Coulomb cone (3D).



$$\beta = \operatorname{argmin}_{\widehat{\beta} \geq 0} v^T D(q) \widehat{\beta} \quad \text{subject to} \quad \left|\left|\widehat{\beta}\right|\right|_1 \leq \mu c_n.$$

Complementarity Formulation of Frictional Constraints

Continuous Cone: $\beta = \text{argmin}_{\widehat{\beta}} v^T \widehat{D}(q)\widehat{\beta}$ subject to $\left|\left|\widehat{\beta}\right|\right| \leq \mu c_n$.

Discretized Cone: $\beta = \text{argmin}_{\widehat{\beta}\geq 0} v^T D(q)\widehat{\beta}$ subject to $\left|\left|\widehat{\beta}\right|\right|_1 \leq \mu c_n$.

Optimality Conditions: There exists a Lagrange multiplier $\lambda \geq 0$ such that

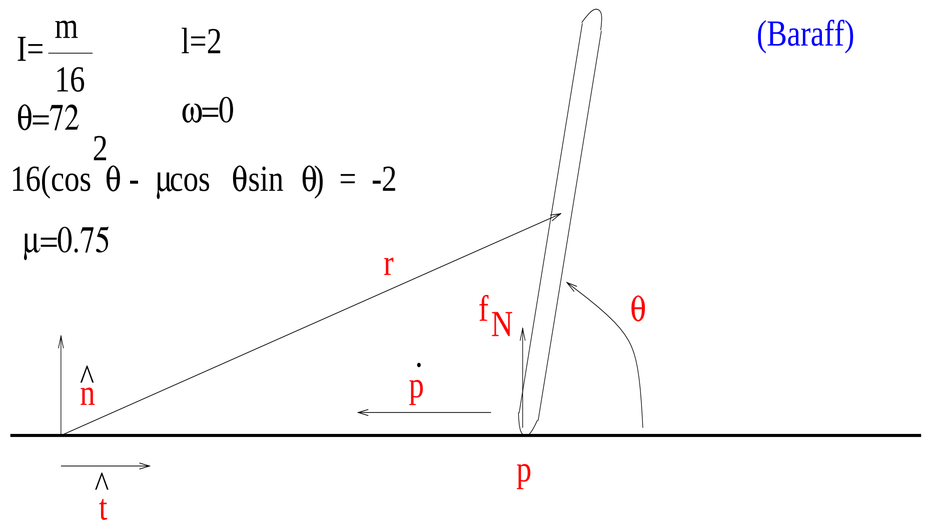
$$\lambda e + D^T v \geq 0 \quad \textbf{complementary to} \quad \beta \geq 0$$
$$\mu c_n - e^T \beta \geq 0 \quad \textbf{complementary to} \quad \lambda \geq 0$$

Here $e = [1, 1, \ldots, 1]^T$. The Lagrange multiplier $\lambda \approx |v_T|$, the approximations approaches equality as the polygone approaches the circular cone.

Acceleration Formulation

$$
\begin{aligned}
M(q)\frac{d^2q}{dt^2} - \sum_{i=1}^{m}\nu^{(i)}c_\nu^{(i)} - \sum_{j=1}^{p} \quad \Big(n^{(j)}(q)c_n^{(j)} \quad &+ D^{(j)}(q)\beta^{(j)}\Big) = k(t,q,\frac{dq}{dt}) \\
\Theta^{(i)}(q) = 0, \qquad\qquad & i = 1\ldots m \\
\Phi^{(j)}(q) \geq 0, \quad \textbf{compl. to} \quad & c_n^{(j)} \geq 0, \quad j = 1\ldots p \\
\beta = \text{argmin}_{\widehat{\beta}^{(j)}} v^T D(q)^{(j)}\widehat{\beta}^{(j)} \quad \textbf{subject to} \quad & \left|\left|\widehat{\beta}^{(j)}\right|\right| \leq \mu^{(j)} c_n^{(j)},\ j = 1\ldots p
\end{aligned}
$$

We use the Coulomb Friction model, nondiscretized. In 2 dimensions the polygonal model and the Coulomb Friction model are equivalent.



$$p = r - \frac{l}{2}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}$$

Constraint: $\hat{n}p \geq 0$ (defined everywhere).

$$\hat{n}\ddot{p} = -g + f_N\left(\frac{1}{m} + \frac{l}{2I}(\cos^2(\theta) - \mu\sin(\theta)\cos(\theta))\right)$$

$$\hat{n}\ddot{p}_a = -g - \frac{f_N}{m}$$

Painleve Paradox: No classical solutions!

Approaching Frictional Inconsistency

Asssume that the system has a classical solution. Formulate the Euler method, half-explicit in velocities, with polyhedral approximation to the friction cone. Linearize the geometrical constraints.

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j \in \mathcal{A}} (n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)}) = hk$$

$$\nu^{(i)^T} v^{l+1} = 0, \qquad i = 1..m$$

$$\rho^{(j)} = n^{(j)^T} v^{l+1} \geq 0, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\sigma^{(j)} = \lambda^{(j)} e^{(j)} + D^{(j)T} v^{l+1} \geq 0, \quad \textbf{compl. to} \quad \beta^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\zeta^{(j)} = \mu^{(j)} c_n^{(j)} - e^{(j)^T} \beta^{(j)} \geq 0, \quad \textbf{compl. to} \quad \lambda^{(j)} \geq 0, \quad j \in \mathcal{A}.$$

Here $\nu^{(\mathbf{i})} = \nabla\Theta^{(\mathbf{i})}$, $n^{(j)} = \nabla\Phi^{(j)}$. h is the time step. The set $\mathcal{A}$ consists of the active constraints. Forces are replaced by impulses!

Matrix Form of the Integration Step

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & 0 \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} -Mv^{(l)} - hk \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

Linear Complementarity Problems (LCP)

$s = \mathcal{M}x + q,\ s \geq 0,\ x \geq 0,\ s^T x = 0.$

- Examples: Linear and Quadratic Programming.
- Important classes of matrices: **PSD** ($x^T \mathcal{M} x \geq 0,\ \forall x$) and **copositive** ($x^T \mathcal{M} x \geq 0,\ \forall x \geq 0$).
- LCP's involving copositive matrices do not have a solution in general.
- Let $\mathcal{M}$ be copositive. If, $x \geq 0$ and $x^T \mathcal{M} x = 0$ implies $q^T x \geq 0$, then the **LCP** has a solution that can be found by Lemke's algorithm.

Theorem

Consider a (mixed) **LCP** of the form

$$\begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = \begin{pmatrix} M & -F & -H \\ F^T & 0 & 0 \\ H^T & 0 & N \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} + \begin{pmatrix} -k \\ 0 \\ 0 \end{pmatrix}$$

$$s \geq 0; \quad \lambda \geq 0; \quad \lambda^T s = 0.$$

If M is a positive definite matrix, N a copositive matrix $\left(x \geq 0 \Rightarrow x^T N x \geq 0\right)$ then the above **LCP** has a solution. Lemke's algorithm will always find a solution λ of the **LCP** obtained by eliminating x and y. A solution (x, y, λ) of the original **LCP** can be recovered by solving for x and y in the first two rows of the mixed **LCP** .

The time-stepping method is guaranteed to have a solution!

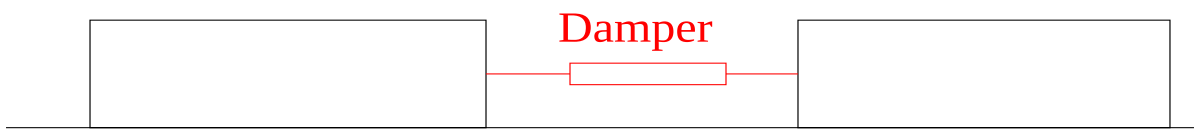
Accommodating Stiffness

- The scheme is based on an explicit Euler scheme and as such cannot accomodate stiffness well (such as systems with very large damping or elastic forces).

- A stiff method should also accomodate the case where there are no contacts and joints. So it should also apply to

$$\begin{aligned} \frac{dq}{dt} &= v; \\ M(q)\frac{dv}{dt} &= k(q, v). \end{aligned}$$

- However, we are still interested in an explicit scheme since otherwise the scheme for the case including contacts would translate into a nonlinear complementarity problem.



Example of a run on a stiff problem

- Identical objects, of mass 1 and with $\mu = 0.4$.
- Initial distance between objects is 3.
- An External force $F = 20cos(t)$ acts on the object in the left.
- Time step 0.05, integration interval 10.
- The damper exerts a force $F_D = \delta\left(-\dot{x}_1 + \dot{x}_2\right)$ on the first object and $-F_D$ on the second object.

Results for $\delta = 20$

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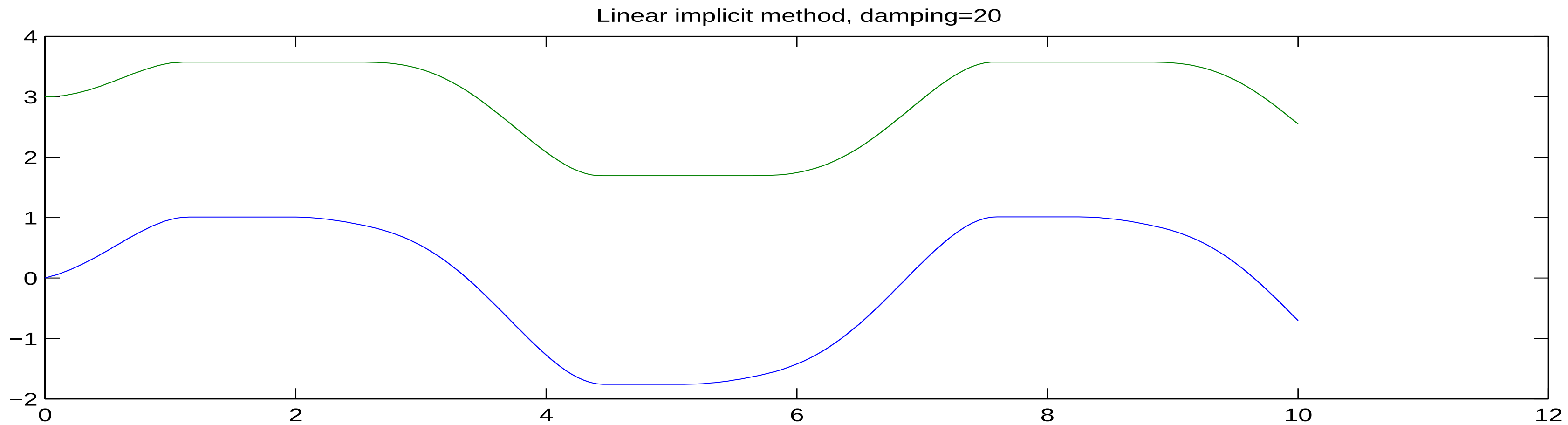
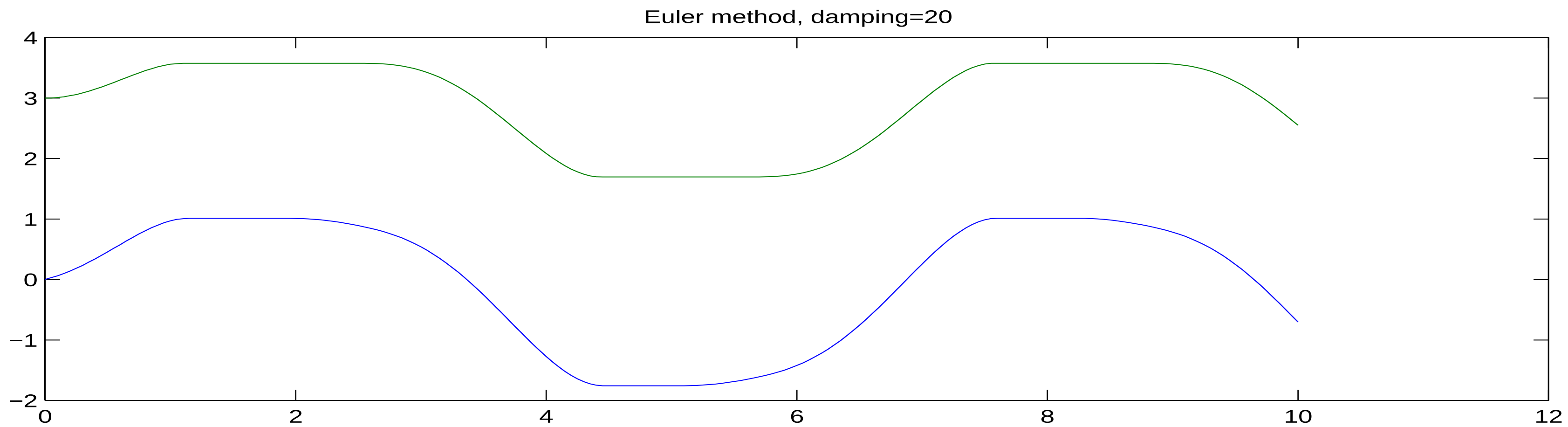
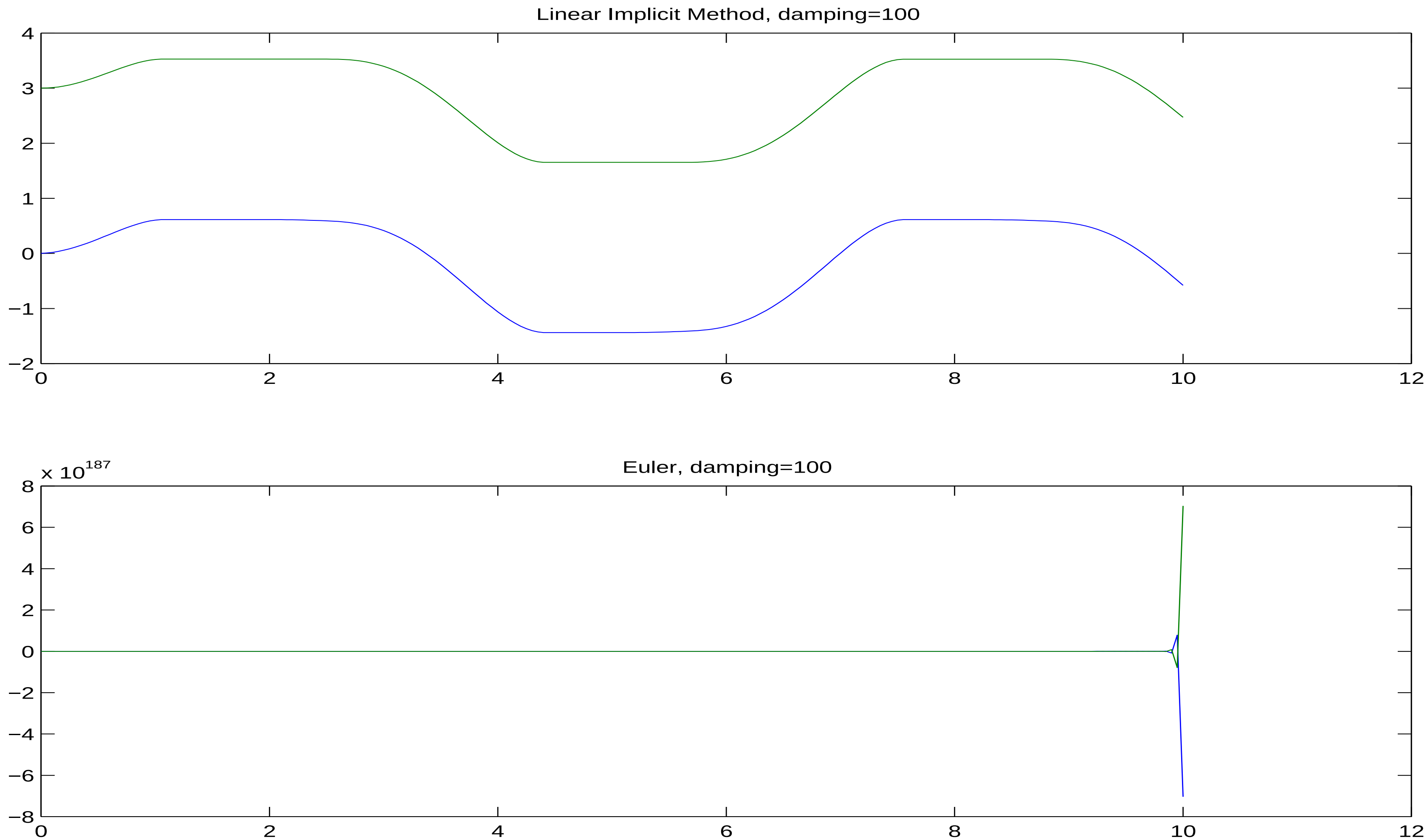




Results for $\delta = 100$, note instability

- 

Linearly Implicit Schemes

$$\begin{aligned}
q^{(n+1)} &= q^{(n)} + h\, v^{(n+1)}, \\
M\left(q^{(n)}\right) \frac{v^{(n+1)} - v^{(n)}}{h} &= k\left(q^{(n)}, v^{(n)}\right) + h\, \nabla_q k\left(q^{(n)}, v^{(n)}\right) v^{(n+1)} \\
&+ \nabla_v k\left(q^{(n)}, v^{(n)}\right)\left(v^{(n+1)} - v^{(n)}\right),
\end{aligned}$$

or, after solving for $v^{(n+1)}$,

$$\begin{aligned}
q^{(n+1)} &= q^{(n)} + h\, v^{(n+1)}, \\
v^{(n+1)} &= \left[M\left(q^{(n)}\right) - h^2\, \nabla_q k\left(q^{(n)}, v^{(n)}\right) - h\nabla_v k\left(q^{(n)}, v^{(n)}\right)\right]^{-1} \times \\
&\left[M\left(q^{(n)}\right) v^{(n)} + h\, k\left(q^{(n)}, v^{(n)}\right) - h\nabla_v k\left(q^{(n)}, v^{(n)}\right) v^{(n)}\right]
\end{aligned}$$

Well-posedness of the method

- Define:

$$\widehat{M} = \left[M\left(q^{(n)}\right) - h^2\,\nabla_q k\left(q^{(n)}, v^{(n)}\right) - h\nabla_v k\left(q^{(n)}, v^{(n)}\right)\right]$$

- Stiff method: replace in the Euler formulation M by $\widehat{M}$ (k by $\widehat{k}$)!

- To ensure consistency by applying the theorem, it will be essential to have $\widehat{M} \succ 0$ and not only invertible.

- If $k(q, v) = -\nabla U(q) - \Gamma(v)$, where $\Gamma(v)$ is a damping-type force, then near an equilibrium point one could expect $\nabla_{qq}U(q) \succeq 0$ and $\nabla_v \Gamma(v) \succeq 0$.

- However, positive definiteness of $\hat{M}$ cannot generally be ensured for moderate values of h when the linear system has eigenvalues with a large negative real part.

Damping and elastic forces

- Most stiff forces in rigid multibody dynamics originate in springs and dampers attached between two points of the multibody system.

- For that case, we have $k(t, q, v) = k_s(t, q, v) + k_1(t, q, v)$, where

$$k_s(t, q, v) = -\sum_{i=1}^{n_\gamma} \gamma_i \phi^{(i)}(q) \nabla_q \phi^{(i)}(q) - \sum_{j=1}^{n_\delta} \delta_j \nabla_q \psi^{(j)}(q) \left(\nabla_q \psi^{(j)^T}(q) v \right)$$

  Here γ_i, $i = 1, \ldots, n_\gamma$ are spring constants and δ_j, $j = 1, \ldots, n_\delta$ are the damper constants. $\phi^{(i)}(q)$ and $\psi^{(j)}(q)$ describe distances between points in the system. $k_1(t, q, v)$ are the nonstiff forces.

- We can then approximate, for the purpose of the linearly implicit method

$$\nabla_q k\,(t, q, v)) \approx -\textstyle\sum_{i=1}^{n_\gamma} \gamma_i \nabla_q \phi^{(i)}(q) \nabla_q \phi^{(i)^T}(q),$$
$$\nabla_v k\,(t, q, v)) \approx -\textstyle\sum_{j=1}^{n_\delta} \delta_j \nabla_q \psi^{(j)}(q) \nabla_q \psi^{(j)^T}(q).$$

Linearly Implicit LCP

$$\begin{bmatrix} \widehat{M} & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & 0 \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} -Mv^{(l)} - h\widehat{k} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

$$\widehat{M} = M + h^2 \sum_{i=1}^{n_\gamma} \gamma_i \nabla_q \phi(q) \nabla_q \phi^T(q) + h \sum_{i=1}^{n_\delta} \delta_i \nabla_q \psi(q) \nabla_q \psi^T(q) \succ 0.$$

Properties of the linearly implicit scheme

- The scheme continues to be well defined for any values of h: the LCP is solvable.

- As $\delta \to \infty$, and $\gamma \to \infty$ the solution to the linearly implicit LCP approaches the solution of the nonstiff LCP that has the additional equality constraints $\nabla_q \phi^{(i)}(q^{(l)})^T v^{(l+1)} = -h\phi^{(i)}(q^{(l)})$ and $\nabla_q \psi^{(j)}(q^{(l)}) v^{(l+1)}(q) = 0$, whenever the limit system has a pointed friction cone. Stiff links behave like joints, for large stiffness parameters!

- Denoting $\widehat{w} = \left(v^{(l)} + khM(q^{(l)})^{-1} k_1(t^{(l)}, q^{(l)}, v^{(l)})\right)^T$, we have

$$v^{(l+1)^T} M(q^{(l)}) v^{(l+1)} + \sum_{i=1}^{n_\gamma} \gamma_i \left(\phi^{(i)^T}(q^{(l)}) + h\nabla_q \phi^{(i)}(q^{(l)})^T v^{(l+1)}\right)^2 \leq$$

$$+\widehat{w}^T M(q^{(l)}) \widehat{w} + \sum_{i=1}^{n_\gamma} \gamma_i \left(\phi^{(i)}(q^{(l)})\right)^2.$$

This ensures the stability of the linear model, as in the unconstrained case.

Collision Assumptions

- The collision within a system of bodies consists of
  - * Compression Phase: interpenetration is prevented by compression impulses from each constraint involved in the collision (even joints).
  - * Decompression Phase: A proportion of e_i (elasticity coefficient) from the normal compression impulse is restituted to the system by each contact constraint Φ_i (Poisson hypothesis). Interpenetration is prevented by decompression impulses.
- The compression/decompression phases following an imminent interpenetration detection are simultaneous for all the bodies involved.

Impact Model: Compression Phase

Collision are instantaneous. Since we have a force-velocity approach, compression can be interpreted as a regular time-step with $h = 0$. Same solvability results apply.

$$
\begin{aligned}
M(v^c - v^-) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{c(i)} - \quad & \textstyle\sum_{j=1}^{p} (n^{(j)} c_n^{c(j)} && + D^{(j)} \beta^{c(j)}) = 0 \\
\nu^{(i)^T} v^c = 0, \quad & && i = 1..m \\
n^{(j)^T} v^c \geq 0, \quad & \textbf{compl to} && c_n^{c(j)} \geq 0, \quad j = 1..p \\
\lambda^{c(j)} e^{(j)} + D^{(j)T} v^c \geq 0, \quad & \textbf{compl. to} && \beta^{c(j)} \geq 0, \quad j = 1..p \\
\mu^{(j)} c_n^{c(j)} - e^{(j)^T} \beta^{c(j)} \geq 0, \quad & \textbf{compl. to} && \lambda^{c(j)} \geq 0, \quad j = 1..p
\end{aligned}
$$

Impact Model: Decompression Phase

Poisson Hypothesis $F^r = \sum_{j=1}^{p} e_j n^{(j)} c_n^{c(j)}$.

$$M(v^+ - v^c) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{x(i)} - \quad \sum_{j=1}^{p} (n^{(j)} c_n^{x(j)} \quad + D^{(j)} \beta^{x(j)}) = F^r$$

$$\begin{array}{rcc}
{\nu^{(i)}}^T v^+ = 0, & & i = 1..m \\
{n^{(j)}}^T v^+ \geq 0, & \textbf{compl to} & c_n^{x(j)} \geq 0, \quad j = 1..p \\
\lambda^{x(j)} e^{(j)} + D^{(j)T} v^+ \geq 0, & \textbf{compl. to} & \beta^{x(j)} \geq 0, \quad j = 1..p \\
\mu^{(j)} c_n^{x(j)} - {e^{(j)}}^T \beta^{x(j)} \geq 0, & \textbf{compl. to} & \lambda^{x(j)} \geq 0, \quad j = 1..p
\end{array}$$

Decompression Solution for a Particular Case

Assumptions

- **(a)** The contacts are frictionless.
- **(b)** All new contacts generated by collision have the same elasticity coefficient ϵ .
- **(c)** The elasticity coefficients characterizing the other contacts are less than ϵ, $e_j \leq \epsilon$, $1 \leq j \leq p$.
- **(d)** The pre-collision velocities satisfy the contact constraints exactly, $(n^{(j)}(q^-))^T v^- = 0$.

Solution and Properties

- **(a)** Just the compression phase is solved by **LCP**.
- **(b)** $v^+ = (1+\epsilon)v^c - v^-$, $c_n^{x(i)} = (\epsilon - e_i)c_n^{c(i)}$.
- **(c)** ${v^+}^T M v^+ \leq v^- M v^-$, the kinetic energy does not increase after the collision (desirable, but not guaranteed for other cases).
- **(d)** This decompression resolution can be used as a general strategy where computational efficiency is required.

Algorithm

$v = v^0$, $q = q^0$; $time = 0$;

while $(time < T)$

$q_{new} = q + hv$;

Find $(v_{new}, \tilde{c}_\nu, \tilde{c}_n, \tilde{\beta}, \tilde{\lambda}) \in \mathcal{L}(v, hk)$ (k at (q, v), the rest at q_{new});

if (no collision detected between $time$ and $time + h$)

$time = time + h$, $q = q_{new}$, $v = v_{new}$;

else

Estimate the collision data $time_{new}$, q_{new} and v^-;

Find $(v^c, \tilde{c}^c_\nu, \tilde{c}^c_n, \tilde{\beta}^c, \tilde{\lambda}^c) \in \mathcal{L}(v^-, 0)$;

Find $(v^+, \tilde{c}^x_\nu, \tilde{c}^x_n, \tilde{\beta}^x, \tilde{\lambda}^x) \in \mathcal{L}(v^c, F^r))$ (or $v^+ = (1+e)v^c - v^-$);

$time = time_{new}$, $v = v^+$, $q = q_{new}$;

end if

end while

LCP contact list

- The initial point is assumed to be feasible for all constraints.
- At each regular (non-collision) step the contact list consists of the union of the set of contacts that the LCP has decided to maintain ($v_n = 0$) at the previous step with the set of contacts that exhibit interpenetration ($\Phi^i(q) < 0$).
- When a collision is detected ($\Phi^i(q)$ changes sign from $+$ to $-$), the contacts for which impact is imminent are added to the contact list.
- The decompression phase uses the same contact list as the compression phase.

Conclusions

- We present a complementarity-based model for multi-rigid body with contact and friction that is guaranteed to be solvable for the most common types of stiff forces.
- The model is based on a discretization of the friction cone and can be as close to the Coulomb model as desired.
- Stiffness is accomodated by means of a linearly implicit scheme for the case of damping forces. In the limit, stiff links behave like joints.
- If the mass matrix $M(q^{(l)})$ is constant and the elastic forces are linear, then the velocity stays bounded at all times. This recovers the analogue of the stability result for differential equations.
- These conclusions were validated with several simulations, where PATH was used to solve the LCP.

Future work

- Higer order schemes between collisions and discontinuities. Extrapolation is atractive since it comes with a minor loss of stability and it adapts very well to this context.
- Alternative friction models, that solve convex subproblems, while maintaining most physical properties of this model. We are currently working on a mixed penalty complementarity framework.
- Interface this approach with enhanced geometrical approaches that compute signed distance functions and feasible configuration fast.
- If a projection is used, how can energy balance be maintained?
- Can a fixed timestep scheme be used, which solves only one LCP per iteration?

Simulations for the cannonball arrangement, h=0.05

Problem	Bodies	Contacts	μ	CPU time (s)
1	10	21	0.2	0.04
2	10	21	0.8	0.03
3	21	52	0.2	0.28
4	21	52	0.8	0.20
5	36	93	0.2	0.81
6	36	93	0.8	0.82
7	55	146	0.2	2.10
8	55	146	0.8	2.07
9	210	574	0.0	0.80
10	210	574	0.2	174.29
11	210	574	0.8	FAIL