A Dynamic Smagorinsky Model for the Spectral Element Method

The notes below are extensions of the material in the thesis of Philipp Schlatter, ETH Dissertation 16000.

1 Basic Model Equations

The equations of motion for the resolved field, $\bar{u}_i$ and $\bar{p}$ are

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} + \frac{1}{Re}\nabla^2 \bar{u}_i \quad (1)$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad (2)$$

where overbar denotes the *filtered* (or resolved) field that is computed on the computational grid. Following the standard approach in LES, we do not explicitly apply a filter to the computed field but instead assume that the restriction of our computed field to the numerical grid constitutes the filtering procedure. The extra stress term is given by

$$\tau_{ij} := \overline{u_i u_j} - \bar{u}_i \bar{u}_j, \quad (3)$$

and needs to be modeled because $\overline{u_i u_j}$ involves the unknown *subgrid-scale* (SGS) quantities, u_i and u_j. The Smagorinsky closure for (3) is

$$\tau_{ij} - \frac{\delta_{ij}}{3}\tau_{kk} \approx -2(C_s\bar{\Delta})^2 |\bar{S}| \bar{S}_{ij}, \quad (4)$$

with the resolved strain-rate tensor

$$\bar{S}_{ij} := \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \quad (5)$$

having norm $|\bar{S}| = (2\bar{S}_{ij}\bar{S}_{ij})^{\frac{1}{2}}$.

Following the Germano *et al.* dynamic Smagorinsky (DS) SGS model we define

$$M_{ij} := \tilde{\bar{\Delta}}^2 |\tilde{\bar{S}}| \tilde{\bar{S}}_{ij} - \bar{\Delta}^2 \widetilde{|\bar{S}|\bar{S}_{ij}} \quad (6)$$

$$= \bar{\Delta}^2 \left(\alpha^2 |\tilde{\bar{S}}| \tilde{\bar{S}}_{ij} - \widetilde{|\bar{S}|\bar{S}_{ij}}\right), \quad (7)$$

where we have introduced a test filter indicated by tilde and $\alpha := \tilde{\bar{\Delta}}/\bar{\Delta}$ is the ratio of the test to grid filter widths. We further define

$$L_{ij} := \tilde{\bar{u}}_i \tilde{\bar{u}}_j - \widetilde{\bar{u}_i \bar{u}_j}. \quad (8)$$

The Lilly determination of the Smagorinsky constant leads to the closure

$$C_{\text{dyn}} := \frac{1}{2}\frac{< M_{ij}L_{ij} >}{< M_{lk}M_{lk} >}, \quad (9)$$

where $< \; . \; >$ indicates some type of smoothing process such as filtering, planar averaging, or Lagrangian averaging. For $C_{\text{dyn}} > 0$, we take

$$C_s := \sqrt{C_{\text{dyn}}}, \quad (10)$$

otherwise $C_s := 0$.

As our point of departure for the SEM implementation, we rewrite (1) as

$$\frac{D\bar{u}_i}{Dt} = -\frac{\partial \hat{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\hat{\nu}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right), \tag{11}$$

$$\tag{12}$$

which can be solved using the SE stress formulation developed by Ho (89). Here,

$$\hat{p} := \bar{p} - \frac{1}{3}\tau_{kk}, \tag{13}$$

$$\hat{\nu} := \left(\frac{1}{Re} + C_{\text{dyn}}\bar{\Delta}^2|\bar{S}|\right), \tag{14}$$

and closure amounts to determining $C_{\text{dyn}}\bar{\Delta}^2$. Defining $\hat{M}_{ij} := \frac{1}{\bar{\Delta}^2}M_{ij}$, we can rewrite (9) as

$$C_{\text{dyn}}\bar{\Delta}^2 = \frac{1}{2}\frac{< \hat{M}_{ij}L_{ij} >}{< \hat{M}_{lk}\hat{M}_{lk} >}, \tag{15}$$

which depends only on the ratio of filter widths α and has no explicit dependence on $\bar{\Delta}$.

2 Choice of Test Filter

Elimination of an explicit dependence on $\bar{\Delta}$ in (15) resolves a significant source of ambiguity for the application of the DS model in complex domains when using general purpose discretizations. Determination of the filter ratio is relatively straightforward. For example, if one constructs a test filter that projects onto half of the number of modes (in each space dimension, leading to an eightfold reduction in three dimensions), $\alpha = 2$. With the spectral element method, we have the possibility of projecting from the Nth-order local basis functions onto a basis of order $\tilde{N}$, with corresponding $\alpha = N/\tilde{N}$.

A open question is how best to define the test filter, F such that $\tilde{\bar{u}} := F(\bar{u})$. Choices abound, and it would seem natural to develop an approach such that the test filter mimics the action of the Galerkin projection (i.e., the implicit filter) on which the original discretization is based, albeit at a larger scale. For the velocity field, several reasonable approaches are possible. The first would be a simple Galerkin projection of the form: *Find* $\tilde{\bar{u}}_i \in X^{\tilde{N}}$ *such that*

$$(v, \tilde{\bar{u}}_i) = (v, \bar{u}_i) \ \forall\, v \in X^{\tilde{N}}, \tag{16}$$

where (f, g) is a suitable inner product, e.g., the standard L^2 inner product $(f, g) := \int_\Omega f g \, dV$. A more elaborate scheme would involve choosing $X^{\tilde{N}}$ to be restricted to the manifold of divergence-free velocity fields. Still more logical, and undoubtedly related to the variational multiscale method of Hughes and co-workers, would be to choose the inner-product to be related to the unsteady Stokes operator that is used to project the nonlinear dynamics onto the discrete approximation space for $\bar{u} \in X_0^N$, with X_0^N replaced by $X_0^{\tilde{N}}$. Such an approach unambiguously gives proper definition of the boundary conditions, divergence-free conditions, and function continuity requirements for the test-filtered field.

Note that, for filtering nonlinear terms, the Galerkin procedures such as (16) can be implemented in fully dealiased form at relatively low cost given that all nonlinear terms in (11) are evaluated in fully dealiased form, that is, on a grid associated with polynomial degree $M \approx \frac{3}{2}N$.

Once available, it's not expensive to collocate the quantities that have already been interpolated using J_M and project them locally onto the $\tilde{N}$ grid by applying $J^T_{\tilde{N}}$. A full projection involving global (albeit diagonal) mass matrices, however, would require simultaneously storing all 30 filtered fields on the M-mesh, for all elements, and would be too expensive from a storage standpoint.

Unfortunately, while readily defined for the velocity field, the projection operators associated with the more elaborate function spaces considered above are not unambiguously defined when computing, say, $q_{ij} := \widetilde{|\bar{S}|\bar{S}_{ij}}$, because the continuity and boundary condition requirements in this case are not well-defined. Inspection of the preceding derivations reveals that the origins of q_{ij} are in the Smagorinsky ansatz and not in well-defined or well-established physical processes. It seems likely that the DS model is more sensitive to consistency in the choice of test-filter process to all quantities, rather than in matching the precise relationship between the test and grid filters. A reasonable starting point, therefore, would be to apply the Boyd filter within each spectral element either using a sharp cut-off or a steeply damped filter for $k > \tilde{N}$.

2.1 Commutativity of the Filter

It should be relatively straightforward numerically and analytically, in the context of one- or two-dimensional SEM model problems, to establish that the $\tilde{N}$-based filters commute with differentiation up to order $\tilde{N}$ for any reasonable choice of filter.