

# Filter-based stabilization of spectral element methods

Paul FISCHER<sup>a</sup>, Julia MULLEN<sup>b</sup>

<sup>a</sup> Mathematics and Computer Science Div., Argonne National Laboratory, Argonne, IL, USA 60439

<sup>b</sup> Division of Applied Mathematics, Brown Univ., Providence RI, USA 02912

(Proposée le 19 Juillet 1999)

---

**Abstract.** *We present a simple filtering procedure for stabilizing the spectral element method (SEM) for the unsteady advection-diffusion and Navier-Stokes equations. A number of example applications are presented, along with basic analysis for the advection-diffusion case.*

*Stabilisation par filtrage pour la méthode des éléments spectraux*

**Résumé.** Nous présentons une procédure simple de filtrage pour la stabilisation de la méthode des éléments spectraux (SEM) appliquée à des équations de convection-diffusion et de Navier-Stokes. Cette procédure est mise en oeuvre sur un grand nombre d'exemples, et une analyse élémentaire est réalisée sur un cas de convection-diffusion.

---

## 1. Introduction

We consider spectral element solution of the incompressible Navier-Stokes equations in  $\mathbb{R}^d$ ,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} \text{ in } \Omega, \quad \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \quad (1)$$

with prescribed boundary and initial conditions for the velocity,  $\mathbf{u}$ . Here,  $p$  is the pressure and  $Re = \frac{UL}{\nu}$  the Reynolds number based on characteristic velocity and length scales.

A well-known difficulty in numerical treatment of (1) is the enforcement of the divergence-free constraint on  $\mathbf{u}$ , particularly at high Reynolds numbers. The  $\mathbb{P}_N - \mathbb{P}_{N-2}$  spectral element method (SEM) introduced in [9] addresses this problem through the use of compatible velocity and pressure spaces that are free of spurious modes. The method attains exponential convergence in space and second- or third-order accuracy in time. Despite these advantages, we have in the past encountered stability problems that have mandated very fine resolution for applications at moderate to high Reynolds numbers ( $10^3$ – $10^4$ ). Here, we demonstrate a simple filtering procedure that largely cures the instability and allows one to recover the full advantages of the SEM.

## 2. Discretization and Filter

The filter is applied at the end of each step of the Navier-Stokes time integration (described in detail in [7]). The temporal discretization is based on the high-order operator-splitting methods developed in [10]. The convective term is expressed as a material derivative, which is discretized using a stable second-order BDF scheme, leading to a linear symmetric Stokes problem to be solved implicitly at each step. The subintegration of the convection term permits timestep sizes,

$\Delta t$ , corresponding to convective CFL numbers of 2–5, thus significantly reducing the number of (expensive) Stokes solves.

The Stokes discretization is based on the variational form *Find*  $(\mathbf{u}, p) \in X_N \times Y_N$  such that

$$\frac{1}{Re}(\nabla \mathbf{u}, \nabla \mathbf{v})_{GL} + \frac{3}{2\Delta t}(\mathbf{u}, \mathbf{v})_{GL} - (p, \nabla \cdot \mathbf{v})_G = (\mathbf{f}, \mathbf{v})_{GL}, \quad (\nabla \cdot \mathbf{u}, q)_G = 0, \quad (2)$$

$\forall (\mathbf{v}, q) \in X_N \times Y_N$ . The inner products  $(\cdot, \cdot)_{GL}$  and  $(\cdot, \cdot)_G$  refer to the Gauss-Lobatto-Legendre (GL) and Gauss-Legendre (G) quadratures associated with the spaces  $X_N := [Z_N \cap H_0^1(\Omega)]^d$  and  $Y_N := Z_{N-2}$ , respectively. Here,  $Z_N := \{v \in L^2(\Omega) | v|_{\Omega^k} \in \mathbb{P}_N(\Omega^k)\}$ , where  $L^2$  is the space of square integrable functions on  $\Omega$ ;  $H_0^1$  is the space of functions in  $L^2$  that vanish on the boundary and whose first derivative is also in  $L^2$ , and  $\mathbb{P}_N(\Omega^k)$  is the space of functions on  $\Omega^k$  whose image is a tensor-product polynomial of degree  $\leq N$  in the reference domain,  $\hat{\Omega} := [-1, 1]^d$ . For  $d = 2$ , a typical element in  $X_N$  is written

$$\mathbf{u}(\mathbf{x}^k(r, s))|_{\Omega^k} = \sum_{i=0}^N \sum_{j=0}^N \mathbf{u}_{ij}^k h_i^N(r) h_j^N(s), \quad (3)$$

where  $\mathbf{u}_{ij}^k$  is the nodal basis coefficient;  $h_i^N \in \mathbb{P}_N$  is the Lagrange polynomial based on the GL quadrature points,  $\{\xi_j^N\}_{j=0}^N$  (the zeros of  $(1 - \xi^2)L'_N(\xi)$ , where  $L_N$  is the Legendre polynomial of degree  $N$ ); and  $\mathbf{x}^k(r, s)$  is the coordinate mapping from  $\hat{\Omega}$  to  $\Omega^k$ . We assume  $\Omega = \cup_{k=1}^K \Omega^k$  and that  $\hat{\Omega}^k \cap \hat{\Omega}^l$  for  $k \neq l$  is either an entire edge, a single vertex, or void. Function continuity ( $\mathbf{u} \in H^1$ ) is enforced by ensuring that nodal values on element boundaries coincide with those on adjacent elements. For  $Y_N$ , a tensor-product form similar to (3) is used, save that the interpolants are based on the G points since interelement continuity is not enforced.

Insertion of the SEM basis into (2) yields a discrete Stokes system to be solved at each step:

$$H \tilde{\mathbf{u}} - D^T \underline{p}^n = B \underline{\mathbf{f}}^n, \quad D \tilde{\mathbf{u}} = 0; \quad \underline{\mathbf{u}}^n = F_\alpha \tilde{\mathbf{u}},$$

where we have introduced the stabilizing filter,  $F_\alpha$ , to be described below. Here,  $H = \frac{1}{Re}A + \frac{1}{\Delta t}B$  is the discrete equivalent of the Helmholtz operator,  $(-\frac{1}{Re}\nabla^2 + \frac{1}{\Delta t})$ ;  $-A$  is the discrete Laplacian;  $B$  is the mass matrix associated with the velocity mesh;  $D$  is the discrete divergence operator, and  $\underline{\mathbf{f}}^n$  accounts for the explicit treatment of the nonlinear terms. The filter,  $F_\alpha$ , is applied on an element-by-element basis once the velocity-pressure pair  $(\tilde{\mathbf{u}}, \underline{p}^n)$  has been computed.

The filter is constructed as follows. Let  $i_n$  be the one-dimensional interpolation operator at the nodes  $\{\xi_j^n\}_{j=0}^n$  over  $\mathbb{P}_n[-1, 1]$ . Then in the square, we can define  $i_n^r$  (resp.  $i_n^s$ ) as being the interpolation operator in the  $r$  (resp.  $s$ ) direction. The filter (in the square) is then  $F_\alpha = \alpha I_{N-1} + (1 - \alpha)Id$ , where  $I_{N-1} = i_{N-1}^r \circ i_{N-1}^s$  and  $Id$  is the identity operator. The interpolation-based procedure ensures that interelement continuity is preserved; and, because the nodal basis points  $\xi_i^N$  interlace  $\xi_i^{N-1}$ ,  $F_\alpha$  will tend to dampen high-frequency oscillations. Moreover, spectral convergence is not compromised, because the interpolation error will go to zero exponentially fast as  $N \rightarrow \infty$  for smooth  $u$ . This operator is stable both in  $L^2$  and  $H^1$  norms (that are natural norms for this equation) as can be found in [2] (13.27–28). We note that  $\alpha = 1$  corresponds to a full projection onto  $\mathbb{P}_{N-1}$ , effectively yielding a sharp cutoff in modal space, whereas  $0 < \alpha < 1$  yields a smoother, preferable decay [3, 6, 8].

### 3. Applications

We have used the filtering procedure on a number of high Reynolds number applications where the standard  $\mathbb{P}_N - \mathbb{P}_{N-2}$  method would not converge (for reasonable values of  $K$  and  $N$ ). These

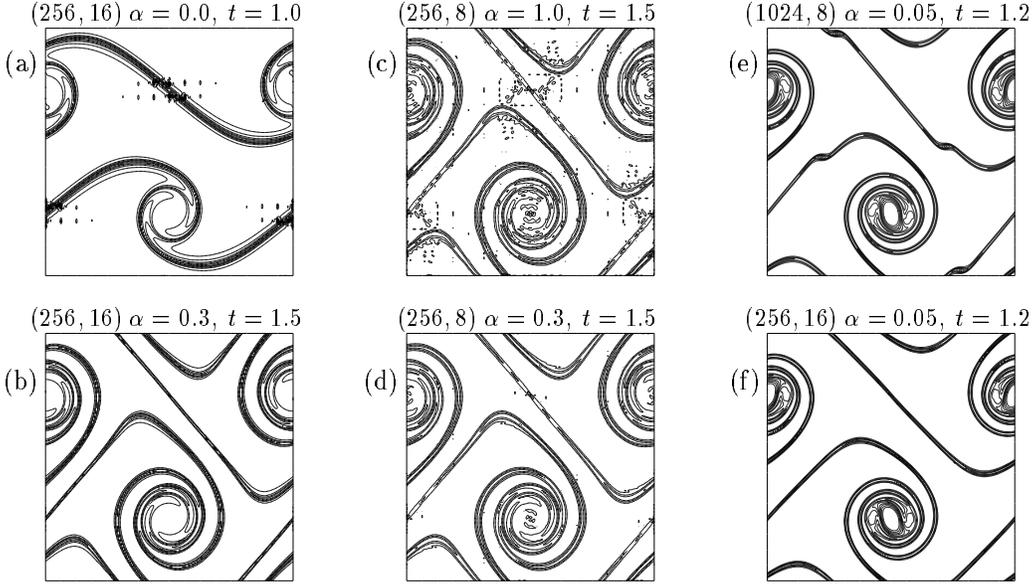


Figure 1: Vorticity for different  $(K, N)$  pairings: (a–d)  $\rho = 30$ ,  $Re = 10^5$ , contours from  $-70$  to  $70$  by  $140/15$ ; (e–f)  $\rho = 100$ ,  $Re = 40,000$ , contours from  $-36$  to  $36$  by  $72/13$ . (cf. Fig. 3c in [4]).

have included the regularized driven cavity at  $Re = 5000$ , transitional channel flow at  $Re_h = 8000$ , and hairpin vortex formation in a boundary layer at  $Re_\delta = 1200$ . The examples below demonstrate the benefits of the filter on some well-known test problems.

**Example 1.** Figure 1 shows results for the shear layer roll-up problem studied in [1, 4]. Doubly-periodic boundary conditions are applied on  $\Omega := [0, 1]^2$ , with initial conditions

$$u = \tanh(\rho(y - 0.25)) \text{ for } y \leq 0.5, \quad u = \tanh(\rho(0.75 - y)) \text{ for } y > 0.5, \quad v = 0.05 \sin(2\pi x).$$

Each case consists of a  $16 \times 16$  array of elements, save for (e), which is  $32 \times 32$ . The time step size is  $\Delta t = .002$  in all cases, corresponding to CFL numbers in the range of 1 to 5. Without filtering, we are unable to simulate this problem at any reasonable resolution. In (a), we see the results just prior to blow up for the unfiltered case with  $N = 16$ , corresponding to an  $n \times n$  grid with  $n = 256$ . Unfiltered results for  $N = 8$  ( $n = 128$ ) and  $N = 32$  ( $n = 512$ ) are similar. Filtering with  $\alpha = 0.3$  yields dramatic improvement for  $n = 256$  (b) and  $n = 128$  (d). Although full projection ( $\alpha = 1$ ) is also stable, it is clear by comparing (c) with (d) that partial filtering ( $\alpha < 1$ ) is preferable. Finally, (e) and (f) correspond to the difficult “thin” shear layer case [4]. The spurious vortices in (e) are eliminated in (f) by increasing the order to  $N = 16$  at fixed resolution ( $n = 256$ ). Note that an even number of contours was chosen to avoid the dynamically insignificant zero contour.

**Example 2.** The spatial and temporal accuracy of the filtered SEM is verified by reconsidering the Orr-Sommerfeld problem studied in [7]. The growth rates of a small-amplitude ( $10^{-5}$ ) Tollmien-Schlichting wave superimposed on plane Poiseuille channel flow at  $Re = 7500$  are compared with the results of linear theory. The errors (see (41) in [7]) at time  $t = 60$  given in Table 1 reveal exponential convergence in  $N$  for both the filtered and unfiltered cases. It is also clear that  $O(\Delta t^2)$  and  $O(\Delta t^3)$  convergence is obtained for the filtered case, but that the unfiltered results are unstable for the third-order scheme. In this case, the stability provided by the filter permits the use of higher-order temporal schemes, thereby allowing a larger time step for a given accuracy.

Table 1: Spatial and Temporal Convergence, Orr-Sommerfeld Problem

N	$\Delta t = 0.00325$		$N = 17$	2nd-order		3rd-order	
	$\alpha = 0.0$	$\alpha = 0.2$	$\Delta t$	$\alpha = 0.0$	$\alpha = 0.2$	$\alpha = 0.0$	$\alpha = 0.2$
7	0.23641	0.27450	0.20000	0.12621	0.12621	171.370	0.02066
9	0.00173	0.11929	0.10000	0.03465	0.03465	0.00267	0.00268
11	0.00455	0.01114	0.05000	0.00910	0.00911	161.134	0.00040
13	0.00004	0.00074	0.02500	0.00238	0.00238	1.04463	0.00012

#### 4. Analysis and Conclusion

The stabilizing role of the filter is illustrated by considering a time marching approach to solving the advection-diffusion equation,  $u_x = \nu u_{xx} + f$ ,  $u(0) = u(1) = 0$ , which was studied in the context of bubble-stabilized spectral methods in [5]. Discretization by SEM/CN-AB3 yields

$$H\tilde{\underline{u}} = H_R\underline{u}^n + C\left(\frac{23}{12}\underline{u}^n - \frac{16}{12}\underline{u}^{n-1} + \frac{5}{12}\underline{u}^{n-2}\right) + B\underline{f}, \quad \underline{u}^{n+1} = F_\alpha\tilde{\underline{u}}, \quad (4)$$

where  $H = (\frac{\nu}{2}A + \frac{1}{\Delta t}B)$  and  $H_R = (-\frac{\nu}{2}A + \frac{1}{\Delta t}B)$  are discrete Helmholtz operators and  $C$  is the convection operator. The fixed point of (4) satisfies

$$(-\nu A + C + H(F_\alpha^{-1} - I))\underline{u} = B\underline{f}. \quad (5)$$

The  $\Delta t$  dependence in (5) can be eliminated by assuming that  $1 \simeq \text{CFL} := \Delta t/\Delta x \simeq \Delta t N^2$ .

For any Galerkin formulation,  $C$  is skew symmetric and therefore singular if the number of variables is odd (the spurious mode being  $L_N - L_0$ ). The eigenvalues of  $(F_\alpha^{-1} - I)$  are  $\{0, 0, \dots, 0, \frac{\alpha}{1-\alpha}\}$  (the non-zero eigenmode being  $\phi_N(x) := \frac{2N-1}{N(N-1)}(1-x^2)L'_{N-1}(x) = L_N - L_{N-2}$ ). The stabilizing term,  $H(F_\alpha^{-1} - I)$ , thus prevents (5) from blowing up as  $\nu \rightarrow 0$  by suppressing the unstable mode. We note that this mode corresponds to a single element in the filter basis suggested in [3]. One can easily suppress more elements in this basis in order to construct smoother filters as suggested, for example, in [3, 6, 8]. However, our early experiences and asymptotic analysis ( $\nu \rightarrow 0$  in (5)) indicate that slight suppression of just the  $N$ th mode is sufficient to stabilize the  $\mathbb{P}_N - \mathbb{P}_{N-2}$  method at moderate to high Reynolds numbers.

#### References

- [1] J. B. Bell, P. Collela, and H. M. Glaz, A second-order projection method for the incompressible Navier-Stokes equations, *J. Comp. Phys.*, 85, (1989) 257–283.
- [2] C. Bernardi and Y. Maday, Spectral Methods, in Handbook of Numerical Analysis P. G. Ciarlet and J. L. Lions, eds., North-Holland (1999).
- [3] J. P. Boyd, Two comments on filtering for Chebyshev and Legendre spectral and spectral element methods, *J. Comp. Phys.*, 143, (1998) 283–288.
- [4] D. L. Brown and M. L. Minion, Performance of under-resolved two-dimensional incompressible flow simulations, *J. Comp. Phys.*, 122, (1995) 165–183.
- [5] C. Canuto and G. Puppo, Bubble stabilization of spectral Legendre methods for the advection-diffusion equation, *Comput. Methods Appl. Mech. Engrg.*, 118, (1994) 239–263.
- [6] W. S. Don and D. Gottlieb, Spectral simulation of supersonic reactive flows, *SIAM J. Numer. Anal.* 35 (1998) 2370–2384.
- [7] P. F. Fischer, An overlapping Schwarz method for spectral element solution of the incompressible Navier-Stokes equations, *J. Comp. Phys.*, 133, (1997) 84–101.
- [8] Y. Maday, S. Ould Kaber, and E. Tadmor, Legendre pseudospectral viscosity method for nonlinear conservation laws, *SIAM J. Numer. Anal.*, 30, (1993) 321–342.
- [9] Y. Maday, and A. T. Patera, Spectral element methods for the Navier-Stokes equations in State of the Art Surveys in Computational Mechanics, A.K. Noor, ed., ASME, New York, (1989) 71–143.
- [10] Y. Maday, A. T. Patera, and E. M. Rønquist, An operator-integration-factor splitting method for time-dependent problems: Application to incompressible fluid flow, *J. Sci. Comput.*, 5(4), (1990) 310–37.