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Synthesis of H_2 optimal static structured controllers: primal and dual formulations

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***Abstract*— We consider the design of H_2 optimal static structured feedback gains for large-scale interconnected systems. The design of distributed controllers with access to measurements of a small number of the subsystems imposes particular sparsity constraints on the feedback gains. For this nonconvex constrained optimal control problem, we study both the primal and dual formulations to obtain optimality bounds. We exploit the sparsity structure present in large-scale systems by implementing an efficient quasi-Newton algorithm to solve the primal problem. We employ the subgradient method to solve the dual problem and obtain a lower bound for the optimal value of the performance index. Surprisingly, in many problems of practical interest, the upper bounds from solving primal problems and the lower bounds from solving dual problems are almost identical, suggesting the lack of duality gap in these applications and that the globally optimal structured gains have in fact been attained.**

***Index Terms*— Decentralized control, large-scale systems, sparse matrices, static feedback gains, zero duality gap.**

I. Introduction

We consider the design of H_2 optimal static structured feedback gains for large-scale interconnected systems. The control objective is to minimize the H_2 norm of the closed-loop system under the information constraints. In particular, we consider the scenario that each controller only has access to measurements of a small number of the subsystems. This constraint imposes particular *sparsity* structure on the static feedback gain. Although we consider static controllers, the methods developed in this work can be extended to the design of fixed-order dynamic controllers by the well-known system augmentation technique [1], [2].

The closely related optimal static output feedback problem has been studied extensively over the last four decades [3]–[10]. This problem was originally introduced in early seventies [3], it is known to be nonconvex, and many techniques for computation of local optima have been proposed ever since. Broadly speaking, these iterative methods fall into two categories. First, the general-purpose minimization methods include, Newton's method [6], quasi-Newton method [5], [9] and trust region method [10]. Second, the special-purpose alternating methods include Levine-Athans method [3], Anderson-Moore method [4], and their variants [7].

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The advent of linear matrix inequalities (LMIs) has sparked renewed interest in static (or fixed-order) output feedback design [2], [11]–[17]. It is well-known that the unstructured H_2 state-feedback problem is equivalent to a convex problem [18] even though it appears to be nonconvex in the feedback gain. However, the structured H_2 problem cannot be transformed into a convex problem [14], [18] and consequently it cannot be cast into an LMI [14], [19]. The aforementioned direct iterative methods [3]–[10] are designed for small or medium size unstructured feedback problems. This implies that these methods can experience computational difficulties when applied to large-scale structured feedback problems.

This paper has two important contributions:

- *We develop efficient quasi-Newton method that is capable of solving large-scale problems by exploiting the underlying sparsity structure.*
- *We formulate and solve the dual problem to obtain a lower bound to the optimal value of the performance index.*

The nonconvexity of the structured H_2 feedback problem implies that the solution obtained by the quasi-Newton method is in general a local minimum, which in effect provides an upper bound to the global optimal value. It is a standard fact in optimization theory that the solution of the dual problem provides a lower bound to the global optimal value [20], [21]. Perhaps surprisingly, for many worked examples, the upper bounds from solving the primal problems and the lower bounds from solving the dual problems correspond to each other, indicating that the global optimality is attained.

The paper is organized as follows. We formulate the optimal control problem and give the necessary conditions for optimality in Section II. We develop an efficient quasi-Newton method for the primal problem in Section III. We formulate and solve the dual problem in Section IV. We present the computational results for a mass-spring system on a discrete spatial lattice in Section V, and offer concluding remarks in Section VI.

II. Problem formulation and necessary conditions for optimality

Consider the following control problem

$$\begin{aligned} \dot{\psi} &= A\,\psi + B_1\,d + B_2\,u \\ z &= C_1\,\psi + D\,u \\ y &= C_2\,\psi, \qquad\qquad u = -F\,y, \end{aligned}$$

where $C_1 = \begin{bmatrix} Q^{1/2} & 0 \end{bmatrix}^T$ and $D = \begin{bmatrix} 0 & R^{1/2} \end{bmatrix}^T$. Matrix F denotes the static feedback gain subject to structural constraints which dictate the zero entries of F. Let the subspace $\mathcal{S}$ encapsulate these constraints and let us assume that there is a stabilizing $F \in \mathcal{S}$. Upon closing the loop, the above problem can equivalently be written as

$$\begin{aligned} \dot{\psi} &= (A - B_2 F C_2)\,\psi + B_1\, d \\ z &= \begin{bmatrix} Q^{1/2} \\ -R^{1/2} F C_2 \end{bmatrix} \psi. \end{aligned}$$

Note that d denotes exogenous signals and that the performance output z encapsulates both the amplitude of the state and that of the control input. We now consider the following optimal control problem:

- *Find the matrix $F \in \mathcal{S}$ such that $\|H\|_2^2$ is minimized, where H is the transfer function from d to z and $\|\cdot\|_2$ is the H_2 norm.*

Explicitly, this structured H_2 (SH2) problem is given by

$$\begin{aligned} \text{minimize} \quad & J = \operatorname{trace}\left(P B_1 B_1^T\right) \\ \text{subject to} \quad & (A - B_2 F C_2)^T P + P\,(A - B_2 F C_2) \\ & \quad = -\left(Q + C_2^T F^T R F C_2\right), \ F \in \mathcal{S}. \end{aligned} \tag{SH2}$$

We are interested in *sparsity* constraints on F. A sparse matrix is populated primarily with zeros and the pattern of the non-zero entries describes the communication architecture of the distributed controller. We focus particularly on cases where F is a banded matrix, i.e., it is only non-zero on its main block-diagonal and a relatively small number of block sub-diagonals.

In general, the H_2 norm of the closed-loop system is not convex in the feedback gain, i.e., $J(F)$ is not a convex function of F. Moreover, the set of all stabilizing state feedback gains is not a convex set [18]. On the other hand, the structural constraint is linear, namely,

$$F_1 \in \mathcal{S},\ F_2 \in \mathcal{S} \ \Rightarrow \ a_1 F_1 + a_2 F_2 \in \mathcal{S},$$

and thus it is a convex constraint in the feedback gain.

We next state the necessary conditions for optimality of (SH2).

Proposition 1 (Necessary conditions for optimality): In order for matrix $F \in \mathcal{S}$, with $A - B_2 F C_2$ Hurwitz, to be optimal for the problem (SH2), it is necessary that it satisfies the following set of equations:

$$\begin{aligned} & (A - B_2 F C_2)^T P + P\,(A - B_2 F C_2) \\ & \quad = -\left(Q + C_2^T F^T R F C_2\right), \end{aligned} \tag{NC1}$$

$$(A - B_2 F C_2)\, L + L\,(A - B_2 F C_2)^T = -B_1 B_1^T, \tag{NC2}$$

$$(R F C_2 L C_2^T) \circ I_{\mathcal{S}} = (B_2^T P L C_2^T) \circ I_{\mathcal{S}}, \tag{NC3}$$

where $\circ$ denotes the element-wise multiplication of matrices.

Proof: We form the Lagrangian

$$\begin{aligned} \mathcal{L}(F, P, L) = {} & \operatorname{trace}\left(P B_1 B_1^T\right) + \\ & \operatorname{trace}\left((A_{\text{cl}}^T P + P A_{\text{cl}} + C_{\text{cl}}^T C_{\text{cl}})^T L\right), \end{aligned}$$

where $A_{\text{cl}} = A - B_2 F C_2$ and $C_{\text{cl}}^T C_{\text{cl}} = Q + C_2^T F^T R F C_2$. Setting $\partial \mathcal{L} / \partial L = 0$ and $\partial \mathcal{L} / \partial P = 0$ gives the first two equations (NC1) and (NC2), respectively. Since F is constrained to the set $\mathcal{S}$ the optimality condition for F is $(\partial \mathcal{L} / \partial F) \circ I_{\mathcal{S}} = 0$, which gives the final equation (NC3). We omit algebraic manipulations for sake of brevity.

The matrix $I_{\mathcal{S}}$ in (NC3) denotes the *structural identity* of $\mathcal{S}$ under the elementwise multiplication, i.e., $F \circ I_{\mathcal{S}} = F$ for a structured matrix $F \in \mathcal{S}$. Specifically, the ijth entry of $I_{\mathcal{S}}$ is given by

$$I_{\mathcal{S}ij} = \begin{cases} 1, & \text{if } F_{ij} \text{ is a free variable;} \\ 0, & \text{if } F_{ij} = 0 \text{ is required.} \end{cases}$$

For the subspaces of diagonal and tridiagonal matrices $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, $I_{\mathcal{S}_1}$ is the identity matrix and $I_{\mathcal{S}_2}$ is the tridiagonal matrix with entries equal to one on its main diagonal, first upper and first lower subdiagonals.

It is noteworthy that (SH2) problem includes other related problems as special cases. As aforementioned, the optimal fixed-order H_2 problems can be formulated as (SH2) by the well-known system augmentation technique [1], [2]. Moreover, by removing the constraint $F \in \mathcal{S}$ from (SH2) the optimal static output feedback problem [3]–[10] is recovered, for which, (NC3) simplifies to

$$R F C_2 L C_2^T = B_2^T P L C_2^T. \tag{NC3'}$$

Many algorithms have been proposed to solve the set of coupled equations (NC1), (NC2) and (NC3') for the static output feedback problem [3]–[10]. We are particularly interested in large-scale problems with the sparsity constraints on the feedback gains, for which we develop an efficient quasi-Newton method in the next section.

III. Descent method for primal problem

In this section, we develop a descent method to solve large-scale (SH2) problem with sparsity constraint on the feedback gain. In particular, we develop an efficient implementation of computing the quasi-Newton (Broyden-Fletcher-Goldfarb-Shanno or BFGS) direction for sparse matrix variables.

Descent methods are iterative algorithms aimed at solving the optimization problem (SH2). Specifically, given an initial stabilizing feedback gain $F^0 \in \mathcal{S}$, a descent method generates a minimizing sequence $\{F^i \in \mathcal{S}\}$ as follows

$$F^{i+1} = F^i + s^i \tilde{F}^i,$$

where $\tilde{F}^i$ is the *descent direction* and s^i is the step-size. We consider two descent directions: negative gradient and quasi-Newton.

The BFGS method is widely considered as the best quasi-Newton method currently known [20]. However, the standard BFGS method is designed for optimization variables in the form of a vector. We develop an efficient scheme to compute BFGS direction for optimization variables assuming the form of sparse matrices.

The gradient direction for the problem (SH2) is given by

$$\nabla J(F) = \left(2\,(R F C_2 - B_2^T P) L C_2^T\right) \circ I_{\mathcal{S}}, \tag{1}$$

where L and P are the solutions of the following Lyapunov equations

$$\begin{aligned}(A - B_2 F C_2)\, L \;+\; L\,(A - B_2 F C_2)^T \;&=\; -\,B_1 B_1^T,\\ (A - B_2 F C_2)^T P \;+\; P\,(A - B_2 F C_2) \;&=\\ &-(Q + C_2^T F^T R F C_2).\end{aligned} \tag{2}$$

The above equations are derived by substituting F in (SH2) with $\{F + \varepsilon \delta F,\, F,\, \delta F\} \in \mathcal{S}$. Subsequently, J can be written as a series expansion in ε, in which the coefficient of the ε term is

$$\operatorname{trace}\left(\left(2\, C_2 \bar{L}(C_2^T F^T R - P B_2)\right)\delta F\right).$$

This gives the gradient term (1), where L and P satisfy (2).

Let a sparse matrix $F_{m\times p} \in \mathcal{S}$, where m is the number of inputs and p is the number of outputs. Let

$$\operatorname{vec} F \;:=\; \left[\, F_1^T \;\, F_2^T \;\, \ldots \;\, F_p^T \,\right]^T$$

where F_i is the ith column of F. Let $\operatorname{sqz}(\operatorname{vec} F)$ remove all zero entries of $\operatorname{vec} F$, which results in a vector whose length is equal to the number of nonzero entries of F. For the 2×2 diagonal matrix example

$$F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix},$$

we have $\operatorname{vec} F = [\, f_1 \;\, 0 \;\, 0 \;\, f_2 \,]^T$ and

$$\operatorname{sqz}(\operatorname{vec} F) = [\, f_1 \;\, f_2 \,]^T.$$

In other words, $v_F := \operatorname{sqz}(\operatorname{vec} F)$ is the compressed vector obtained by eliminating the zero entries of $\operatorname{vec} F$. Clearly, given v_F, one can readily construct $F \in \mathcal{S}$ by the reverse procedure. Specifically, let z_i be the number of zero entries between the ith and $(i+1)$th non-zero entry of the vector $\operatorname{vec} I_{\mathcal{S}}$. Vector $\operatorname{vec} F$ can be obtained by inserting z_i zeros between the ith and the $(i+1)$th entry of v_F. Thus, the structured matrix F is determined by

$$F = [\, v_{F1} \; v_{F2} \; \ldots \; v_{Fp} \,]$$

where v_{Fi} is the vector from the $((i-1)m+1)$th entry to (im)th entry of v_F. Similarly, we form the compressed gradient $v_g := \operatorname{sqz}(\operatorname{vec} \nabla J(F))$.

Given two compressed vectors v_{F_k} and $v_{F_{k+1}}$ and their respective compressed gradients v_{g_k} and $v_{g_{k+1}}$, the BFGS update [22] to approximate the Hessian inverse of the objective function with respect to v_F is given by

$$H_{k+1} = (I - \rho_k s_k y_k^T)\, H_k\, (I - \rho_k y_k s_k^T) + \rho_k s_k s_k^T \tag{3}$$

where

$$\rho_k = 1/(y_k^T s_k), \quad s_k = v_{F_{k+1}} - v_{F_k}, \quad y_k = v_{g_{k+1}} - v_{g_k}.$$

The BFGS direction of the compressed vector is given by

$$\tilde{v}_{k+1} = -H_{k+1} v_{g_{k+1}}.$$

To initialize the BFGS method, one can choose an initial condition v_{F_0} and use a gradient step to find v_{F_1}. The initial Hessian inverse is the identity matrix. The BFGS direction in the form of structured sparse matrix is then readily obtained by the aforementioned procedure.

Descent method to solve (SH2)

Given stabilizing $F^0 \in \mathcal{S}$;

for $i = 0, 1, 2, \ldots$, **repeat**

1) compute descent direction $\tilde{F}^i$;
2) use step-size rule to determine s^i;
3) update $F^{i+1} \,=\, F^i \,+\, s^i \tilde{F}^i$;

until $\|\nabla J(F^i)\|_F < \epsilon$

For the step-size rule, we employ the backtracking line search [21], [22]. Note that the closed-loop stability in descent method is guaranteed by the step-size selection.

Step-size rule

Let $\alpha = 0.3,\; \beta = 0.5,\; s^i = 1$ (see [21, p. 464]);

until: both conditions are satisfied

- $J(F^i + s^i\, \tilde{F}^i) \,<\, J(F^i) \,+\, \alpha\, s^i \operatorname{trace}(\nabla J(F^i)^T \tilde{F}^i)$;
- $A_{\rm cl}^i \,=\, A \,-\, B_2\,(F^i + s^i\, \tilde{F}^i)\, C_2$ is a Hurwitz matrix.

repeat: $s^i \;:=\; \beta s^i$

IV. Dual problem

In this section, we study the dual problem of (SH2). In optimization theory, the study of the dual problem can be motivated from various aspects [20], [21]. In particular, the solution of the dual problem provides a lower bound to the global optimal value of the *primal problem* (SH2). Since an upper bound can be obtained by finding a local minimum of (SH2) via the quasi-Newton method, by solving the dual problem we then effectively have an estimate of the global optimal value of (SH2). The difference between the primal and the dual optimal values determines the duality gap [20], [21]. For a strictly feasible convex problem, the duality gap is zero [20], [21]; for a nonconvex problem the duality gap is not zero in general. However, in our computations, several problems of practical interest have been solved with negligible duality gaps (for an illustration, see example in Section V). Specifically, the primal and dual optimal values in these problems correspond to each other, indicating that the global minima have in fact been attained.

A. Dual problem formulation

We next formulate the dual problem of (SH2). We transform the structured output feedback (SH2) problem to an equivalent structured state feedback problem. This transformation plays an important role in finding the good initial condition in the minimization of Lagrangian in Section IV-B. More importantly, it allows us to generalize the algebraic Riccati equation in the standard state feedback H_2 theory to a system of a Lyapunov equation and a Riccati equation; these two equations are coupled by the dual variables associated with the structural constraint.

Consider the change of coordinates $\phi = T\psi$ such that

$$C_{2T} \;=\; C_2\, T^{-1} \;=\; [\, I_{p\times p} \;\; O_{p\times(n-p)} \,].$$

Note that this change of coordinates is not unique; namely, since

$$\begin{aligned} C_2 &= C_{2T}\,T \\ &= [\,I_{p\times p}\;\; O_{p\times(n-p)}\,]\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{bmatrix} \\ &= [\,T_{11}\;\; T_{12}\,]. \end{aligned}$$

Hence, any $[\,T_{21}\;\; T_{22}\,]$ such that

$$T = \begin{bmatrix} C_2 \\ T_{21}\;\; T_{22} \end{bmatrix}$$

is a full-rank matrix gives the desired coordinate transformation. One choice is $[\,T_{21}\quad T_{22}\,] = \mathrm{null}(C_2)^T$, where $\mathrm{null}(C_2)$ denotes the orthogonal column vectors that span the nullspace of C_2.

Since the H_2 norm is invariant under coordinate transformation, (SH2) in new coordinates is given by

$$\begin{array}{ll} \text{minimize} & J \,=\, \mathrm{trace}\left(P_T B_{1T} B_{1T}^T\right) \\ \text{subject to} & (A_T - B_{2T} F C_{2T})^T\, P_T \,+\, P_T\,(A_T - B_{2T} F C_{2T}) \\ & \quad =\, -\left(C_{1T}^T C_{1T} \,+\, C_{2T}^T F^T R F C_{2T}\right),\;\; F \in \mathcal{S}, \end{array}$$

where

$$\begin{array}{lll} A_T \,=\, TAT^{-1}, & P_T \,=\, (T^{-1})^T P T^{-1}, & \\ B_{1T} \,=\, TB_1, & B_{2T} \,=\, TB_2, & C_{1T} \,=\, C_1 T^{-1}. \end{array}$$

By defining $K := F\,C_{2T} = [\,F\;\; O_{m\times(n-p)}\,]$, we obtain the equivalent

$$\begin{array}{ll} \text{minimize} & J \,=\, \mathrm{trace}\left(P B_1 B_1^T\right) \\ \text{subject to} & (A - B_2 K)^T\, P \,+\, P\,(A - B_2 K) \\ & \quad =\, -\left((T^{-1})^T\, Q\, T^{-1} \,+\, K^T R K\right), \\ & K = [\,K_1\;\; K_2\,],\;\; K_1 \in \mathcal{S},\;\; K_2 = 0, \end{array}$$

where we drop the subscript $_T$ for notational convenience. The structural constraint $F \in \mathcal{S}$ for the output feedback gain is preserved in $K_1 \in \mathcal{S}$ under the coordinate transformation. The extra constraint $K_2 = 0$ is a consequence of the implicit constraint imposed by the output matrix C_2.

One can now define the structure on state feedback gain, e.g., as $K \in \mathcal{K}$ with $\mathcal{K} = [\,\mathcal{S}\quad O_{m\times(n-p)}\,]$. With a slight abuse of notation, however, we will still use $\mathcal{S}$ as the subspace of the structured gains and the structure will be clear from the context.

To formulate the Lagrange dual problem, we need to determine a more convenient representation of the algebraic constraints on K. To this end, we rewrite the constraint $K \in \mathcal{S}$ in the form of an equivalent set of trace constraints. Let $E_{ij} := e_i e_j^T$, with e_i denoting the ith standard basis vector [23]. The constraint $k_{ij} = 0$ can be written as

$$k_{ij} \,=\, \mathrm{trace}\,(e_i^T K e_j) \,=\, \mathrm{trace}\,(E_{ij}^T K) = 0.$$

Thus, the structural constraint on K is equivalent to a set of trace constraints

$$\mathrm{trace}\,(E_{ij}^T K) \,=\, 0, \qquad (i,j) \in \mathcal{S}_{ij}^c,$$

where set $\mathcal{S}_{ij}^c$ of the index pairs $\{(i,j)\}$ identify the zero entries of K. Similarly, set $\mathcal{S}_{ij}$ identifies the nonzero entries of $K \in \mathcal{S}$. For example, $\mathcal{S}_{ij}^c = \{(1,2),(2,1)\}$ and $\mathcal{S}_{ij} = \{(1,1),(2,2)\}$ correspond to the diagonal structure on 2×2 matrices.

Thus, we have

$$\begin{array}{ll} \text{minimize} & J \,=\, \mathrm{trace}\left(P B_1 B_1^T\right) \\ \text{subject to} & (A - B_2 K)^T\, P \,+\, P\,(A - B_2 K) \\ & \quad =\, -\left((T^{-1})^T\, Q\, T^{-1} \,+\, K^T R K\right), \\ & \mathrm{trace}\,(E_{ij}^T K) \,=\, 0, \qquad (i,j) \in \mathcal{S}_{ij}^c. \end{array}$$

We are now ready to employ the standard Lagrange multiplier theory to form Lagrangian and the corresponding dual problem [20], [21]. The Lagrangian is given by

$$\mathcal{L}(K,\nu) = \mathrm{trace}\left(P(K) B_1 B_1^T\right) \,+\, \sum_{(i,j)\in\mathcal{S}_{ij}^c} \nu_{ij}\,\mathrm{trace}(E_{ij}^T K),$$

where $\nu_{ij} \in \mathbb{R}$ is the Lagrange multiplier associated with constraint $\mathrm{trace}\,(E_{ij}^T K) = 0$. By introducing

$$E_\nu := \sum_{(i,j)\in\mathcal{S}_{ij}^c} \nu_{ij} E_{ij},$$

Lagrangian can be written compactly as

$$\mathcal{L}(K,\nu) \,=\, \mathrm{trace}\left(P(K) B_1 B_1^T\right) \,+\, \mathrm{trace}\,(E_\nu^T K), \tag{4}$$

where $P(K)$ is the unique positive definite solution of the Lyapunov equation

$$(A - B_2 K)^T\, P \,+\, P\,(A - B_2 K) \,=\, -\left(Q_T \,+\, K^T R K\right)$$

with $Q_T = (T^{-1})^T\, Q\, T^{-1}$.

The dual function evaluated at the dual variable $\nu := \{\nu_{ij},\, (i,j) \in \mathcal{S}_{ij}^c\}$ is given by the minimum of the Lagrangian, i.e.,

$$g(\nu) \;:=\; \underset{K}{\text{minimize}} \;\; \mathcal{L}(K,\nu).$$

Note that K must be a stabilizing gain for the minimization of Lagrangian to be meaningful. Also note that the dual variable ν is subject to an implicit constraint $\nu \in \mathcal{D}$ such that Lagrangian is bounded below. In other words, the dual function is defined on the set

$$\mathcal{D} \,=\, \{\,\nu \,|\, g(\nu) > -\infty\,\}. \tag{5}$$

For a given dual variable ν, the dual function is always a lower bound to the global optimal value of the primal problem (SH2). The best (greatest) lower bound is obtained by solving the so-called dual maximization problem

$$\text{maximize} \quad g(\nu) \qquad\quad \text{subject to} \quad \nu \in \mathcal{D}. \tag{D}$$

The dual problem is always convex [20], [21] regardless of the nonconvexity of the primal problem. In particular, it can be shown [20] that the domain $\mathcal{D}$ is convex and the dual function $g(\nu)$ is concave over $\mathcal{D}$.

B. Alternating method

In this section, we determine the necessary conditions for the Lagrangian optimality in the form of two coupled equations. These coupled equations for the structured problem generalize the standard algebraic Riccati equation for the unstructured problem. We propose an alternating method to solve the coupled equations to minimize Lagrangian.

One approach to the Lagrangian minimization problem is to employ the quasi-Newton method developed in Section III. Alternatively, we study the necessary conditions for the optimality of Lagrangian. The gradient of Lagrangian

$$\mathcal{L}(K,\nu) \;=\; J(K) \;+\; \mathrm{trace}(E_\nu^T K)$$

is given by

$$\begin{aligned}\nabla\mathcal{L}(K,\nu) &= \nabla J(K) \;+\; E_\nu \\ &= 2\,(RK - B_2^T P)\,L \;+\; E_\nu,\end{aligned} \tag{6}$$

where L, P are the solutions of the Lyapunov equations

$$\begin{aligned}&(A - B_2K)\,L \;+\; L\,(A - B_2K)^T \;=\; -\,B_1B_1^T,\\ &(A - B_2K)^T\,P \;+\; P\,(A - B_2K) \;=\; -(Q_T + K^TRK).\end{aligned}$$

Thus, the necessary conditions for Lagrangian optimality are given by

$$\begin{aligned}&2(RK - B_2^TP)L + E_\nu \;=\; 0,\\ &(A - B_2K)\,L \;+\; L\,(A - B_2K)^T \;=\; -\,B_1B_1^T,\\ &(A - B_2K)^T\,P \;+\; P\,(A - B_2K) \;=\; -(Q_T + K^TRK).\end{aligned} \tag{7}$$

Assuming the pair $(A - B_2K, B_1)$ is controllable, L is positive definite and hence invertible. Thus,

$$K \;=\; R^{-1}B_2^TP - (1/2)R^{-1}E_\nu L^{-1}. \tag{8}$$

Substituting (8) into the two Lyapunov equations in (7) yields

$$\begin{aligned}&(A - B_2R^{-1}B_2^TP)L + L(A - B_2R^{-1}B_2^TP)^T + M_\nu = 0,\\ &A^TP + PA - PB_2R^{-1}B_2^TP +\\ &\qquad\qquad Q_T + (1/4)L^{-1}E_\nu^TR^{-1}E_\nu L^{-1} = 0,\end{aligned} \tag{9}$$

where

$$M_\nu \;=\; B_1B_1^T \;+\; (1/2)(B_2R^{-1}E_\nu \;+\; E_\nu^TR^{-1}B_2^T).$$

Recall that the standard algebraic Riccati equation (ARE) is

$$A^TP \;+\; PA \;-\; PB_2R^{-1}B_2^TP + Q_T \;\;=\; 0 \tag{10}$$

and the *centralized* state feedback gain is $K_c = R^{-1}B_2^TP$ with the unique solution $P > 0$ of the ARE (10). Note that Eqs. (9) generalize the ARE (10) to two coupled equations: one Lyapunov equation in L and one ARE in P. These two equations are coupled through the dual variables in E_ν. The centralized feedback gain K_c is recovered by setting $E_\nu = 0$ in Eq. (8); and the ARE (10) for unstructured problems is recovered by setting $E_\nu = 0$ in the ARE for P in (9). Furthermore, since (A, Q_T) is observable and $L_k^{-1}E_\nu^TR^{-1}E_\nu L_k^{-1}$ is positive semi-definite, the pair

$$(\,A,\; Q_T \;+\; (1/4)L_k^{-1}E_\nu^TR^{-1}E_\nu L_k^{-1}\,)$$

is also observable. Therefore, there exists a unique positive definite solution P of the ARE in Eqs. (9).

Denoting $Z = L^{-1}$, and pre-multiplying and post-multiplying Z to the Lyapunov equation of L in Eqs. (9) yields the following *coupled algebraic Riccati equations* (CAREs)

$$\begin{aligned}&(A - B_2R^{-1}B_2^TP)^TZ + Z(A - B_2R^{-1}B_2^TP) +\\ &\qquad\qquad ZM_\nu Z = 0,\\ &A^TP + PA - PB_2R^{-1}B_2^TP +\\ &\qquad\qquad Q_T + (1/4)ZE_\nu^TR^{-1}E_\nu Z = 0.\end{aligned}$$

Related CAREs from the study of linear-quadratic Nash game have been studied in [24]–[27]. However, a direct numerical method to solve the coupled Riccati equations is currently unknown [26]. In view of this, we next consider an alternating method to solve the coupled equations (9).

Alternating method

Given $E_\nu, L_0 > 0$;

for $k = 0, 1, \ldots,$ **repeat**

1) solve for positive definite P_k

$$\begin{aligned}&A^TP_k \;+\; P_kA \;-\; P_kB_2R^{-1}B_2^TP_k \;+\\ &\qquad Q_T \;+\; (1/4)L_k^{-1}E_\nu^TR^{-1}E_\nu L_k^{-1} \;=\; 0;\end{aligned}$$

2) solve for L_{k+1}

$$\begin{aligned}&(A - B_2R^{-1}B_2^TP_k)L_{k+1} + L_{k+1}(A - B_2R^{-1}B_2^TP_k)^T\\ &\qquad\qquad + M_\nu \;=\; 0;\end{aligned}$$

3) if $\|\mathcal{R}_k\|_F < \epsilon$, where

$$\begin{aligned}\mathcal{R}_k \;=\; &A^TP_k \;+\; P_kA \;-\; P_kB_2R^{-1}B_2^TP_k \;+\\ &Q_T \;+\; (1/4)L_{k+1}^{-1}E_\nu^TR^{-1}E_\nu L_{k+1}^{-1};\end{aligned}$$

stop and return L_{k+1}, P_k as the solutions.

end

Similar alternating methods to solve coupled Lyapunov equations (NC1), (NC2) and (NC3') arising from the optimal static output feedback problem have been proposed in [3], [4]. Although there is no proof for convergence of the alternating method, we consider it as an alternative to the quasi-Newton method for large-scale Lagrangian minimization problems.

C. Subgradient method

In this section, we employ the standard subgradient method [20] to solve the dual problem. The reason for employing the subgradient method is that the dual function is not differentiable at the optimal point if there exists a nonzero duality gap [20, section 6.1]. Since, in general, (SH2) is a nonconvex problem (implying possibility of duality gap), the nondifferentiability issue of the dual function cannot be ignored. Another motivation for use of subgradient method is that the subgradient is obtained at essentially no computational cost; it only requires evaluation of the dual function [20]. This is advantageous in computations since the number of dual variables $\{\nu_{ij}\}$ is large due to the sparsity of K. Furthermore, the subgradient method is simple and robust. Its only drawback is a slow convergence rate [20].

Let K_ν be the minimizer of Lagrangian (4) for given dual variable $\nu = \{\nu_{ij}\}$. Then it can be shown [20, section 6.1] that *one* subgradient direction of ν_{ij} is given by

$$\tilde{\nu}_{ij} \;=\; \mathrm{trace}\,(E_{ij}^T K_\nu) \;=\; k_{\nu ij}.$$

The subgradient can be rewritten compactly as

$$\tilde{E}_\nu \;:=\; \sum_{(i,j)\in\mathcal{S}_{ij}^c} \tilde{\nu}_{ij} E_{ij} \;=\; \sum_{(i,j)\in\mathcal{S}_{ij}^c} k_{\nu_{ij}}\, E_{ij} \;=\; K_\nu \circ I_{\mathcal{S}}^c.$$

We next give the subgradient algorithm and then provide more details about the implementation.

Subgradient method to solve dual problem (D)
Given $\nu^0 = 0$, $E_\nu^0 = 0$, $K_\nu^0 = K_c$, solve for L_ν^0

$$(A - B_2 K_c)\, L_\nu^0 \;+\; L_\nu^0\,(A - B_2 K_c)^T \;=\; -\,B_1 B_1^T;$$

for $i = 0, 1, \ldots,$ **repeat**

1) form subgradient $\tilde{E}_\nu^i \;=\; K_\nu^i \circ I_{\mathcal{S}}^c$; update $E_\nu^{i+1} \;=\; E_\nu^i \;+\; s^i\,\tilde{E}_\nu^i$,
where $s^i = \eta(J(K_p^\star) - g(\nu^i))/\|\tilde{E}_\nu^i\|_F^2$ with constant $0 < \eta < 2$;
2) if the quasi-Newton method is employed for Lagrangian minimization, use K_ν^i as initial condition to obtain minimizer K_ν^{i+1} for $\mathcal{L}(K, \nu^{i+1})$;
if the alternating method is employed to solve Eq. (9), use L_ν^i as initial condition to obtain solution L_ν^{i+1}, P_ν^{i+1}; and form

$$K_\nu^{i+1} = R^{-1} B_2^T P_\nu^{i+1} - (1/2) R^{-1} E_\nu^{i+1} L_\nu^{i+1}.$$

In general, there is no formal stopping criterion for subgradient method [28]. If the dual function is differentiable, then the subgradient is exactly the gradient direction. Then, $\|\tilde{E}_\nu^i\|_F^2 < \epsilon$ can be used as a the stopping criterion. In this case, the minimizer $K_\nu^\star$ of the Lagrangian $\mathcal{L}(K, \nu^\star)$ for the optimal dual variable $\nu^\star$ satisfies

$$\|\tilde{E}_\nu^\star\|_F^2 \;=\; \|K_\nu^\star \circ I_{\mathcal{S}}^c\|_F^2 < \epsilon.$$

Also note that differentiability of the dual function at optimal point implies that there is no duality gap. Hence, if the subgradient vanishes at the optimal point, it provides numerical evidence of zero duality gap.

Remark 1: The subgradient method is closely related to the so-called *multiplier method* [1], [20], which was introduced to alleviate the difficulty of finding an initial structured stabilizing feedback gain in [1]. By minimizing the *augmented* Lagrangian [1], [20], the multiplier method converges to a local minimum of (SH2). In general, the multiplier method is effective for small size problems. However, the optimization variable in the augmented Lagrangian minimization is also a full matrix, which limits its efficiency for large-scale problems.

V. An Example

This section contains a mass-spring example to illustrate the efficiency of quasi-Newton (BFGS) method to solve primal problem (SH2). We also present results obtained by solving dual problem (D) using subgradient method. We implement the alternating method to minimize Lagrangian in each subgradient step. One surprising fact is that duality gap is very small and the optimal feedback gains from solving the primal and dual problems are also very close to each other. In other words, solutions of primal and dual problems provide tight estimates on the global optimal value. This indicates that the feedback gains resulting from the primal problem can be effectively considered as the global optimal feedback gains.

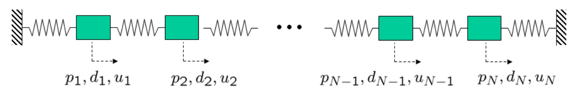

Fig. 1. Mass-spring system.

We consider a mass-spring system consisting of N masses and $N+1$ springs on a line, Fig. 1. The dynamics of the ith mass m_i is given by

$$m_i \ddot{p}_i \;+\; k_{i-1}(p_i - p_{i-1}) \;+\; k_i(p_i - p_{i+1}) \;=\; u_i \;+\; d_i,$$

where p_i represents the displacement from a reference position of the ith mass and k_i is the spring constant of the ith spring. We assign unit values to $\{m_i, k_i\}$. The first and the last masses are connected to fixed boundaries; hence, $p_0 \equiv 0$, $\dot{p}_0 \equiv 0$, $p_{N+1} \equiv 0$, $\dot{p}_{N+1} \equiv 0$. By selecting the state variables $\psi_1 := \mathrm{col}\,\{p_i\}$ and $\psi_2 := \mathrm{col}\,\{\dot{p}_i\}$, $i = \{1, 2, \ldots, N\}$, the state-space representation is determined by

$$A \;=\; \begin{bmatrix} O_N & I_N \\ T_N & O_N \end{bmatrix}, \quad B_1 \;=\; B_2 \;=\; \begin{bmatrix} O_N \\ I_N \end{bmatrix},$$

where $T_N := \mathrm{toeplitz}([\;-2\;\;1\;\;0\;\;\cdots\;\;0\;])$ of size $N \times N$ and O_N is $N \times N$ zero matrix. We consider structured state-feedback design with $C_2 = I$, and Q, R also being identity matrices. The state feedback consists of position and velocity feedback gains $K \;=\; [\,K_p,\; K_v\,]$. We consider two cases: (a) the completely decentralized case where both K_p and K_v are diagonal matrices; and (b) the nearest-neighbor interactions where both K_p and K_v are tridiagonal matrices.

We implement the computations in MATLAB R2008a on a personal computer with 3.2GHz CPU and 2.5GB RAM. We solve the primal problems using quasi-Newton method initialized by the truncated state feedback gains. The stopping criterion of quasi-Newton method for (SH2) is $\|\nabla J(K)\|_F \;<\; 10^{-5}$. In the subgradient method, the step-size constant is $\eta = 0.5$ and the stopping criterion is $\|\tilde{E}_\nu\|_F < 10^{-5}$. In Lagrangian minimization, the stopping criterion for the alternating method is $\|\mathcal{R}_k\|_F < 10^{-3}$. The computation results are reported in Table I, where time is in seconds and $J^\star$ is the optimal value. We define

$$\kappa_J \;:=\; \frac{J^\star - g^\star}{J^\star}$$

to measure the duality gaps, where $g^\star$ is the optimal dual value obtained by solving the dual problem.

VI. Concluding Remarks

We consider the optimal design of static structured feedback gains for large-scale interconnected systems. We de-

TABLE I
COMPUTATION RESULTS FOR A MASS-SPRING SYSTEM

	diagonal K_p and K_v		
N	Time(sec)	$J^\star$	κ_J
50	2.0	6.7226×10^1	1.6720×10^{-7}
100	11	1.3464×10^2	8.3315×10^{-8}
200	75	2.6947×10^2	4.3273×10^{-8}
	tridiagonal K_p and K_v		
N	Time(sec)	$J^\star$	κ_J
50	2.5	6.5631×10^1	8.2253×10^{-8}
100	15	1.3139×10^2	8.4465×10^{-8}
200	93	2.6291×10^2	8.5566×10^{-8}

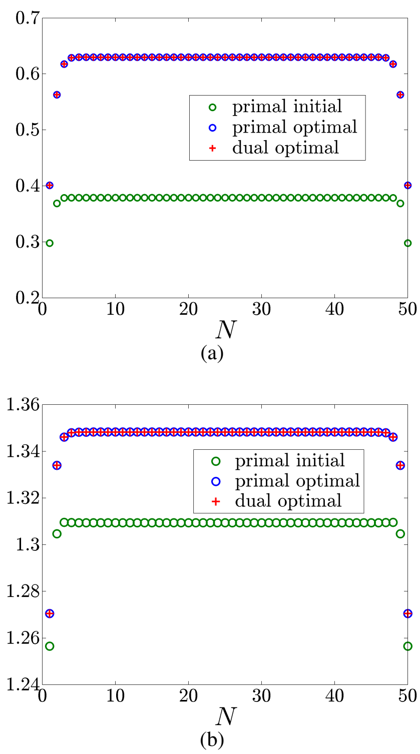

Fig. 2. Comparison of elements on the main diagonals of feedback gains for diagonal K_p and K_v: (a) position feedback gains; (b) velocity feedback gains.

velop efficient implementation of quasi-Newton method for sparse matrices. We formulate and solve the dual problem to obtain a lower bound to the optimal performance index. It is observed that in the mass-spring example, the upper bounds from solving the primal problems and the lower bounds from solving the dual problems correspond to each other, suggesting that the global optimal gains are attained. Similar observation was made in several other examples not reported here.

Our future research will be aimed at identifying the underlying structures that have zero duality gaps. It is also of interest to study (SH2) problem with additional inequality constraints to address practical implementation issues such as actuator saturations. We also intend to apply the developed methods to the vehicular formation problems, which have received considerable attention in recent years but a systematic procedure for the design of optimal localized controllers is yet to be developed.

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