An ADMM algorithm for matrix completion of partially known state covariances

Fu Lin, Mihailo R. Jovanović, and Tryphon T. Georgiou

***Abstract*— We study the inverse problem of reproducing partially known second-order statistics of a linear time invariant system by the least number of possible input disturbance channels. This can be formulated as a rank minimization problem, and for its solution, we employ a convex relaxation based on the nuclear norm. The resulting optimization problem can be cast as a semi-definite program and solved efficiently using general-purpose solvers for small- and medium-size problems. In this paper, we focus on issues and techniques that are pertinent to large-scale systems. We bring in a re-parameterization which transforms the problem into a form suitable for the alternating direction method of multipliers. Furthermore, we show that each iteration of this algorithm amounts to solving a system of linear equations, an eigenvalue decomposition, and a singular value thresholding. An illustrative example is provided to demonstrate the effectiveness of the developed approach.**

***Index Terms*— Alternating direction method of multipliers, convex optimization, low-rank approximation, nuclear norm regularization, singular value thresholding, state covariances, structured matrix completion problems.**

I. Introduction

We consider the problem of reproducing partially known second-order statistics using a linear time invariant (LTI) system with a minimum number of input channels. The motivation for this inverse problem stems from applications in which it is desired to reproduce state covariances generated in experiments or simulations using linear systems driven by stochastic disturbances. This completion of state covariances is closely related to well-known matrix completion problems [1]–[4] with a Lyapunov equality constraint imposed by the underlying linear dynamics.

The search for the least number of possible input disturbance channels gives rise to a rank minimization problem. We employ a convex relaxation based on the nuclear norm [5], [6]. The resulting optimization problem can be cast as a semi-definite program (SDP) and solved efficiently using general-purpose SDP solvers for problems of small and medium size. To deal with large problems, we rely on a re-parameterization that brings the optimization problem into a form suitable for the alternating direction method of multipliers (ADMM). Furthermore, we show that each ADMM iteration amounts to solving a system of linear equations, an eigenvalue decomposition, and a singular value thresholding.

The paper is organized as follows. We formulate the minimum rank covariance completion problem and make connections to well-known matrix completion problems in Section II. We relax the rank using the nuclear norm and provide a re-parameterization suitable for large problems in Section III. We then cast the constrained convex optimization problem into a form suitable for ADMM in Section IV. We give an illustrative example to demonstrate the effectiveness of the developed approach in Section V. We summarize our contributions and discuss future directions in Section VI.

II. Problem formulation

Consider a stochastically forced LTI system

$$\dot{x} \;=\; A\,x \;+\; B\,u \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, and $u(t) \in \mathbb{C}^m$ is a stationary zero-mean stochastic process. For a Hurwitz matrix A and a controllable pair (A, B), it was shown in [7] that the steady-state covariance

$$X \;:=\; \lim_{t\to\infty} \mathcal{E}(x(t)x^*(t)),$$

with $\mathcal{E}(\cdot)$ denoting the expectation operator, satisfies a constraint imposed by the underlying linear dynamics

$$AX \;+\; XA^* \;=\; -\,(BH^* + HB^*). \tag{2}$$

More specifically, a positive semi-definite matrix X is qualified to be the state covariance of the linear system (1) if and only if equation (2) is solvable in terms of $H \in \mathbb{C}^{n\times m}$.

Our objective is to identify input statistics from partially available state covariances. The motivation for this inverse problem stems from applications in which it is desired to reproduce second-order statistics (generated in experiments or simulations) using an LTI system driven by stochastic disturbances.

For a given set of N constraints on a state covariance matrix

$$\mathrm{trace}\,(T_i\,X) \;=\; g_i, \qquad i \;=\; 1,\ldots,N, \tag{3}$$

we are interested in finding B, H, and $X = X^* \succeq 0$ that satisfy the Lyapunov equation (2) and the constraint (3). Here, problem data is specified by (A, T_i, g_i) where matrices T_i relate X to sampled statistics g_i.

The structure of the matrix B determines the way disturbance inputs enter into the state equation; in particular, the

Supported in part by the University of Minnesota Initiative for Renewable Energy and the Environment under Early Career Award RC-0014-11, the NSF under Grant 1027696, and the AFOSR under Grant FA9550-12-1-0319.

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dimension of B dictates the number of input channels. It is thus of interest to seek a linear model with a small number of inputs that is capable of reproducing available data. In this case matrix B also becomes a design parameter. However, since the product of B and H in (2) results in a difficult non-convex equality constraint, we introduce new variable

$$Q := BH^* + HB^*.$$

Since the rank of Q bounds the rank of BH^*, and hence the number of input channels [8], we seek a matrix Q with the minimum rank. Therefore, we consider the following rank minimization problem

$$\begin{array}{ll} \underset{Q,\,X}{\text{minimize}} & \mathbf{rank}\,(Q) \\ \text{subject to} & AX + XA^* + Q = 0 \\ & \text{trace}\,(T_i\,X) = g_i, \quad i = 1,\ldots,N \\ & X \succeq 0. \end{array} \tag{4}$$

Note that the constraint set is convex because it is the intersection of the positive semi-definite cone and the linear subspace determined by the Lyapunov equation and trace constraints. On the other hand, the rank in the objective function is the source of non-convexity.

A. Connections to matrix completion problems

Matrix completion problems have received considerable attention in recent years due to emerging applications, exciting theoretical developments, and efficient optimization algorithms [1]–[4], [6], [9]–[13]. It is therefore instructive to provide connections and highlight differences between (4) and the positive semi-definite matrix completion problem.

Consider the problem of recovering a positive semi-definite matrix from a sample of its entries. This problem can be expressed as

$$\begin{array}{ll} \text{find} & X \succeq 0 \\ \text{subject to} & X_{jl} = M_{jl}, \quad (j,l) \in \Omega \end{array}$$

where Ω is the index set of known entries M_{jl}. The problem of finding a minimum rank matrix that is consistent with the observed data can be formulated as

$$\begin{array}{ll} \underset{X}{\text{minimize}} & \mathbf{rank}\,(X) \\ \text{subject to} & X_{jl} = M_{jl}, \quad (j,l) \in \Omega \\ & X \succeq 0. \end{array} \tag{5}$$

Note that the rank minimization problem (4) with

$$A = -(1/2)I, \qquad T_i = \mathrm{e}_l\,\mathrm{e}_j^T$$

where e_j is the jth element of the canonical basis of $\mathbb{R}^n$, simplifies to the standard matrix completion problem (5). In other words, the rank minimization problem (4) encompasses the following *structured covariance completion* problem

$$\begin{array}{ll} \underset{Q,\,X}{\text{minimize}} & \mathbf{rank}\,(Q) \\ \text{subject to} & AX + XA^* + Q = 0 \\ & X_{jl} = M_{jl}, \quad (j,l) \in \Omega \\ & X \succeq 0. \end{array} \tag{6}$$

It is worth mentioning that problem (4) also accommodates the *output covariance* completion problem. Suppose that instead of having access to some of the entries of the state covariance matrix, we have a sample of entries of the output covariance matrix

$$(CXC^*)_{jl} = M_{jl}, \quad (j,l) \in \Omega$$

with $C \in \mathbb{C}^{p\times n}$ denoting the output matrix. Since these linear constraints can be written as

$$\text{trace}\left(C^*\mathrm{e}_l\mathrm{e}_j^T CX\right) = M_{jl}, \quad (j,l) \in \Omega,$$

we conclude that the *structured output covariance completion* problem

$$\begin{array}{ll} \underset{Q,\,X}{\text{minimize}} & \mathbf{rank}\,(Q) \\ \text{subject to} & AX + XA^* + Q = 0 \\ & (CXC^*)_{jl} = M_{jl}, \quad (j,l) \in \Omega \\ & X \succeq 0 \end{array} \tag{7}$$

is also a special case of (4) with

$$T_i = C^*\mathrm{e}_l\mathrm{e}_j^T C, \qquad g_i = M_{jl}.$$

III. Nuclear norm relaxation

In this section, we relax the rank with the nuclear norm to obtain a convex optimization problem. For the relaxed convex problem, we provide an SDP formulation. Furthermore, we parameterize the constraint set and bring the relaxed formulation into a form that is well-suited to large problems.

The nuclear norm, i.e., the sum of the singular values of a matrix

$$\|Q\|_* := \sum_{i=1}^{n} \sigma_i(Q)$$

provides an effective surrogate for rank minimization problems [5], [6]. In contrast to the non-convex rank function, the nuclear norm is convex and it is the best convex approximation of rank function over the set of matrices with spectral norm no greater than one [6]. Thus, relaxation of rank to nuclear norm in (4) yields a convex optimization problem

$$\begin{array}{ll} \underset{Q,\,X}{\text{minimize}} & \|Q\|_* \\ \text{subject to} & AX + XA^* + Q = 0 \\ & \text{trace}\,(T_i\,X) = g_i, \quad i = 1,\ldots,N \\ & X \succeq 0. \end{array} \tag{MC}$$

A. SDP formulation

For a Hermitian matrix Q, the nuclear norm of Q can be written as

$$\|Q\|_* = \sum_{i=1}^{n} |\lambda_i(Q)|.$$

Therefore, by splitting Q into positive and negative semi-definite parts

$$Q = Q_+ - Q_-, \qquad Q_+ \succeq 0, \qquad Q_- \succeq 0$$

it can be shown [5] that the matrix completion problem (MC) is equivalent to the following semi-definite program

$$\begin{array}{ll} \underset{Q_+,\, Q_-,\, X}{\text{minimize}} & \text{trace}\,(Q_+) \,+\, \text{trace}\,(Q_-) \\ \text{subject to} & AX \,+\, XA^* \,+\, Q_+ \,-\, Q_- \,=\, 0 \\ & \text{trace}\,(T_i\, X) \,=\, g_i, \quad i \,=\, 1, \ldots, N \\ & X \,\succeq\, 0, \quad Q_+ \,\succeq\, 0, \quad Q_- \,\succeq\, 0. \end{array}$$

When the number of states is $n \lesssim 50$, this SDP can be solved using general-purpose SDP solvers. Customized interior point method for *unconstrained* nuclear norm minimization [9] can deal with matrices with a few hundreds of rows and columns. However, the positive semi-definite constraint and the Lyapunov equality constraint prevent us from employing this approach. We next provide an alternative problem formulation that is well-suited to large problems.

B. Null space parameterization

We begin by parameterizing the null space of the linear operator determined by trace constraints. This null space parameterization will allow us to eliminate trace constraints and the Lyapunov equation from (MC).

Since the set of trace constraints

$$\text{trace}\,(T_i\, X) \,=\, g_i, \qquad i \,=\, 1, \ldots, N$$

defines a linear mapping

$$\mathcal{T}(X) \,=\, g \tag{8}$$

from the Hermitian matrix X to the vector $g \,:=\, [\, g_1 \,\cdots\, g_N \,]^T$, it follows that all solutions to (8) can be parameterized using the null space of the linear operator $\mathcal{T}$. Specifically,

$$X \,=\, X_0 \,+\, \sum_{j\,=\,1}^{q} z_j X_j \tag{9}$$

where $X_0 \,=\, X_0^*$ is a particular solution of (8), z_j's are real coefficients, and $X_j \,=\, X_j^*$'s are linearly independent matrices that satisfy

$$\text{trace}\,(T_i\, X_j) \,=\, 0, \qquad i \,=\, 1, \ldots, N, \qquad j \,=\, 1, \ldots, q.$$

Here, q is the dimension of the null space of $\mathcal{T}$.

Similarly, we parameterize Q by substituting (9) into the Lyapunov equation in (MC)

$$Q \,=\, -\,(AX \,+\, XA^*) \,=\, Q_0 \,+\, \sum_{j\,=\,1}^{q} z_j Q_j$$

where

$$Q_0 \,:=\, -\,(AX_0 \,+\, X_0 A^*), \qquad Q_j \,:=\, -\,(AX_j \,+\, X_j A^*).$$

Therefore, the above presented null space parameterization can be used to recast the state covariance completion problem (MC) into

$$\begin{array}{ll} \underset{Q,\, X,\, z}{\text{minimize}} & \|Q\|_* \\ \text{subject to} & Q \,=\, Q_0 \,+\, \displaystyle\sum_{j\,=\,1}^{q} z_j Q_j \\ & X \,=\, X_0 \,+\, \displaystyle\sum_{j\,=\,1}^{q} z_j X_j \\ & X \,\succeq\, 0. \end{array} \tag{10}$$

Here, Q_j and X_j are problem data and $Q = Q^*$, $X = X^*$, and z are the optimization variables. Note that the vector z couples linear constraints in (10). This particular structure will be exploited in Section IV.

Remark 1: One approach to compute a solution X_0 of (8) is to solve the under-determined system of linear equations

$$[\,\text{vec}\,(T_1) \;\cdots\; \text{vec}\,(T_N)\,]^T\, \text{vec}\,(X) \,=\, g$$

where $\text{vec}\,(X)$ denotes the vectorization of the upper triangular part (including the main diagonal) of the Hermitian matrix X. The basis elements X_j's of the null space of $\mathcal{T}$ can be computed via the singular value decomposition of $\mathcal{T}$.

IV. Alternating direction method of multipliers

We next employ the alternating direction method of multipliers (ADMM) to solve (10). This method has been employed effectively in low-rank matrix recovery [11], sparse covariance selection [14], image denoising and magnetic resonance imaging [15], sparse feedback synthesis [16], and many other applications [17]. When applied to (10), we will demonstrate that each ADMM iteration requires an eigenvalue decomposition of X, a singular value thresholding of Q, and the solution to a system of linear equations for z.

Note that problem (10) is not in a form suitable for ADMM since it has three independent variables $\{Q,\, X,\, z\}$ and two coupling linear constraints. To put (10) into the so-called two-block form suitable for ADMM, let us introduce

$$f(Q, X) \,:=\, \|Q\|_* \,+\, \phi(X), \qquad h(z) \,\equiv\, 0$$

where $\phi(X)$ is the indicator function

$$\phi(X) \,=\, \begin{cases} 0, & X \,\succeq\, 0 \\ \infty, & \text{otherwise.} \end{cases}$$

Since f is a convex separable function of Q and X and since h is a trivially convex function of z, the following equivalent formulation to (10) is in the standard form suitable for ADMM [17]

$$\begin{array}{ll} \underset{X,\, Q,\, z}{\text{minimize}} & f(Q, X) \,+\, h(z) \\ \text{subject to} & \begin{bmatrix} \mathcal{Q}(z) \\ \mathcal{X}(z) \end{bmatrix} \,-\, \begin{bmatrix} Q \\ X \end{bmatrix} \,=\, 0 \end{array} \tag{11}$$

where

$$\mathcal{Q}(z) \,:=\, Q_0 \,+\, \sum_{j\,=\,1}^{q} z_j Q_j, \qquad \mathcal{X}(z) \,:=\, X_0 \,+\, \sum_{j\,=\,1}^{q} z_j X_j.$$

We next form the augmented Lagrangian associated with

the constrained problem (11)

$$\mathcal{L}_\rho(Q,X,z,Y,Z) \;=\; \|Q\|_* \;+\; \phi(X) \;+\; \langle Y, \mathcal{Q}(z) \;-\; Q\rangle + \langle Z, \mathcal{X}(z) \;-\; X\rangle \;+\; \frac{\rho}{2}\|\mathcal{Q}(z) \;-\; Q\|_F^2 \;+\; \frac{\rho}{2}\|\mathcal{X}(z) \;-\; X\|_F^2$$

where $Y = Y^*$ and $Z = Z^*$ are Lagrange multipliers, ρ is a positive scalar, $\langle\cdot,\cdot\rangle$ is the inner product of two matrices, and $\|\cdot\|_F$ is the Frobenius norm. The ADMM algorithm uses a sequence of iterations

$$(Q,X)^{k+1} \;:=\; \underset{Q,\,X}{\arg\min}\; \mathcal{L}_\rho(Q,X,z^k,Y^k,Z^k) \tag{12a}$$

$$z^{k+1} \;:=\; \underset{z}{\arg\min}\; \mathcal{L}_\rho(Q^{k+1},X^{k+1},z,Y^k,Z^k) \tag{12b}$$

$$\begin{bmatrix} Y^{k+1} \\ Z^{k+1} \end{bmatrix} \;:=\; \begin{bmatrix} Y^{k} \\ Z^{k} \end{bmatrix} \;+\; \rho \begin{bmatrix} \mathcal{Q}(z^{k+1}) \;-\; Q^{k+1} \\ \mathcal{X}(z^{k+1}) \;-\; X^{k+1} \end{bmatrix} \tag{12c}$$

to find the solution of (11). These iterations terminate when the primal and dual residuals are sufficiently small

$$\text{primal residues:} \quad \begin{cases} \|\mathcal{Q}(z^{k+1}) - Q^{k+1}\|_F \;\le\; \epsilon_1 \\ \|\mathcal{X}(z^{k+1}) - X^{k+1}\|_F \;\le\; \epsilon_2 \end{cases}$$

$$\text{dual residues:} \quad \begin{cases} \|Q^{k+1} - Q^{k}\|_F \;\le\; \epsilon_3 \\ \|X^{k+1} - X^{k}\|_F \;\le\; \epsilon_4 \\ \|z^{k+1} - z^{k}\|_2 \;\le\; \epsilon_5 \end{cases}$$

where ϵ_i's are determined by problem data; see [17, Section 3.3] for details.

A. Solution to the (Q,X)-minimization problem

For fixed $\{z^k,\; Y^k,\; Z^k\}$, the minimization of the augmented Lagrangian $\mathcal{L}_\rho$ with respect to (Q,X) amounts to

$$\underset{Q,\,X}{\text{minimize}} \quad \|Q\|_* \;+\; \phi(X) \;+\; \frac{\rho}{2}\,\|Q \;-\; (\mathcal{Q}(z^k) + \frac{1}{\rho}\,Y^k)\|_F^2 \;+\; \frac{\rho}{2}\,\|X \;-\; (\mathcal{X}(z^k) + \frac{1}{\rho}\,Z^k)\|_F^2.$$

Separability of the objective function with respect to Q and X results in the following two *uncoupled* problems

$$\underset{Q}{\text{minimize}} \;\; \|Q\|_* \;+\; (\rho/2)\|Q \;-\; (\mathcal{Q}(z^k) + (1/\rho)Y^k)\|_F^2 \tag{13}$$

and

$$\underset{X}{\text{minimize}} \;\; \phi(X) \;+\; (\rho/2)\|X \;-\; (\mathcal{X}(z^k) + (1/\rho)Z^k)\|_F^2. \tag{14}$$

The solution to the Q-minimization problem (13) is given by the *singular value thresholding* operator [18]. Specifically, we compute the singular value decomposition of the symmetric matrix

$$\mathcal{Q}(z^k) + (1/\rho)Y^k \;=\; U\,\Sigma\,U^*$$

and apply the soft-thresholding operator to the resulting singular values

$$\mathcal{S}_{1/\rho}(\Sigma) \;=\; \text{diag}\{(\sigma_i - 1/\rho)_+\}$$

where $a_+ \;:=\; \max\{a, 0\}$. Then, the solution to the Q-minimization problem (13) is determined by [18]

$$Q^{k+1} \;=\; U\,\mathcal{S}_{1/\rho}(\Sigma)\,U^*.$$

On the other hand, the solution of the X-minimization problem (14) is determined by the projection of $\mathcal{X}(z^k) + (1/\rho)Z^k$ onto the positive semi-definite cone. Specifically, we compute the eigenvalue decomposition of the matrix

$$\mathcal{X}(z^k) \;+\; (1/\rho)Z^k \;=\; V\,\Lambda\,V^*$$

and truncate the negative eigenvalues

$$\Lambda_+ \;=\; \text{diag}\{(\lambda_i)_+\}$$

to obtain the solution

$$X^{k+1} \;=\; V\,\Lambda_+\,V^*.$$

B. Solution to the z-minimization problem

Problem (12b) is a convex quadratic program

$$\underset{z}{\text{minimize}} \quad \frac{\rho}{2}\,\|\mathcal{Q}(z) \;-\; Q^{k+1}\|_F^2 \;+\; \frac{\rho}{2}\,\|\mathcal{X}(z) \;-\; X^{k+1}\|_F^2 + \langle Y^k, \mathcal{Q}(z) \;-\; Q^{k+1}\rangle \;+\; \langle Z^k, \mathcal{X}(z) \;-\; X^{k+1}\rangle$$

that can be rewritten compactly as

$$\underset{z}{\text{minimize}} \quad (\rho/2)\,z^T H z \;+\; (b^k)^T z.$$

Here, the Hessian H and the gradient b^k are given by

$$\begin{aligned} H_{ij} \;&=\; \text{trace}\,(Q_i Q_j \;+\; X_i X_j) \\ b_i^k \;&=\; \text{trace}\,(Y^k Q_i \;+\; Z^k X_i \;+\; \rho\, Q_i(Q_0 - Q^{k+1}) + \\ &\qquad \rho X_i(X_0 - X^{k+1})). \end{aligned}$$

Therefore, this z-minimization problem (12b) amounts to solving the system of linear equations

$$\rho\, H z \;+\; b^k \;=\; 0$$

whose solution can be computed efficiently using Cholesky factorization and back-solve operations. Note that the Hessian matrix H is independent of ADMM iterations. This implies that we only need to compute the Cholesky factorization of H *once*, and then save the Cholesky factor for the back-solve operations in subsequent z-minimization steps.

C. Computational complexity

Since the Q-minimization problem amounts to a singular value decomposition, it costs $O(n^3)$ operations, where n is the number of states. Similarly, since the X-minimization problem amounts to an eigenvalue decomposition, this step also requires $O(n^3)$ operations. On the other hand, the Cholesky factorization of H takes $O(q^3)$ operations and the subsequent z-minimization step takes $O(q^2)$ operations for the back-solve operations, where

$$q \;:=\; n(n+1)/2 \;-\; N$$

is the dimension of the null space. Therefore, the total computational cost is determined by

$$O(\max\{q^3, kn^3, kq^2\})$$

where k is the number of ADMM iterations.

Without exploiting sparsity in basis representation, the memory requirements are $O(q^2)$ for the Cholesky factor of H and $O(qn^2)$ for the basis of the null space. Note that for the structured covariance completion problem (6), the basis elements are sparse matrices that contain only single nonzero entry.

V. An example

Consider a mass-spring-damper system with 50 masses subject to disturbances that are outputs of a low-pass filter driven by white noise

$$\begin{array}{rrcl} \text{low-pass filter:} & \dot{x}_f & = & -x_f + d \\ & u & = & x_f \\ \text{mass-spring-damper system:} & \dot{x} & = & A\,x + B\,u \end{array}$$

where d is a white stochastic disturbance with zero-mean and unit variance. The state and the input matrices of the mass-spring-damper system are

$$A = \begin{bmatrix} O & I \\ -K & -2K \end{bmatrix}, \quad B = \begin{bmatrix} O \\ I \end{bmatrix}$$

where O and I are zero and identity matrices and $K \in \mathbb{R}^{50\times 50}$ is a symmetric tridiagonal Toeplitz matrix with 2 on the main diagonal and -1 on the first upper- and lower-subdiagonal.

The steady-state covariance Σ of the cascade connection is determined by the solution of the Lyapunov equation

$$\mathcal{A}\Sigma + \Sigma\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = 0 \tag{15}$$

where

$$\mathcal{A} = \begin{bmatrix} A & B \\ O & -I \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B \\ I \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xx_f} \\ \Sigma_{x_f x} & \Sigma_{x_f x_f} \end{bmatrix}.$$

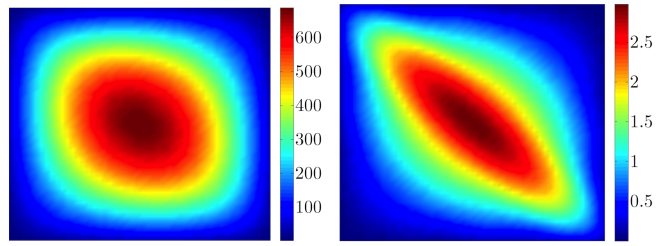
We partition the state covariance Σ_{xx} of the mass-spring-damper system into 2×2 block matrices compatible with the position and velocity of masses

$$\Sigma_{xx} = \begin{bmatrix} \Sigma_{pp} & \Sigma_{pv} \\ \Sigma_{vp} & \Sigma_{vv} \end{bmatrix}.$$

The diagonals of Σ_{pp}, Σ_{vv}, and Σ_{pv} are assumed to be the available data and hence

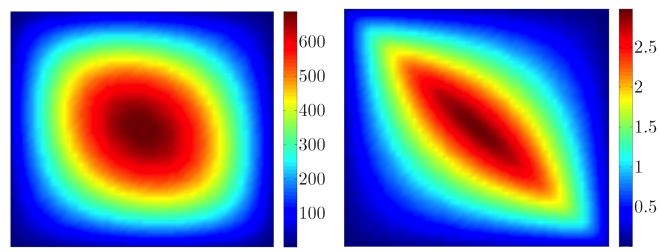
$$g = \begin{bmatrix} \operatorname{diag}(\Sigma_{pp}) \\ \operatorname{diag}(\Sigma_{vv}) \\ \operatorname{diag}(\Sigma_{pv}) \end{bmatrix}.$$

We are interested in completing the state covariance X that are consistent with the diagonals of Σ_{pp}, Σ_{vv}, and Σ_{pv}. To this end, we solve the covariance completion problem (MC).



(a) X_{pp} (b) X_{vv}

Fig. 1: The matrices X_{pp} and X_{vv} resulting from the covariance completion problem (MC) for the mass-spring-damper system. Sampled statistics in (MC) are determined by the diagonals of the matrices Σ_{pp}, Σ_{vv}, and Σ_{pv}.



(a) Σ_{pp} (b) Σ_{vv}

Fig. 2: Position and velocity covariances, Σ_{pp} and Σ_{vv}, resulting from the solution Σ to the Lyapunov equation (15) for the mass-spring-damper system.

Figure 1 shows the completed position and velocity covariances X_{pp} and X_{vv}. We observe their close correspondence to the true covariance matrices Σ_{pp} and Σ_{vv} shown in Fig. 2. In this example, it turns out that both Σ_{pv} and X_{pv} are zero matrices.

We next examine the singular values of Q that results from solving covariance completion problem (MC). As shown in Fig. 3, there are seven relatively large singular values and the rest of singular values are much smaller. On the other hand, Fig. 4 demonstrates that there is no clear-cut in the singular values of $\bar{Q}$ with

$$\bar{Q} = -(A\Sigma_{xx} + \Sigma_{xx}A^*).$$

The approximately low-rank feature of the solution Q illustrates the utility of the covariance matrix completion problem (MC).

VI. Concluding remarks

We study the inverse problem of reproducing partially known second-order statistics using an LTI system with a minimum number of input channels. We formulate a rank minimization problem that encompasses the positive semi-definite matrix completion problem. We employ the nuclear norm relaxation and show that the resulting optimization problem can be formulated as an SDP. Furthermore, we provide a null space parameterization that allows us to cast

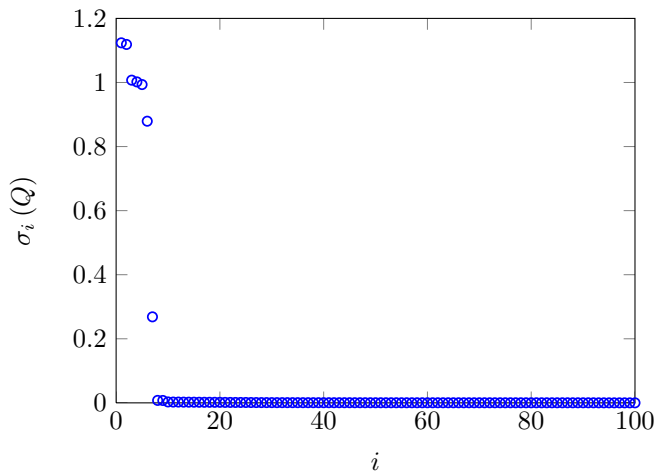


Fig. 3: Singular values of the approximately low-rank solution Q to the covariance matrix completion problem (MC) for the mass-spring-damper system.

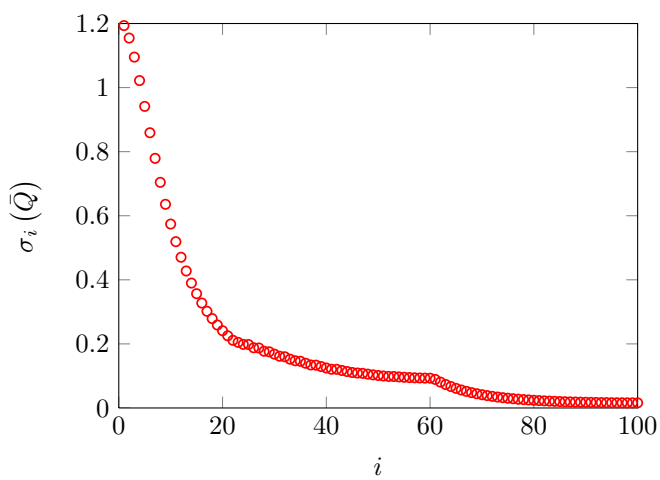


Fig. 4: Singular values of $\bar{Q} = -(A\Sigma_{xx} + \Sigma_{xx}A^*)$ for the mass-spring-damper system.

the problem into a form suitable for the ADMM algorithm. We show that each ADMM iteration amounts to computing an eigenvalue and a singular value decomposition and solving a system of linear equations. An illustrative example is provided to demonstrate the utility of the developed approach.

A. Future directions

Suppose that Q is decomposed into $Q = S + S^*$. It is shown in [8] that the number of positive and negative eigenvalues of Q is bounded by the rank of S. In particular, the maximum of the number of positive and negative eigenvalues of Q coincides with the rank of S; see [8]. In view of this, it is also of interest to consider the following rank minimization problem

$$\begin{array}{ll} \underset{Q_+, Q_-, X}{\text{minimize}} & \max\{\mathbf{rank}(Q_+), \mathbf{rank}(Q_-)\} \\ \text{subject to} & AX + XA^* + Q_+ - Q_- = 0 \\ & \text{trace}(T_i X) = g_i, \quad i = 1, \ldots, N \\ & X \succeq 0, \quad Q_+ \succeq 0, \quad Q_- \succeq 0. \end{array}$$

We intend to develop efficient algorithms for the convex relaxation of this problem in future work.

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