Sparsity-promoting optimal control of distributed systems

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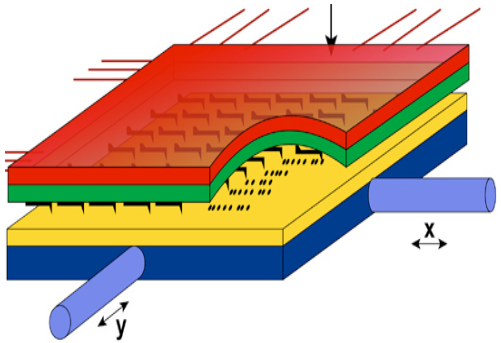
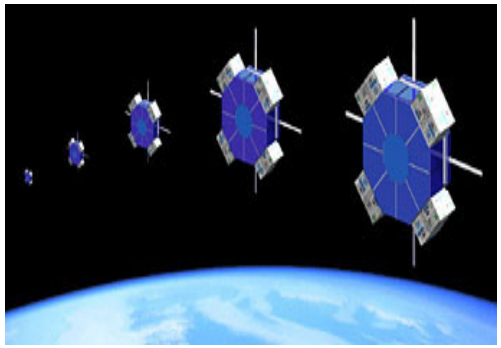
Distributed systems

- OF INCREASING IMPORTANCE IN MODERN TECHNOLOGY

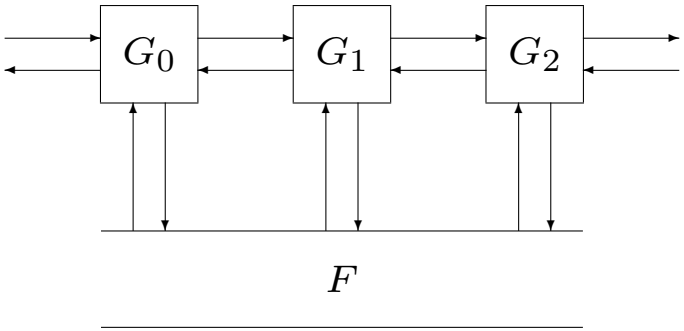
APPLICATIONS:

wind farms	micro-cantilevers	UAV formations satellite constellations
		

- CHALLENGE: DESIGN CONTROLLER ARCHITECTURES

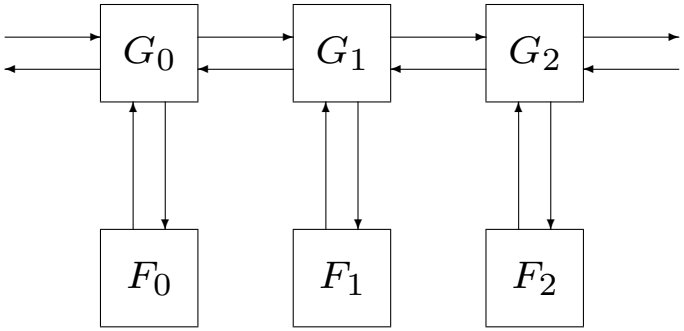
Controller architectures

CENTRALIZED



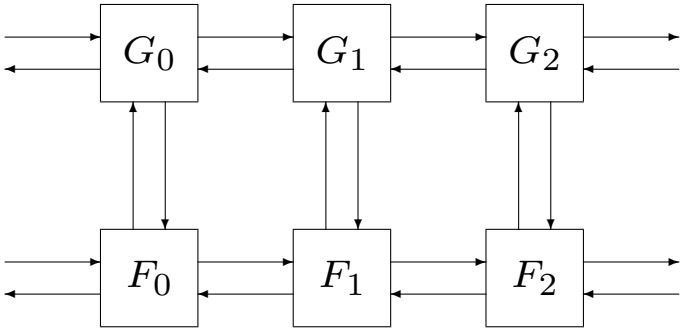
best performance
excessive communication

FULLY DECENTRALIZED



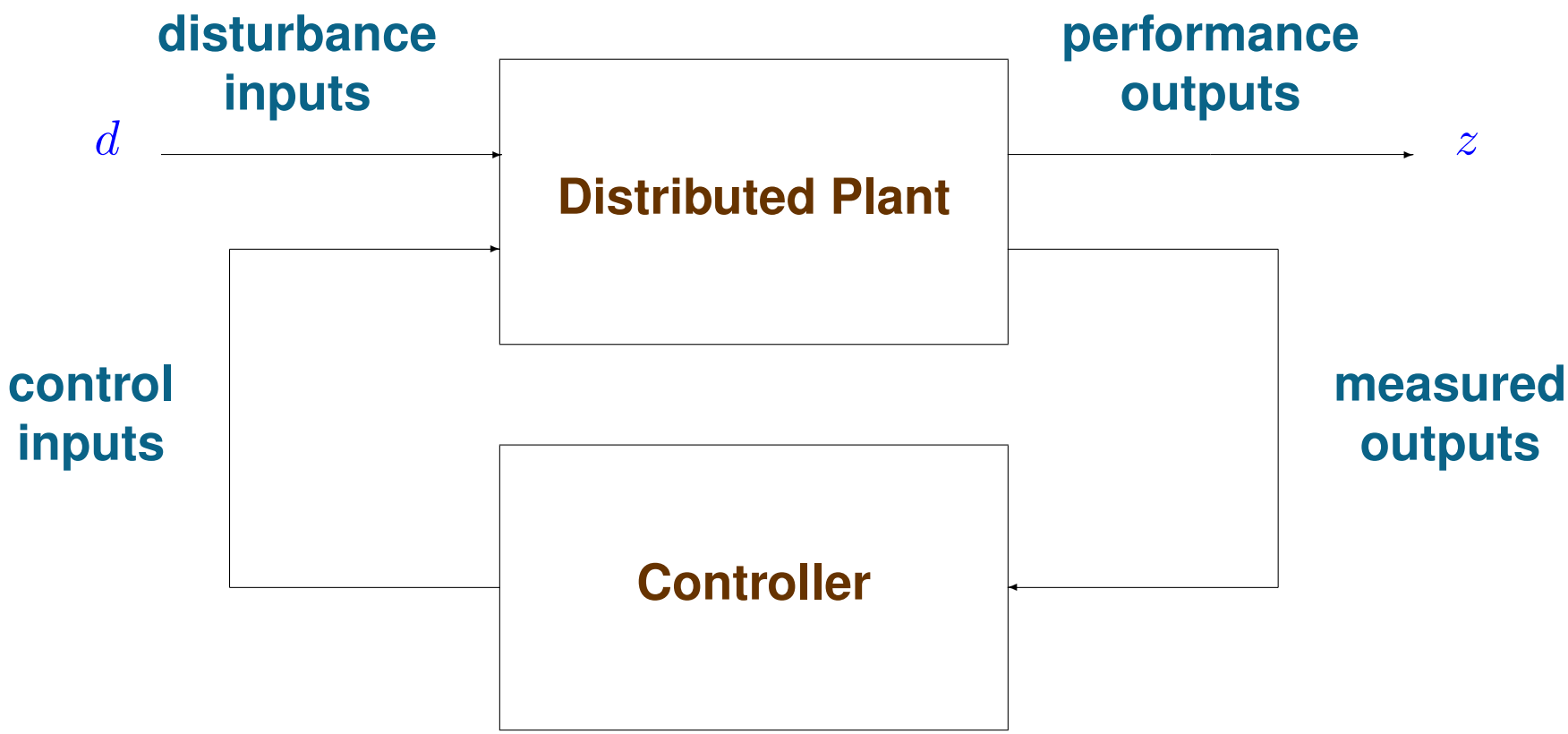
worst performance
no communication

LOCALIZED



many possible architectures

Standard control problem



OBJECTIVE:

strike a balance between $\|d \to z\|$ and number of controller communication links

Sparsity-promoting optimal control

$$
\begin{aligned}
\dot{x} &= A\,x + B_1\,d + B_2\,u \\
z &= \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\
u &= -F\,x \qquad\qquad F - \textbf{state feedback matrix}
\end{aligned}
$$

minimize $\quad J(F) \quad + \quad \gamma \;\; \mathrm{card}(F)$

$\downarrow \qquad\qquad \downarrow$

variance amplification $d \to z$

number of nonzero elements of F

$\gamma > 0$ — quadratic performance vs. sparsity tradeoff

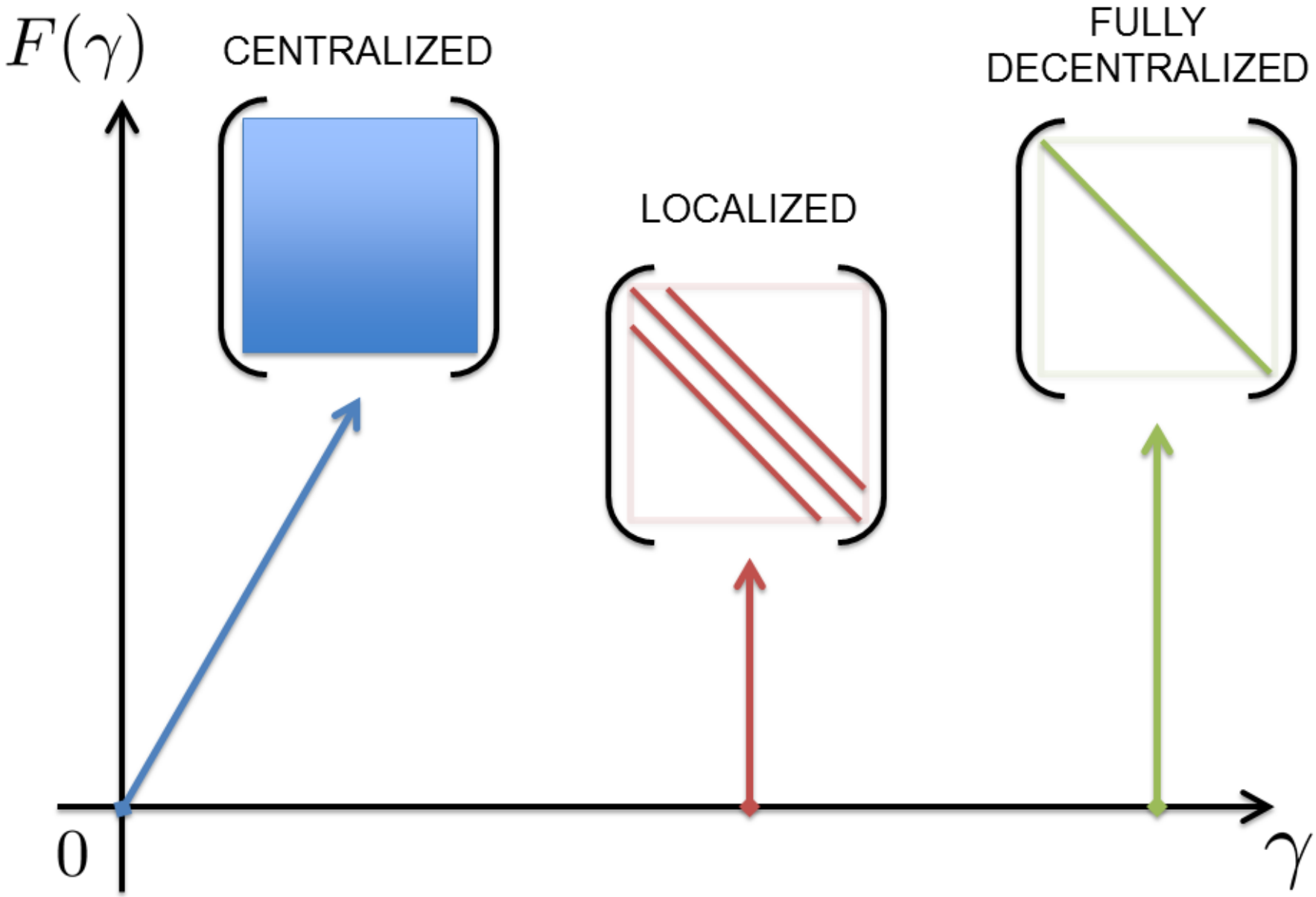
$$\text{minimize} \quad \underbrace{\mathrm{trace}\left(\int_0^\infty \mathrm{e}^{(A-B_2F)^T t}\left(Q + F^T R F\right)\mathrm{e}^{(A-B_2F)t}\,\mathrm{d}t\, B_1 B_1^T\right)}_{J(F)} + \gamma\,\mathbf{card}\,(F)$$

- DIFFICULTIES
  - ⋆ J — non-convex & smooth
  - ⋆ $\mathbf{card}$ — non-convex & non-differentiable

- APPROACHES
  - ⋆ Identify convex problems/relaxations
  - ⋆ Utilize powerful distributed optimization algorithm — ADMM

Parameterized family of feedback gains

$$F(\gamma) \;:=\; \arg\min_{F} \left(J(F) \;+\; \gamma\, \mathbf{card}\,(F) \right)$$



Convex relaxations of $\mathbf{card}\,(F)$

ℓ_1 **norm:** $\displaystyle\sum_{i,\,j} |F_{ij}|$

weighted ℓ_1 **norm:** $\displaystyle\sum_{i,\,j} W_{ij}\,|F_{ij}|, \quad W_{ij} \;>\; 0$

- CARDINALITY VS. WEIGHTED ℓ_1 NORM

$$\{W_{ij} \;=\; 1/|F_{ij}|,\;\; F_{ij} \;\neq\; 0\} \quad \Rightarrow \quad \mathbf{card}\,(F) \;=\; \sum_{i,\,j} W_{ij}\,|F_{ij}|$$

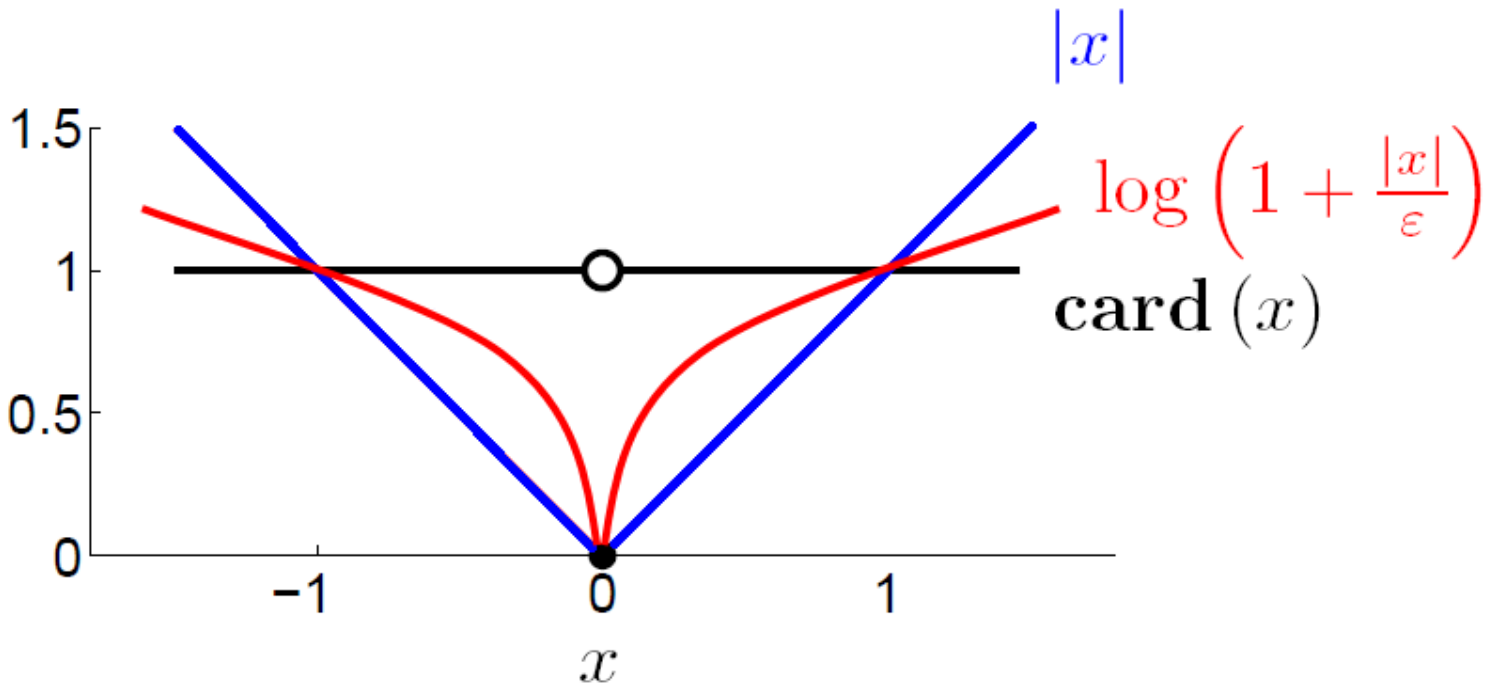
- RE-WEIGHTED SCHEME

$$W_{ij}^{+} \;=\; \frac{1}{|F_{ij}| \;+\; \varepsilon}$$

Candès, Wakin, Boyd, J. Fourier Anal. Appl. '08

A non-convex relaxation of $\mathbf{card}\,(F)$

sum-of-logs function: $\displaystyle\sum_{i,\,j} \log\left(1 + \frac{|F_{ij}|}{\varepsilon}\right), \qquad 0 < \varepsilon \ll 1$



Candès, Wakin, Boyd, J. Fourier Anal. Appl. '08

Alternating direction method of multipliers

$$\text{minimize} \quad J(F) \;+\; \gamma\, g(F)$$

- **Step 1: introduce additional variable/constraint**

$$\boxed{\begin{array}{ll} \text{minimize} & J(F) \;+\; \gamma\, g(G) \\ \text{subject to} & F \;-\; G \;=\; 0 \end{array}}$$

benefit: decouples J and g

- **Step 2: introduce augmented Lagrangian**

$$\mathcal{L}_\rho(F, G, \Lambda) \;=\; J(F) \;+\; \gamma\, g(G) \;+\; \mathrm{trace}\left(\Lambda^T (F \;-\; G)\right) \;+\; \frac{\rho}{2} \|F \;-\; G\|_F^2$$

- **Step 3: use ADMM for augmented Lagrangian minimization**

$$\mathcal{L}_\rho(F, G, \Lambda) \;=\; J(F) \;+\; \gamma\, g(G) \;+\; \mathrm{trace}\left(\Lambda^T (F - G)\right) \;+\; \frac{\rho}{2}\, \|F - G\|_F^2$$

ADMM:

F^{k+1}	$:=$	$\arg\min_F \mathcal{L}_\rho(F, G^k, \Lambda^k)$	$\cdots\cdots$	F-minimization
G^{k+1}	$:=$	$\arg\min_G \mathcal{L}_\rho(F^{k+1}, G, \Lambda^k)$	$\cdots\cdots$	G-minimization
Λ^{k+1}	$:=$	$\Lambda^k + \rho(F^{k+1} - G^{k+1})$	$\cdots\cdots$	Λ-update

- ⋆ F-minimization: efficient descent scheme
- ⋆ G-minimization: elementwise analytical solutions

- **Step 4: Polishing – structured optimal design**

$$\begin{array}{ll} \text{minimize} & J(F) \\ \text{subject to} & F \in \mathcal{S} \end{array}$$

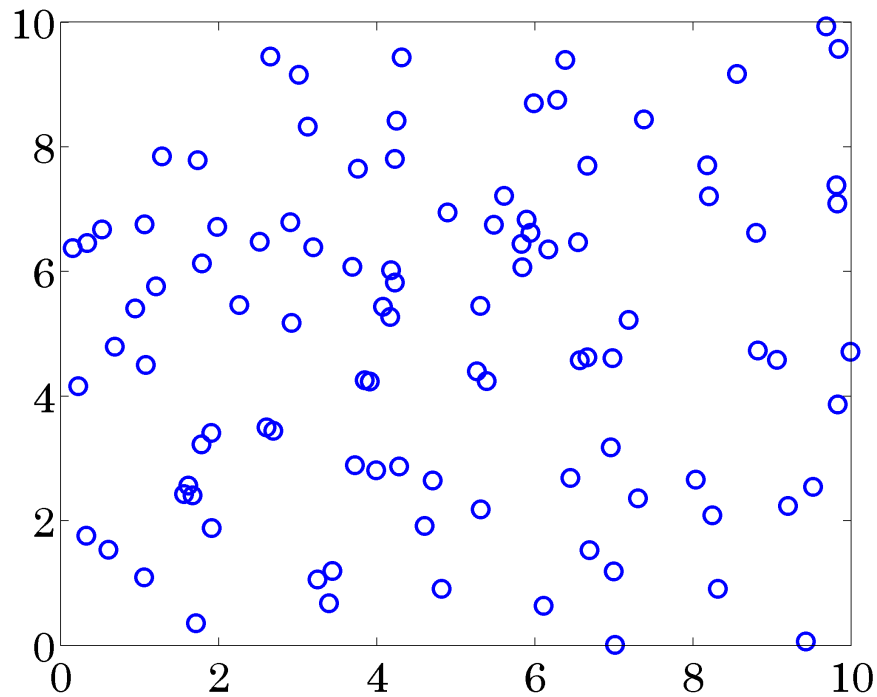
⋆ ADMM $\begin{cases} \text{identifies sparsity patterns } \mathcal{S} \\ \text{provides good initial condition for structured design} \end{cases}$

SOFTWARE

⋆ www.ece.umn.edu/∼mihailo/software/lqrsp/

$>> \texttt{solpath} = \texttt{lqrsp}(\texttt{A}, \texttt{B1}, \texttt{B2}, \texttt{Q}, \texttt{R}, \texttt{options});$

Distributed system with $N = 100$ subsystems



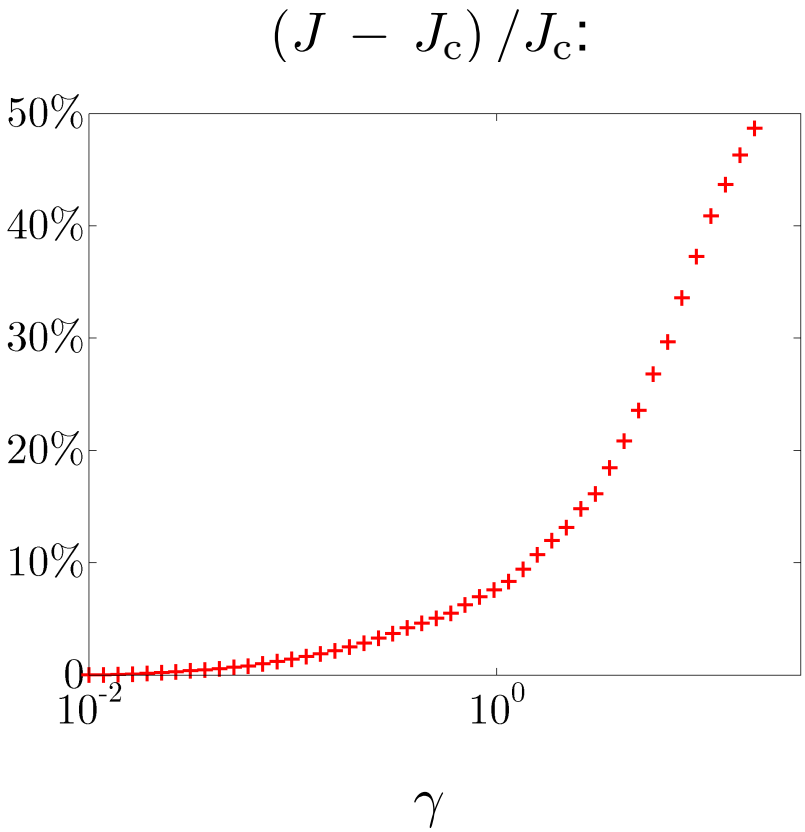
$$\dot{x}_i = \sum_{j=1}^{N} [A]_{ij}\, x_j + [B_1]_{ii}\, d_i + [B_2]_{ii}\, u_i$$

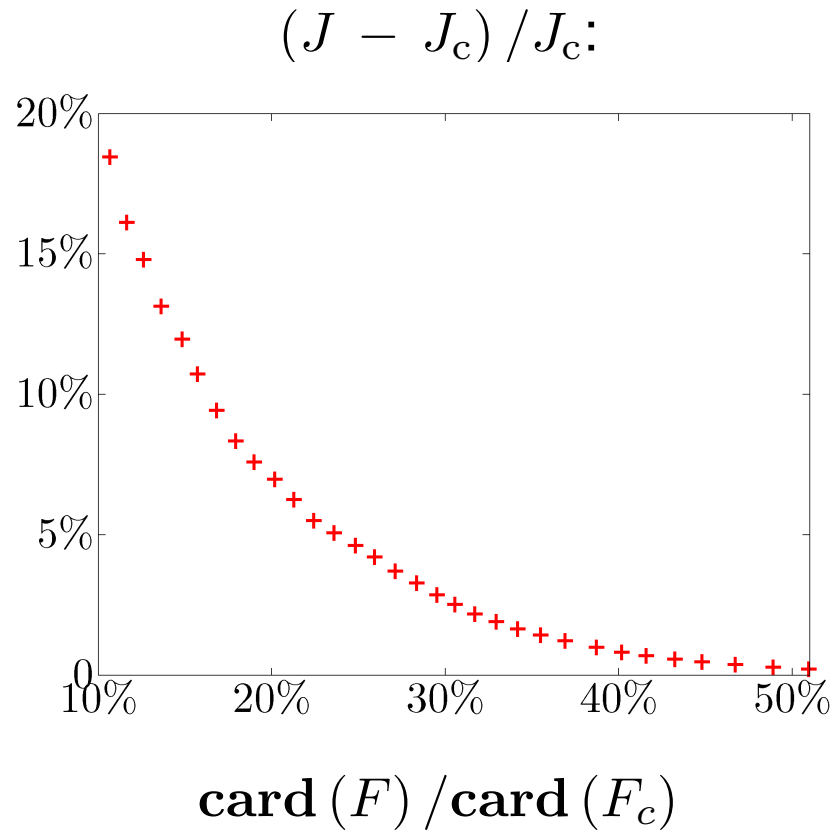
$$[A]_{ii} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad [A]_{ij} = \frac{1}{\mathrm{e}^{\alpha(i,j)}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [B_1]_{ii} = [B_2]_{ii} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\alpha(i,j)$: Euclidean distance between subsystems i and j

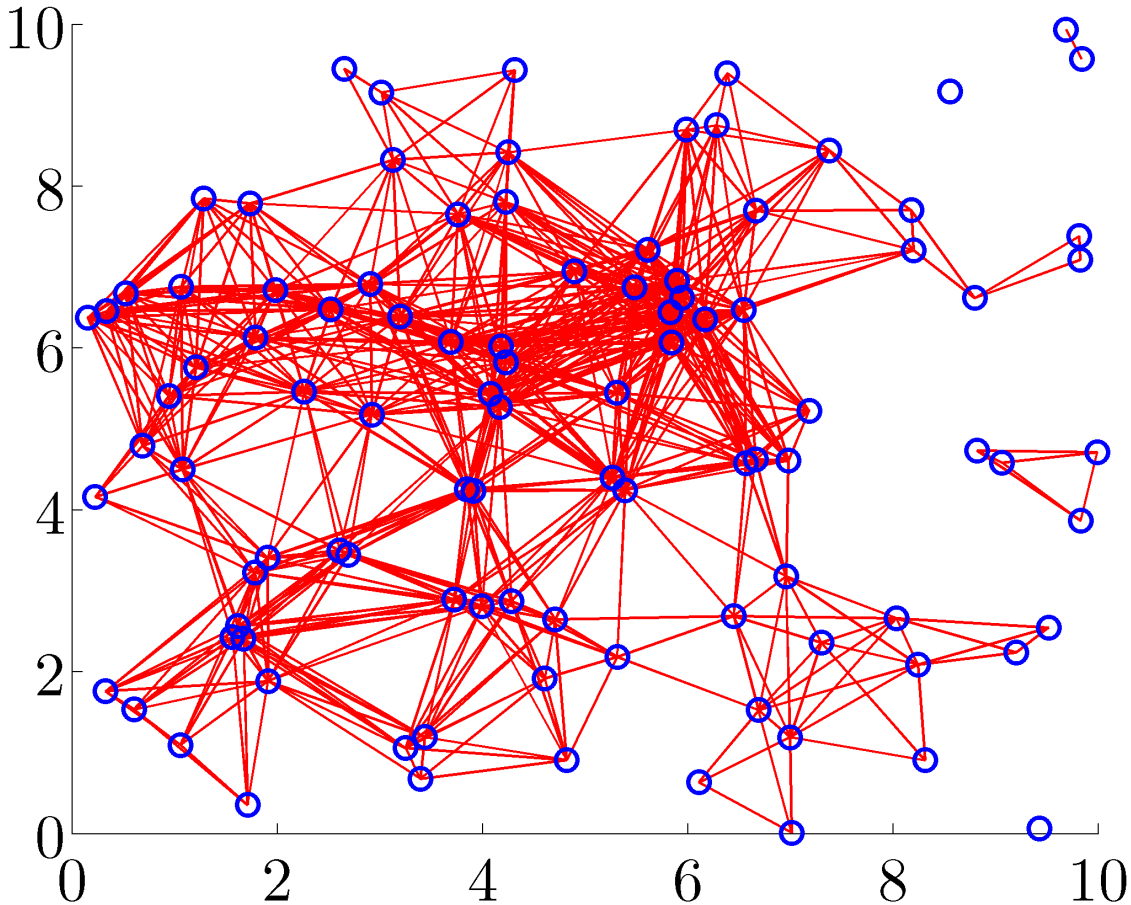
Motee & Jadbabaie, IEEE TAC '08

- **Performance comparison: sparse vs. centralized**





communication graph identified via ADMM:

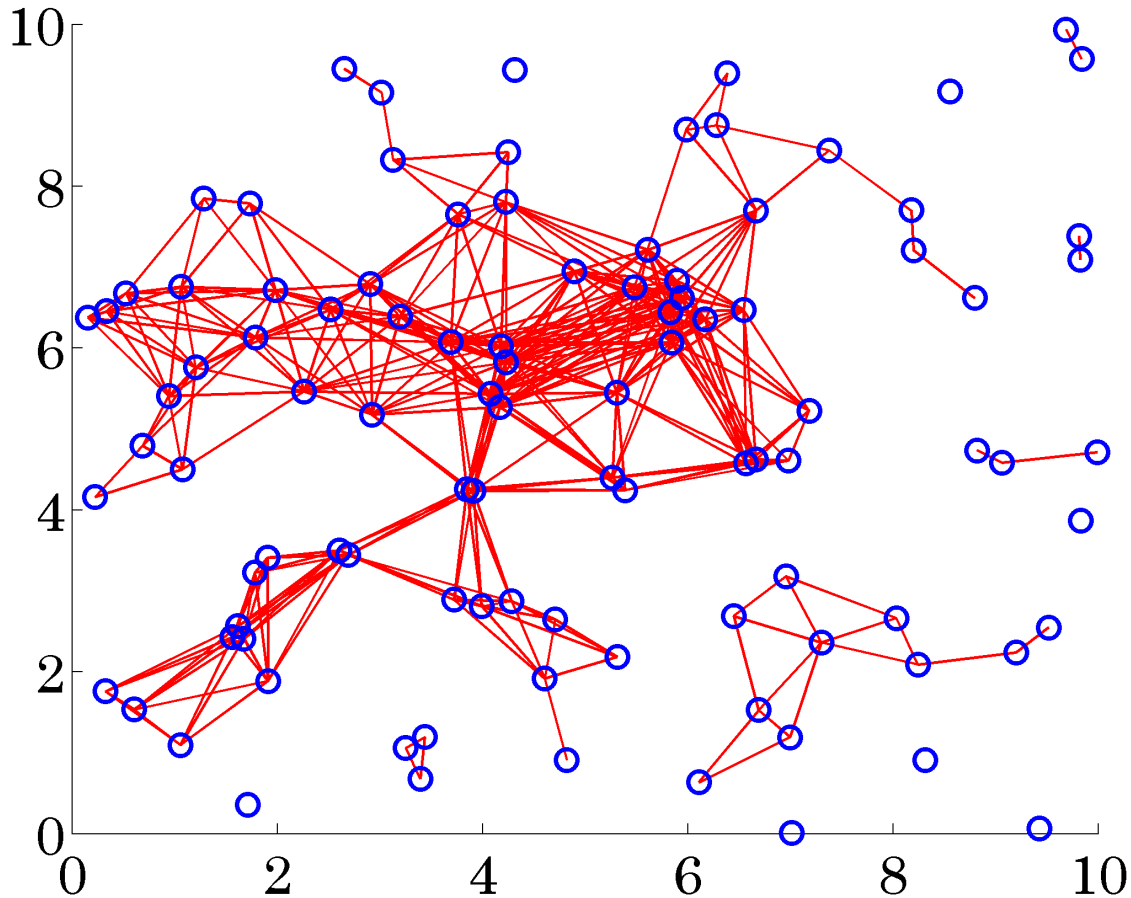


$$\gamma = 5$$

$$\mathbf{card}\,(F)\,/\mathbf{card}\,(F_{\mathrm{c}}) = 8.8\%$$

$$(J - J_{\mathrm{c}})\,/J_{\mathrm{c}} = 24.6\%$$

communication graph identified via ADMM:

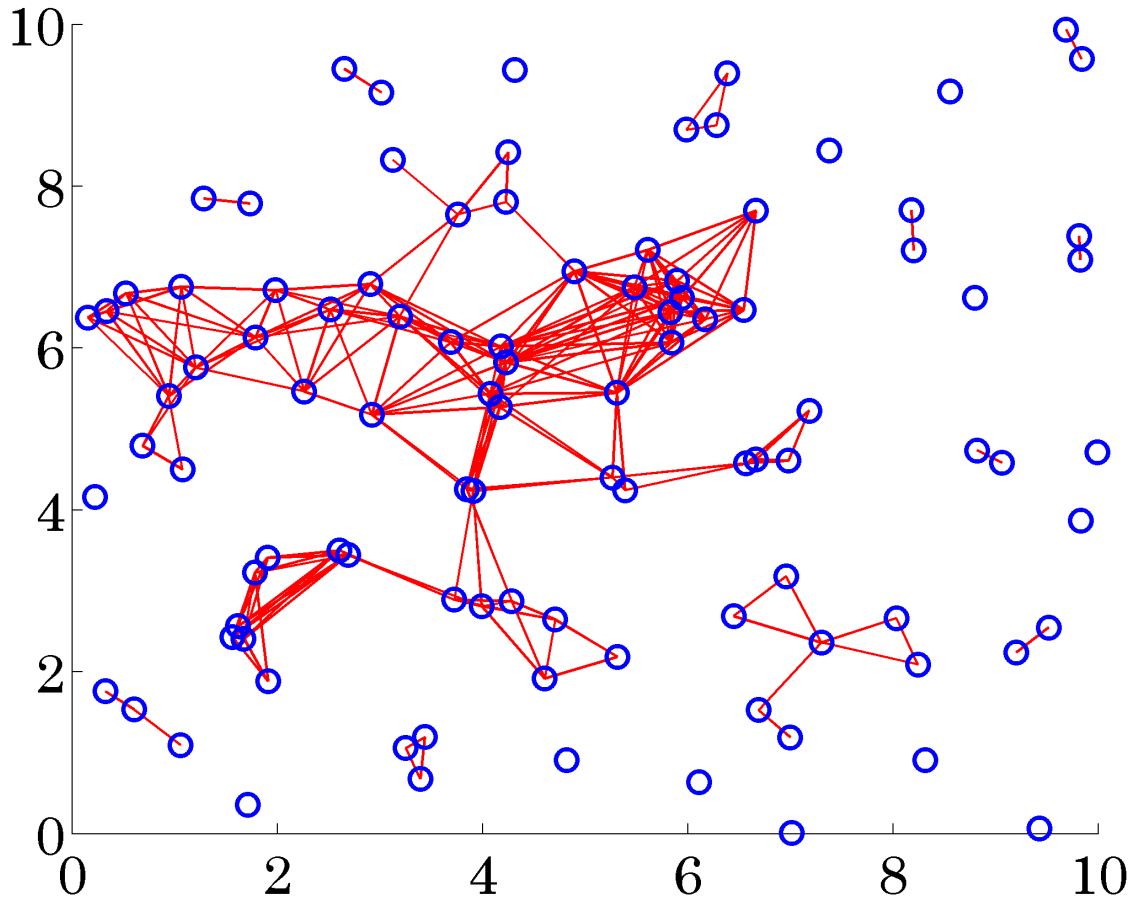


$$\gamma = 11$$

$$\mathbf{card}\,(F)\,/\mathbf{card}\,(F_{\mathrm{c}}) = 5.1\%$$

$$(J - J_{\mathrm{c}})\,/J_{\mathrm{c}} = 40.9\%$$

communication graph identified via ADMM:

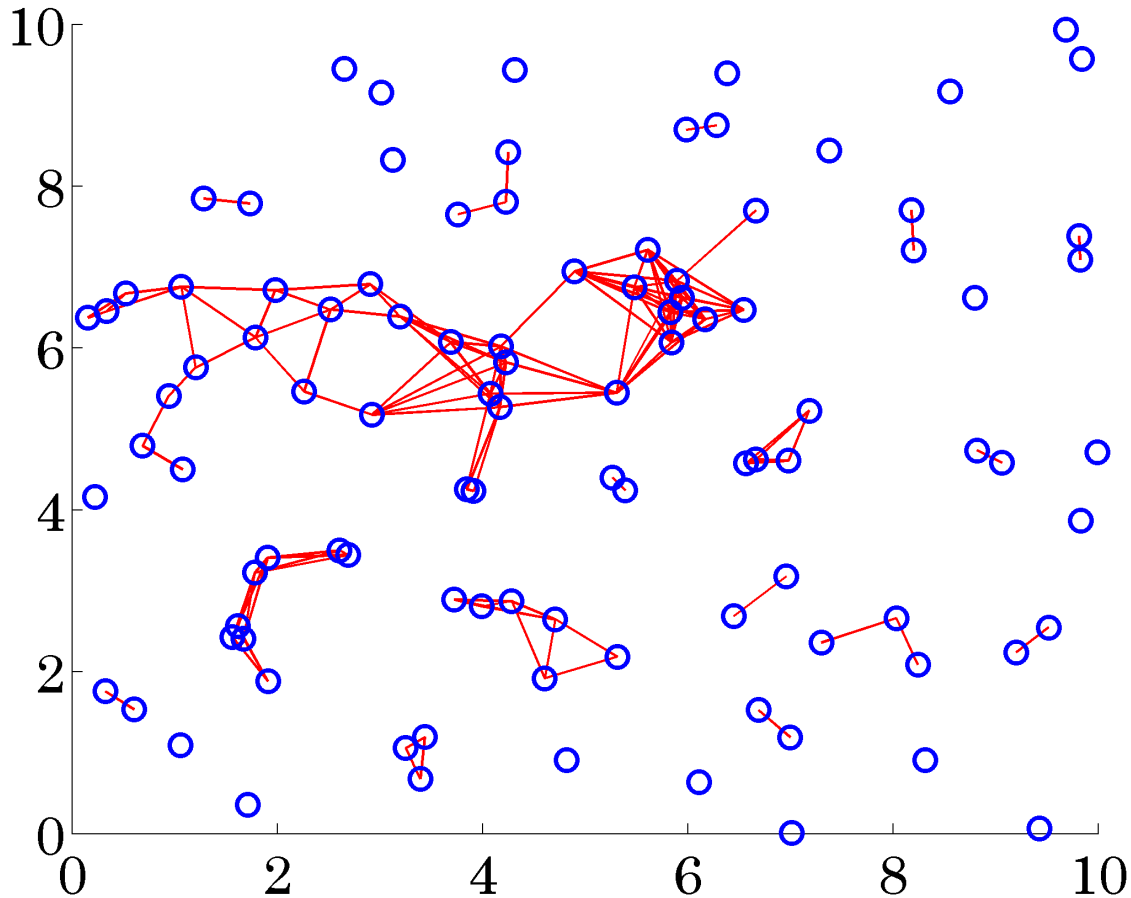


$$\gamma = 18$$

$$\mathbf{card}\,(F)\,/\mathbf{card}\,(F_{\mathrm{c}}) = 3.4\%$$

$$(J - J_{\mathrm{c}})\,/J_{\mathrm{c}} = 48.7\%$$

communication graph identified via ADMM:

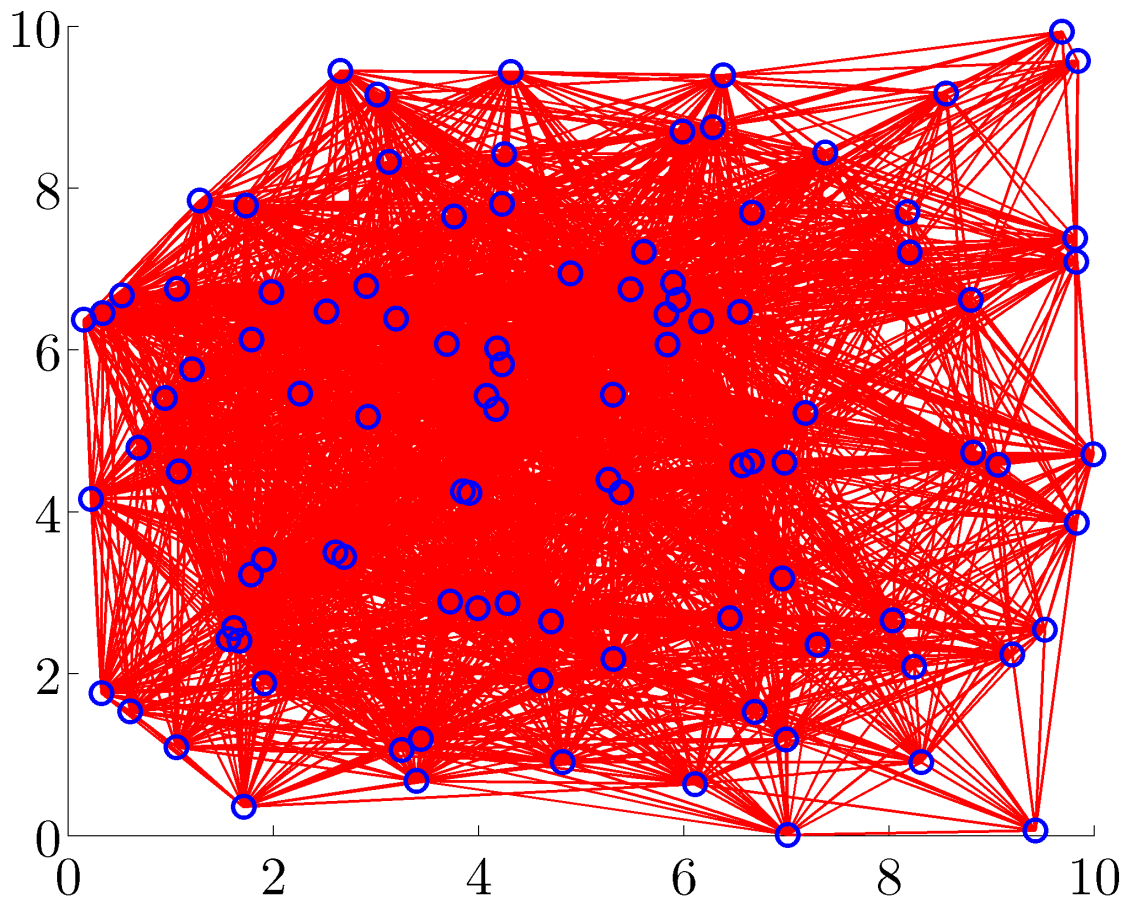


$$\gamma = 30$$

$$\mathbf{card}\,(F)\,/\mathbf{card}\,(F_{\mathrm{c}}) = 2.4\%$$

$$(J - J_{\mathrm{c}})\,/J_{\mathrm{c}} = 54.8\%$$

communication graph of truncated centralized gain:



$$\mathbf{card}\,(F) \;=\; 7380 \;\; (36.9\%)$$

non-stabilizing

Algorithm: details

Separability of G-minimization problem

$$\underset{G}{\text{minimize}} \quad \gamma\, g(G) \;+\; \frac{\rho}{2}\,\|G - V\|_F^2$$

$$V \;:=\; (1/\rho)\Lambda^k \;+\; F^{k+1}$$

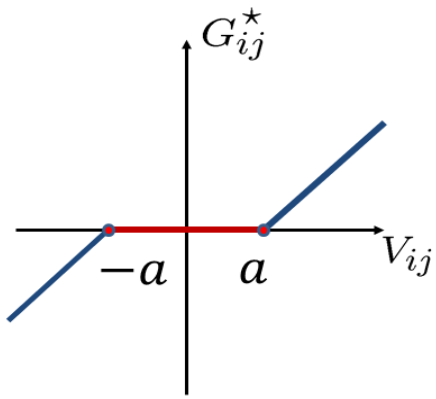
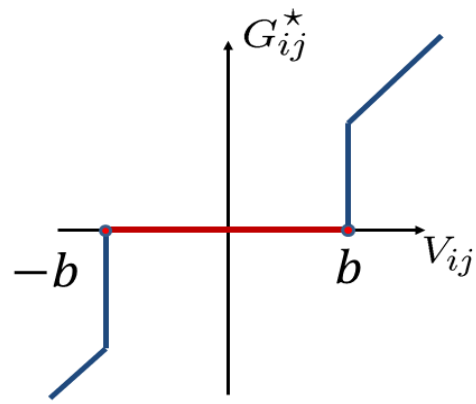
weighted ℓ_1: $$\underset{G_{ij}}{\text{minimize}} \quad \sum_{i,\,j} \left(\gamma\, W_{ij}\, |G_{ij}| \;+\; \frac{\rho}{2}\,(G_{ij} - V_{ij})^2 \right)$$

sum-of-logs: $$\underset{G_{ij}}{\text{minimize}} \quad \sum_{i,\,j} \left(\gamma\, \log\left(1 + \frac{|G_{ij}|}{\varepsilon}\right) \;+\; \frac{\rho}{2}\,(G_{ij} - V_{ij})^2 \right)$$

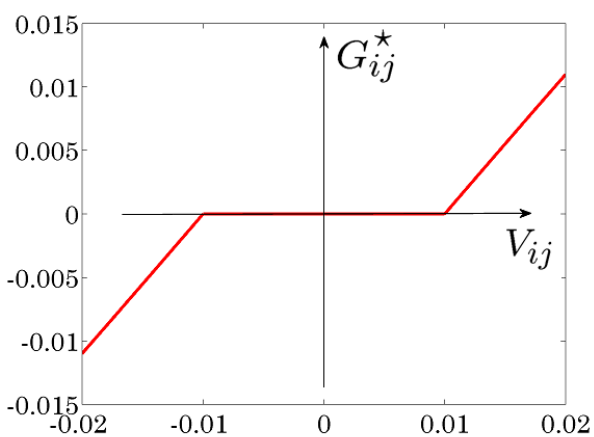
cardinality: $$\underset{G_{ij}}{\text{minimize}} \quad \sum_{i,\,j} \left(\gamma\, \mathbf{card}\,(G_{ij}) \;+\; \frac{\rho}{2}\,(G_{ij} - V_{ij})^2 \right)$$

separability $\Rightarrow$ **element-wise analytical solution**

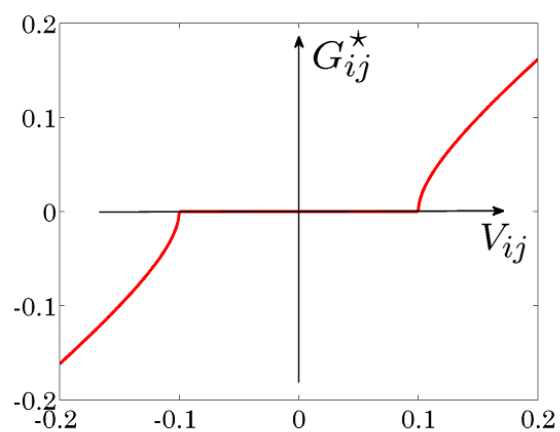
Solution to G-minimization problem

weighted ℓ_1: shrinkage	cardinality: truncation
	
$a = (\gamma/\rho)W_{ij}$	$b = \sqrt{2\gamma/\rho}$

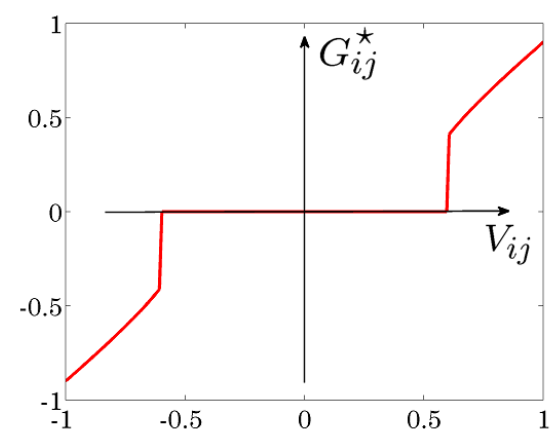
sum-of-logs with $\{\rho = 100, \varepsilon = 0.1\}$



$\gamma = 0.1$



$\gamma = 1$



$\gamma = 10$

Solution to F-minimization problem

$$\underset{F}{\text{minimize}} \quad J(F) + \frac{\rho}{2}\|F - U\|_F^2$$

$$U := G^k - (1/\rho)\Lambda^k$$

NECESSARY CONDITIONS FOR OPTIMALITY:

$$\begin{aligned}
(A - B_2 F)\, L + L\,(A - B_2 F)^T &= -B_1 B_1^T \\
(A - B_2 F)^T P + P\,(A - B_2 F) &= -(Q + F^T R F) \\
F\,L + \rho\,(2R)^{-1} F &= R^{-1} B_2^T P L + \rho\,(2R)^{-1} U
\end{aligned}$$

- ITERATIVE SCHEME

Given F_0 solve for $\{L_1, P_1\} \rightarrow F_1 \rightarrow \{L_2, P_2\} \rightarrow F_2 \quad \cdots$

descent direction $+$ **line search** $\Rightarrow$ **convergence**

Summary

- SPARSITY-PROMOTING OPTIMAL CONTROL
  - ⋆ Design of optimal sparse and block sparse feedback gains

    Lin, Fardad, Jovanović, IEEE TAC '11 (submitted; also: arXiv:1111.6188v1)

- SOFTWARE
  - ⋆ www.ece.umn.edu/∼mihailo/software/lqrsp/

    $>> \texttt{solpath} = \texttt{lqrsp}(\texttt{A}, \texttt{B1}, \texttt{B2}, \texttt{Q}, \texttt{R}, \texttt{options});$

- RELATED WORK
  - ⋆ Optimal synchronization of sparse oscillator networks

    Fardad, Lin, Jovanović, ACC '12

  - ⋆ Optimal dissemination of information in social networks

    Fardad, Zhang, Lin, Jovanović, CDC '12 (submitted)

Design of undirected networks of single-integrators

- Convex optimization problem

$$\begin{array}{ll} \text{minimize} & (1/2)\,\mathrm{trace}(Q^{1/2}(F+\mathbb{1}\mathbb{1}^T/n)^{-1}Q^{1/2}+RF) \;+\; \gamma\,\|W\circ F\|_{\ell_1} \\ \text{subject to} & F\mathbb{1} \;=\; 0, \;\; F \;+\; \mathbb{1}\mathbb{1}^T/n \;\succ\; 0 \end{array}$$

- SDP formulation

$$\begin{array}{ll} \text{minimize} & (1/2)\,\mathrm{trace}(X+RF) \;+\; \gamma\,\mathrm{trace}(\mathbb{1}\mathbb{1}^T Y) \\ \text{subject to} & \begin{bmatrix} X & Q^{1/2} \\ Q^{1/2} & F+\mathbb{1}\mathbb{1}^T/n \end{bmatrix} \succeq 0 \\ & -Y \leq W\circ F \leq Y \\ & F\mathbb{1} \;=\; 0 \end{array}$$