Algorithms for Leader Selection in Large Dynamical Networks: Noise-corrupted Leaders

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joint work with:

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Large dynamical networks

- All around us

Power grid	Internet	Social networks
		

- Interactions cause complex behavior

Northeast blackout 2003

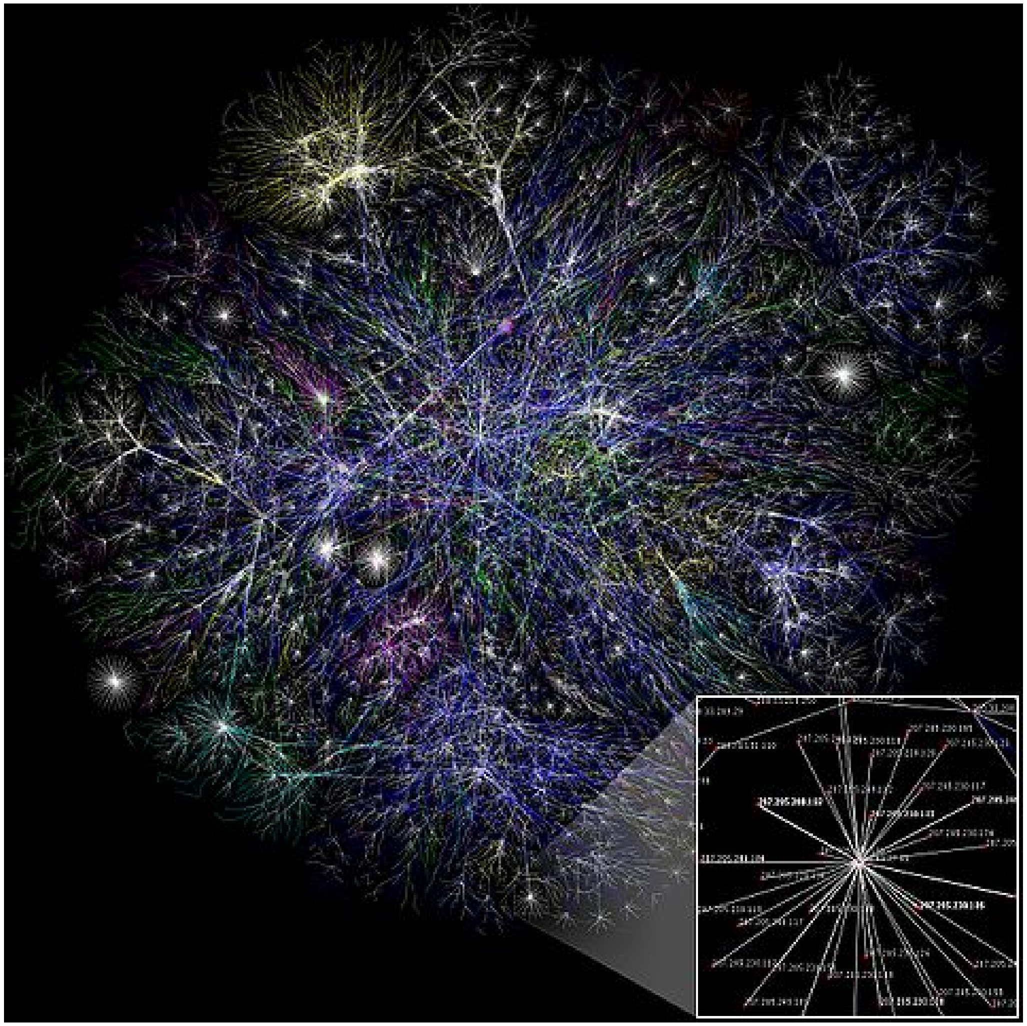
before:

after:

- The blackout was caused by a power plant going offline

US-Canada Power System Outage Task Force Final Report

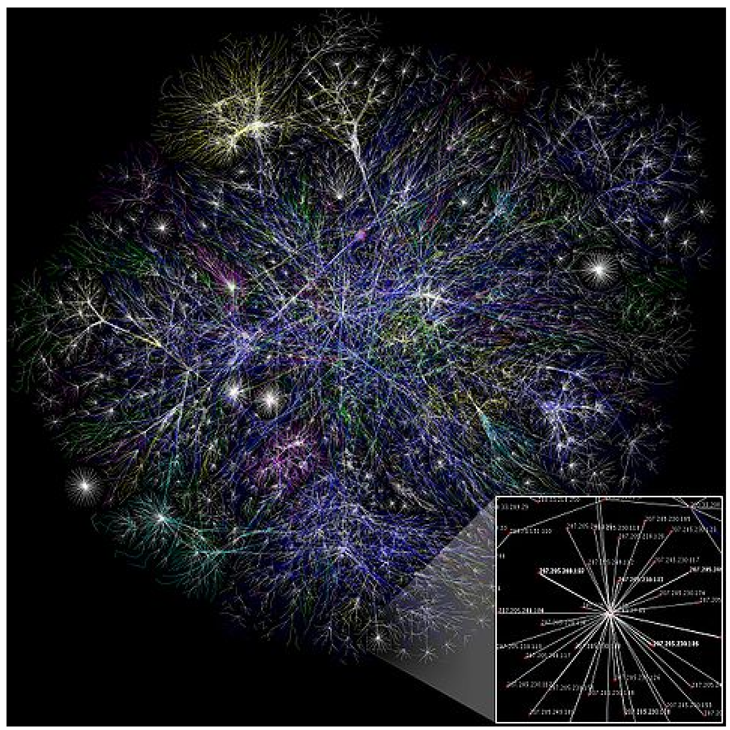
Hierarchical structure of Internet



Opte project (www.opte.org)

- **resilient to random failure** *Cohen et al., Phys. Rev. Lett. '00*
- **vulnerable to removal of high degree nodes** *Cohen et al., Phys. Rev. Lett. '01*


CLEVELAND



Outline

❶ LEADER SELECTION PROBLEM

⋆ **Combinatorial optimization problem**

❷ ALGORITHMS

⋆ Convex relaxation $\Rightarrow$ **lower bound**

⋆ Alternating direction method of multipliers $\Rightarrow$ **upper bound**

❸ EXAMPLE

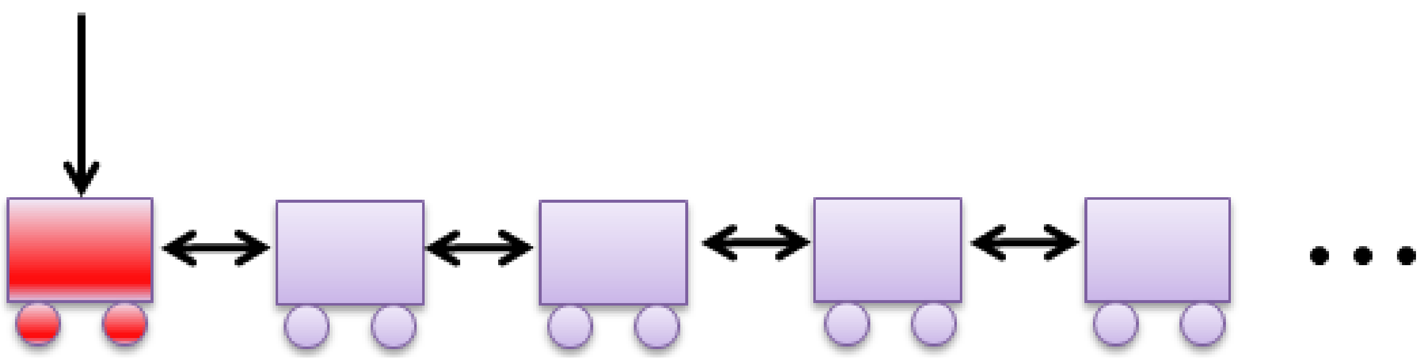
❹ CONCLUDING REMARKS

Leader-follower consensus dynamics

- connected, undirected networks

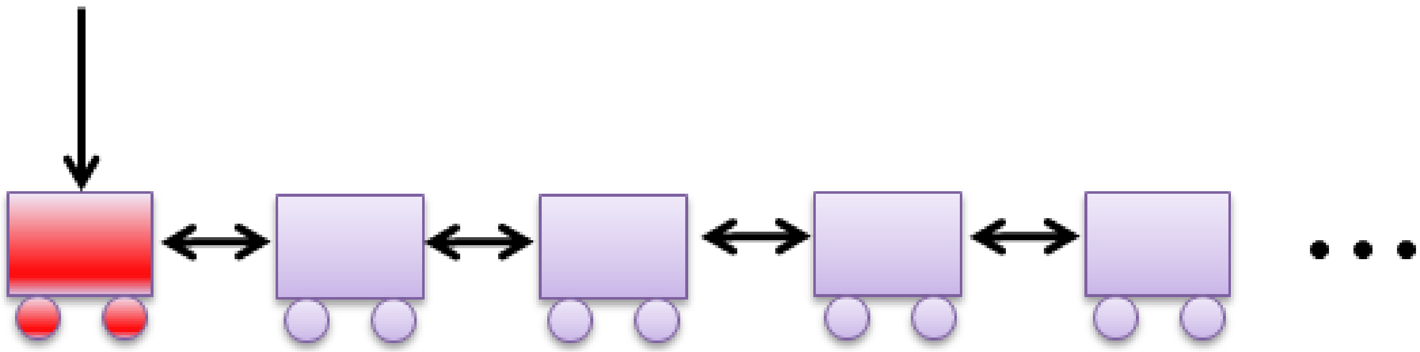
FOLLOWERS: $$\dot{x}_i(t) \;=\; -\sum_{j\,\in\,\mathcal{N}_i} (x_i(t) \;-\; x_j(t)) \;+\; w_i(t)$$

LEADERS: $$\dot{x}_i(t) \;=\; -\sum_{j\,\in\,\mathcal{N}_i} (x_i(t) \;-\; x_j(t)) \;-\; x_i(t) \;+\; w_i(t)$$



Minimum variance leader selection problem

$$\dot{x}(t) \;=\; -\;(L\,+\,H)\,x(t)\;+\;w(t)$$



$$L \;=\; \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \qquad H \;=\; \mathrm{diag}\,(h) \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- assign k leaders to minimize the total steady-state variance

$$V_{ss} \;=\; \lim_{t\,\to\,\infty}\;\mathrm{trace}\left(\mathcal{E}\{x(t)x^T(t)\}\right)$$

$$\begin{array}{lrcl} \text{minimize} & J(h) & = & \operatorname{trace}\left((L + \operatorname{diag}(h))^{-1}\right) \\ \text{subject to} & h_i & \in & \{0,1\}, \quad i = 1,\ldots,N \\ & h^T \mathbb{1} & = & k \end{array}$$

- FEATURES:
  - ⋆ convex objective function, linear constraint
  - ⋆ **difficulty: Boolean constraints**

- APPROACH:
  - ⋆ convex relaxation $\Rightarrow$ lower bound
  - ⋆ alternating direction method of multipliers $\Rightarrow$ upper bound

Convex relaxation

$$\begin{array}{ll} \text{minimize} & J(h) \;=\; \operatorname{trace}\left((L + \operatorname{diag}(h))^{-1}\right) \\ \text{subject to} & h_i \;\in\; [0,1], \quad i \;=\; 1,\ldots,N \\ & h^T \mathbb{1} \;=\; k \end{array}$$

lower bound on J

SEMIDEFINITE PROGRAM:

$$\begin{array}{ll} \text{minimize} & \operatorname{trace}(X) \\ \text{subject to} & \begin{bmatrix} X & I \\ I & L + \operatorname{diag}(h) \end{bmatrix} \succeq 0 \\ & h_i \;\in\; [0,1], \quad i \;=\; 1,\ldots,N \\ & h^T \mathbb{1} \;=\; k \end{array}$$

without exploiting structure: $O(N^6)$

$$\begin{array}{rrcl} \text{minimize} & J(h) & = & \operatorname{trace}\left((L + \operatorname{diag}(h))^{-1}\right) \\ \text{subject to} & h_i & \in & [0,1], \quad i = 1, \ldots, N \\ & h^T \mathbb{1} & = & k \end{array}$$

CUSTOMIZED INTERIOR POINT METHOD:

$$\begin{array}{ll} \text{minimize} & \gamma \operatorname{trace}\left((L + H)^{-1}\right) + \displaystyle\sum_{i=1}^{N} \left(-\log(h_i) - \log(1 - h_i)\right) \\ \text{subject to} & h^T \mathbb{1} = k \end{array}$$

Newton's method: $O(N^3)$

Alternating direction method of multipliers

- **Step 1: introduce indicator function of the constraint set**

$$g(h) \;=\; \begin{cases} 0, & h_i \,\in\, \{0,1\},\;\; h^T \mathbb{1} \;=\; k \\ +\infty, & \text{otherwise} \end{cases}$$

$$\boxed{\text{minimize} \;\; J(h) \;+\; g(h)}$$

- **Step 2: introduce additional variable/constraint**

$$\boxed{\begin{array}{ll} \text{minimize} & J(h) \;+\; g(z) \\ \text{subject to} & h \;-\; z \;=\; 0 \end{array}}$$

benefit: decouples J and g

- **Step 3: introduce augmented Lagrangian**

$$\mathcal{L}_\rho(h, z, \lambda) \;=\; J(h) \;+\; g(z) \;+\; \lambda^T(h \,-\, z) \;+\; \frac{\rho}{2}\,\|h \,-\, z\|_2^2$$

- **Step 4: use ADMM for augmented Lagrangian minimization**

ADMM:

$$\boxed{\begin{array}{rcl} h^{r+1} & := & \underset{h}{\arg\min}\; \mathcal{L}_\rho(h, z^r, \lambda^r) \\[1em] z^{r+1} & := & \underset{z}{\arg\min}\; \mathcal{L}_\rho(h^{r+1}, z, \lambda^r) \\[1em] \lambda^{r+1} & := & \lambda^r \,+\, \rho(h^{r+1} \,-\, z^{r+1}) \end{array}}$$

MANY MODERN APPLICATIONS

Boyd et al., Foundations and Trends in Machine Learning '11

⋆ z-minimization problem – nonconvex, discontinuous

$$\underset{z}{\text{minimize}} \quad g(z) + \frac{\rho}{2}\|z - \bar{z}\|_2^2$$

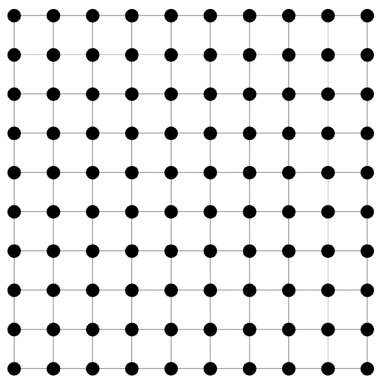
$$\bar{z} := (1/\rho)\lambda^r + h^{r+1}$$

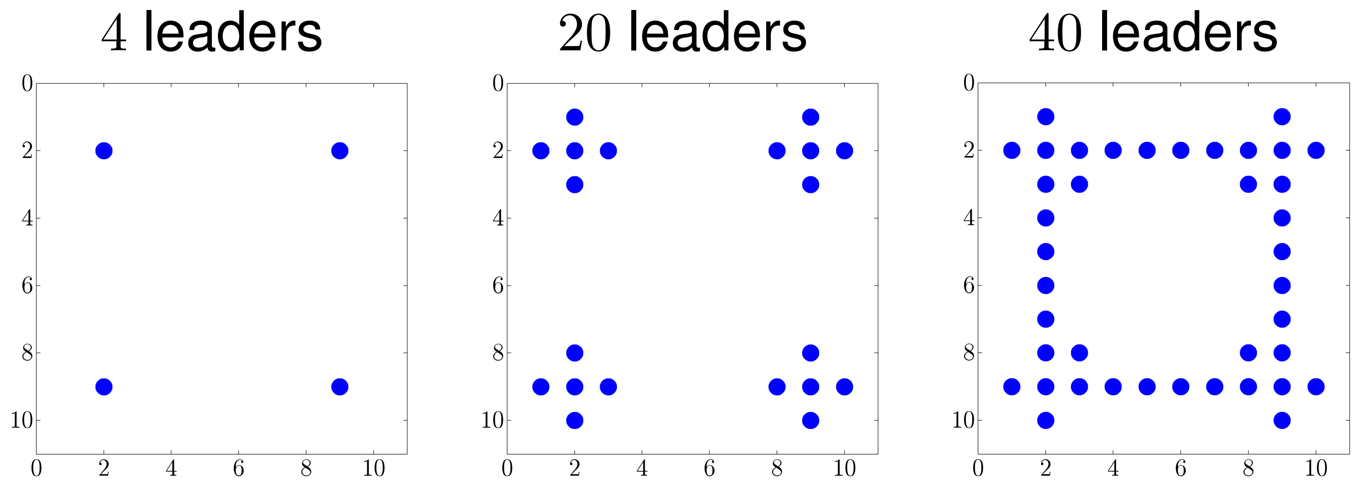
GLOBAL SOLUTION: $z_i = \begin{cases} 1, & \bar{z}_i \geq [\bar{z}]_k \\ 0, & \bar{z}_i < [\bar{z}]_k \end{cases}$

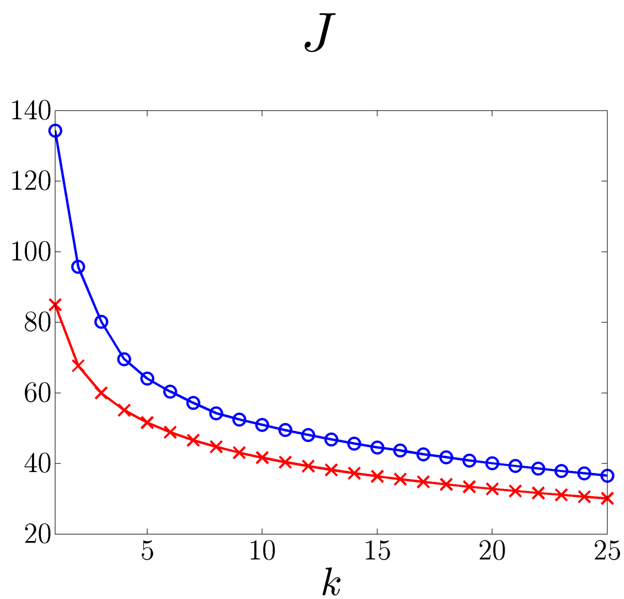
$$k = 2, \quad \bar{z} = \begin{bmatrix} 4.1 \\ 1.5 \\ -6.7 \\ 5.2 \\ 3.9 \end{bmatrix} \Rightarrow z = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Example

- 10×10 lattice







ADMM: upper bound ($\circ$)

convex relaxation: lower bound ($\times$)

Concluding remarks

- Leader selection problem: noise-corrupted leaders

- Ongoing research:
  - ⋆ apply these algorithms to complex large networks
  - ⋆ utilize sparsity structure of Laplacian

- Leader selection problem: noise-free leaders (ThB04.1)
  - ⋆ non-convex objective function and non-convex constraints
  - ⋆ semidefinite relaxation

- Design of optimal sparse and block sparse feedback gains via ADMM

  Lin, Fardad, Jovanović, IEEE TAC '11 (submitted; also: arXiv:1111.6188v1)