Structure identification and optimal design of large-scale networks of dynamical systems

Fu Lin

UNIVERSITY OF MINNESOTA

Electrical and Computer Engineering

Ph.D. defense

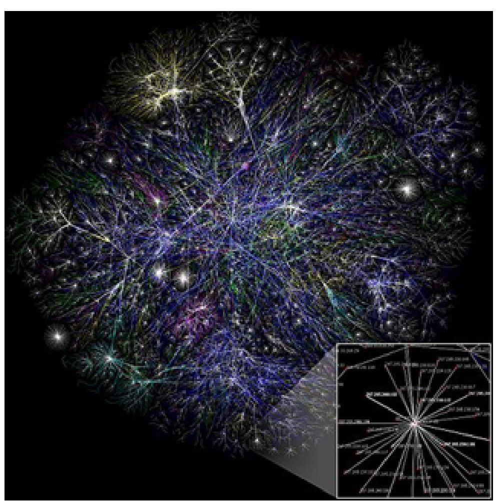
Networks of dynamical systems

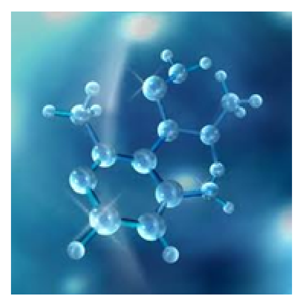
- ALL AROUND US

Power grid	Internet	Social networks
		

- MANY OTHERS:

World Wide Web, networking services, economics, material science, . . .





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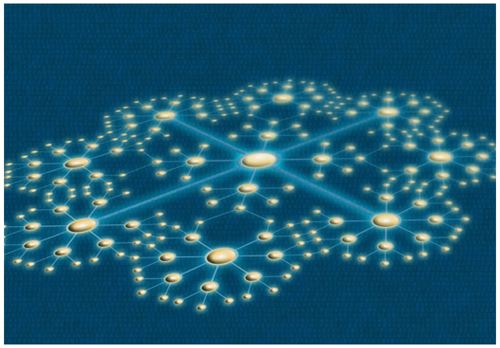
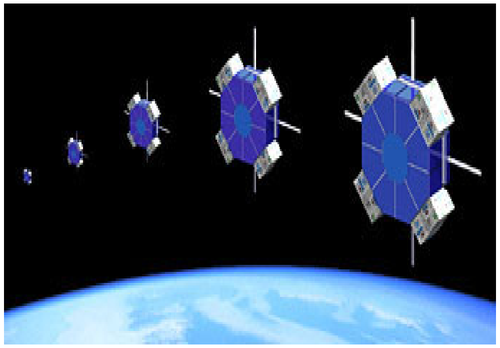
- DRIVEN BY TECHNOLOGICAL ADVANCEMENTS

APPLICATIONS:

wind farms	sensor networks	UAV formations satellite constellations
		

KEY QUESTION: Interplay between network structure and system performance

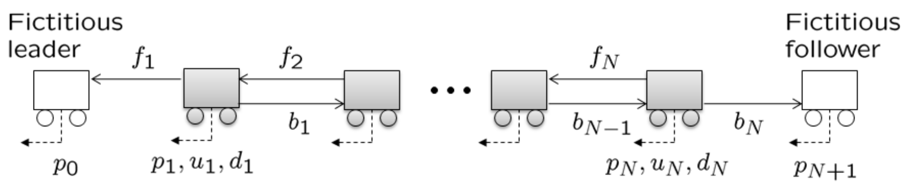
Overview

- **Main topics:**
  - ⋆ Localized control of vehicular formations
  - ⋆ Sparsity-promoting optimal control
  - ⋆ Sparse consensus networks
  - ⋆ Algorithms for leader selection in consensus networks

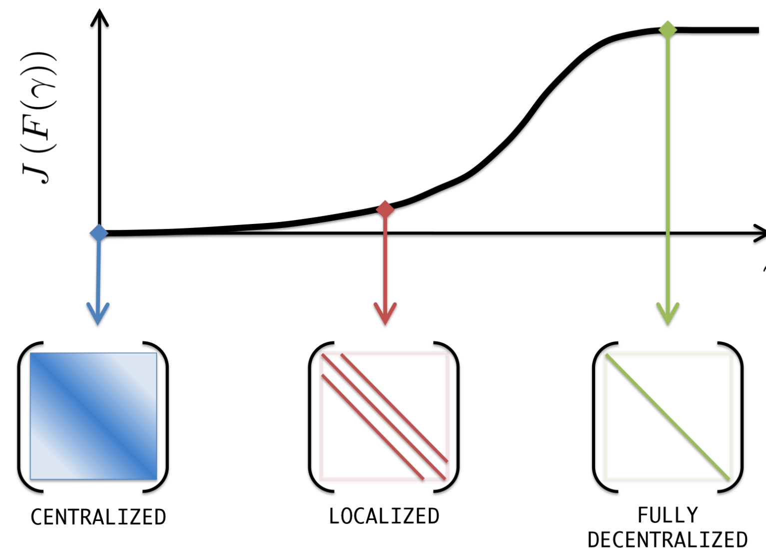
- **Challenges:**
  - ⋆ Networks – combinatorial objects
  - ⋆ Optimization – constrained nonconvex problems

- **Approach:**
  - ⋆ Identify classes of convex problems
  - ⋆ Exploit problem structure to develop efficient algorithms

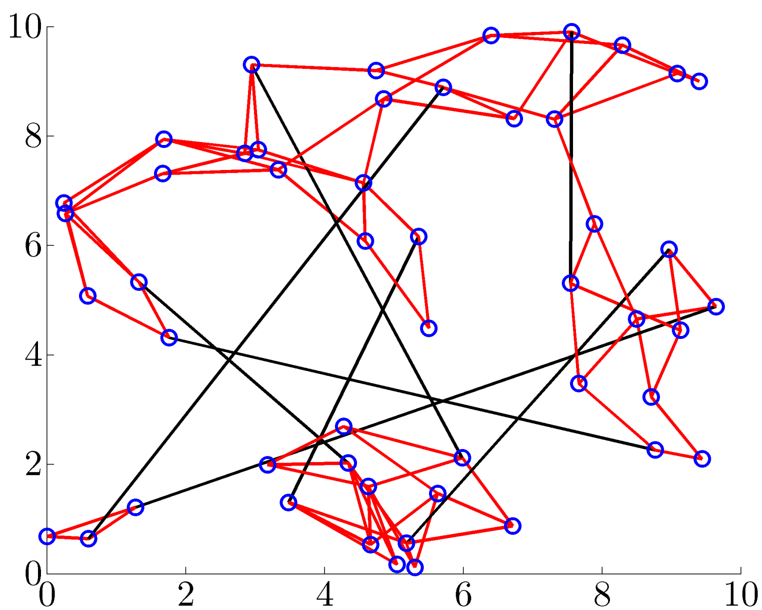
Localized control of vehicular formations



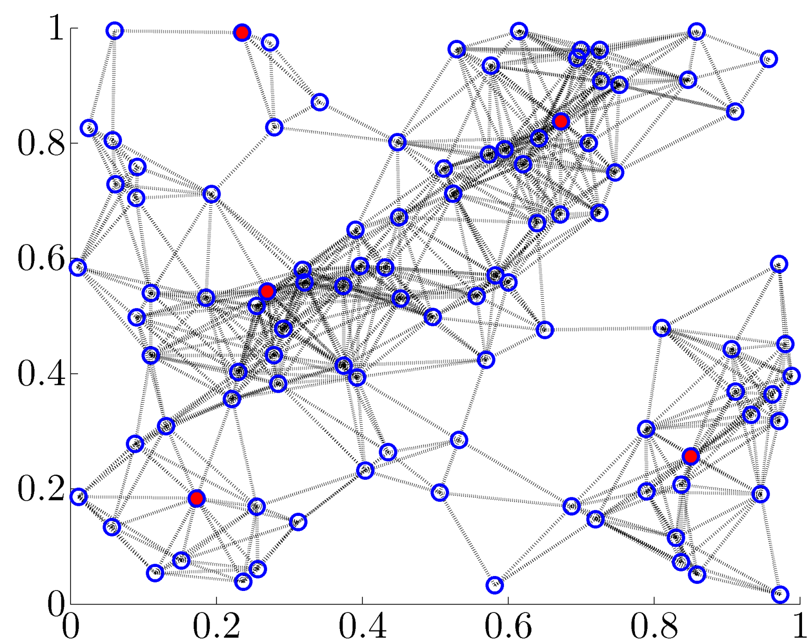
Sparsity-promoting optimal control



Sparse consensus networks



Algorithms for leader selection

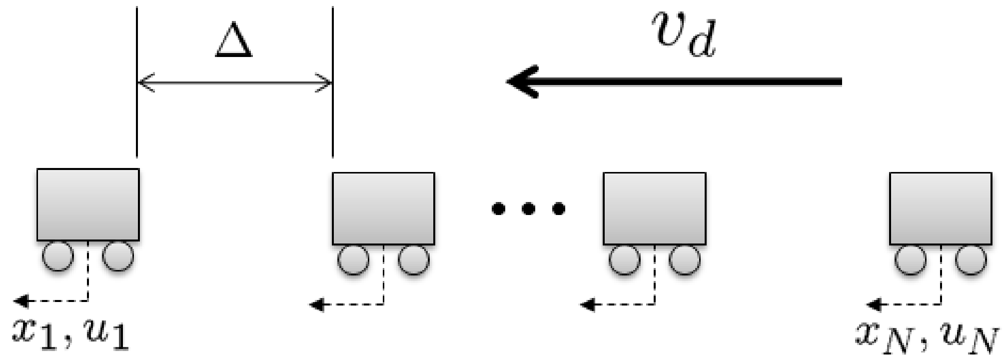


Localized control of vehicular formations

Vehicular formations

AUTOMATED CONTROL OF EACH VEHICLE

tight spacing at desired speeds



KEY ISSUES (ALSO IN: CONTROL OF SWARMS, FLOCKS, FORMATION FLIGHT)

⋆ Is it enough to only look at neighbors?

⋆ Are there any fundamental limitations?

OUR CONTRIBUTIONS

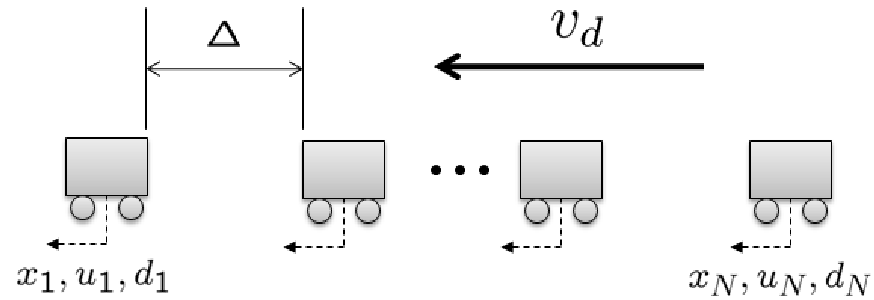
⋆ Design of optimal localized controllers

⋆ Performance limitations of localized controllers

Problem setup

SINGLE INTEGRATOR MODEL

$$\dot{x}_n \;=\; \underset{\substack{\uparrow \\ \text{control}}}{u_n} \;+\; \underset{\substack{\uparrow \\ \text{disturbance}}}{d_n}$$



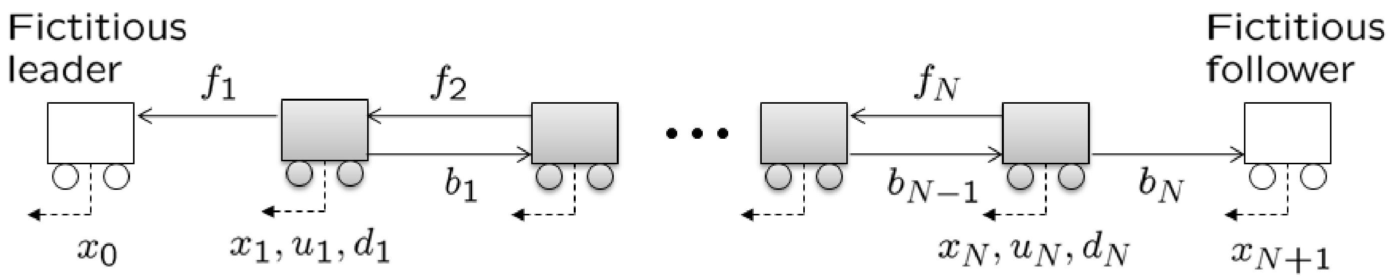
- Desired trajectory: $\begin{cases} \bar{x}_n \;:=\; v_d\, t \;+\; n\Delta \\ \text{constant velocity } v_d \end{cases}$

- Deviations: $\left.\begin{array}{rcl} \tilde{x}_n & := & x_n \;-\; \bar{x}_n \\ \tilde{u}_n & := & u_n \;-\; v_d \end{array}\right\} \;\Rightarrow\; \dot{\tilde{x}}_n \;=\; \tilde{u}_n \;+\; d_n$

- Controls: $\tilde{u} \;=\; -\, K\tilde{x}, \quad K\text{:}$ structured feedback gain

Use nearest neighbor interactions

- Design forward and backward gains



Relative position feedback:

$$\begin{aligned} \tilde{u}_n &= -\, f_n \, (\tilde{x}_n - \tilde{x}_{n-1}) \; - \; b_n \, (\tilde{x}_n - \tilde{x}_{n+1}) \\ \tilde{u} &= -\, K \tilde{x} \; = \; - \left[\begin{array}{cc} F_f & F_b \end{array} \right] \left[\begin{array}{c} C_f \\ C_b \end{array} \right] \tilde{x} \end{aligned}$$

$$K \sim \underbrace{\begin{bmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{bmatrix}}_{F_f} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{C_f} + \underbrace{\begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}}_{F_b} \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{C_b}$$

Structured feedback design

$$\text{state equation:} \quad \dot{\tilde{x}} \;=\; d \,+\, \tilde{u}$$

$$\text{performance output:} \quad z \;=\; \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} \tilde{x} \,+\, \begin{bmatrix} 0 \\ I \end{bmatrix} \tilde{u}$$

$$\text{control input:} \quad \tilde{u} \;=\; -\begin{bmatrix} F_f & F_b \end{bmatrix} \begin{bmatrix} C_f \\ C_b \end{bmatrix} \tilde{x}, \quad F_f,\, F_b \;-\; \text{diagonal}$$

OBJECTIVE:

Design diagonal $\{F_f, F_b\}$ to minimize variance amplification $d \,\to\, z$

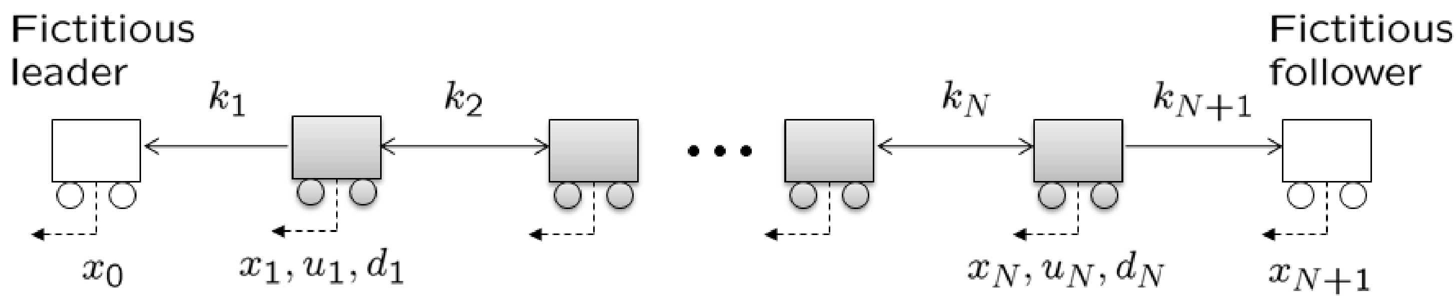
$$\text{minimize} \quad J(F) \;=\; \mathrm{trace}\left(\int_0^\infty \mathrm{e}^{(-FC)^T t}\left(Q \,+\, C^T F^T F C\right)\mathrm{e}^{(-FC)t}\,\mathrm{d}t\right)$$

$$\text{subject to} \quad F \;=\; \left[\,F_f \;\; F_b\,\right], \quad F_f,\; F_b \;-\; \text{diagonal}$$

- CHALLENGE:
  - ⋆ Nonconvex problem

- APPROACH:
  - ⋆ Identify convex problems (symmetric gains)
  - ⋆ Homotopy path (non-symmetric gains)

Design of optimal symmetric gains



symmetric gains:

$$\tilde{u}_n = -k_n(\tilde{x}_n - \tilde{x}_{n-1}) - k_{n+1}(\tilde{x}_n - \tilde{x}_{n+1})$$

$$\tilde{u} = -K\tilde{x}, \quad K = K^T \succ 0$$

CONVEX PROBLEM:

$$\boxed{\begin{array}{ll} \text{minimize} & J(K) = \operatorname{trace}\left(K + Q K^{-1}\right) \\ \text{subject to} & K = K^T \succ 0, \quad K - \text{tridiagonal} \end{array}}$$

can be formulated as an SDP

Design of optimal non-symmetric gains

- HOMOTOPY PATH

$$\begin{array}{ccc} \begin{array}{c}\textbf{uniform gain}\\ F_0 \,=\, [\, I \;\; I \,]\end{array} & & \begin{array}{c}\textbf{optimal gain}\\ F^\star \,=\, [\, F_f^\star \;\; F_b^\star \,]\end{array} \\ \Downarrow & & \Uparrow \\ \begin{array}{c}\textbf{inversely optimal}\\ \textbf{w.r.t. weight } Q_0\end{array} & \xrightarrow[\varepsilon \,\in\, [0,1]]{Q(\varepsilon) \,=\, (1 - \varepsilon)\, Q_0 \,+\, \varepsilon\, Q_d} & \begin{array}{c}\textbf{desired}\\ \textbf{weight } Q_d\end{array} \end{array}$$

Step 1: Find Q_0 inversely optimal w.r.t. F_0

Step 2: Perturbation analysis $0 < \varepsilon \ll 1$

Step 3: Followed by homotopy

- SPATIALLY UNIFORM $(K_0 = C_f + C_b)$

  inversely optimal wrt $Q_0 = K_0^2$

$$\left.\begin{aligned} -P_0^2 + Q_0 &= 0 \\ K_0 &= P_0 \end{aligned}\right\}$$

Necessary conditions for optimality

$$\begin{aligned} (-K)\,L + L\,(-K)^T &= -I \\ (-K)^T P + P\,(-K) &= -(Q + K^T K) \\ \big((K - P)LC^T\big) \circ \left[\begin{array}{cc} I & I \end{array}\right] &= 0 \end{aligned}$$

- PERTURBATION ANALYSIS

$$0 < \varepsilon \ll 1, \quad P = \sum_{n=0}^{\infty} \varepsilon^n P_n, \quad L = \sum_{n=0}^{\infty} \varepsilon^n L_n, \quad K = \sum_{n=0}^{\infty} \varepsilon^n K_n$$

Conveniently coupled equations

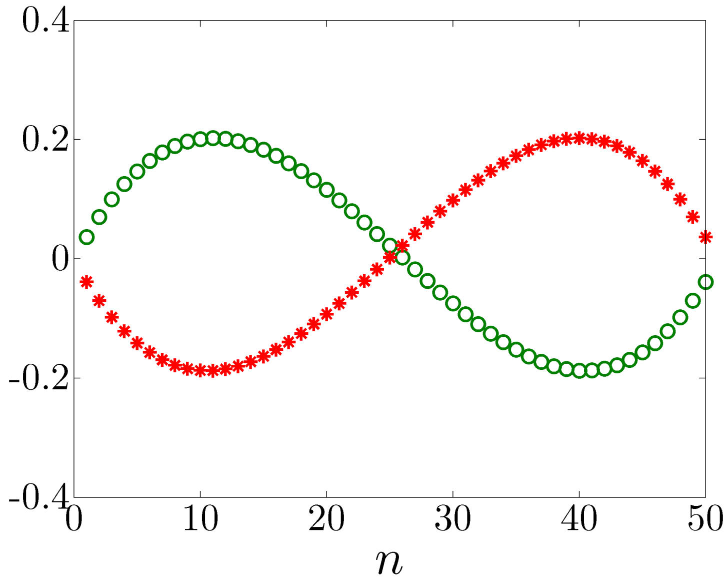
$$O(1): \begin{cases} -P_0^2 + Q_0 = 0 \\ K_0 = P_0 \\ -K_0 L_0 - L_0 K_0 = -I \end{cases}$$

$$O(\varepsilon): \begin{cases} -K_0 P_1 - P_1 K_0 = -(Q_d - Q_0) \\ \left[\left(\begin{bmatrix} F_{f1} & F_{b1} \end{bmatrix} \begin{bmatrix} C_f \\ C_b \end{bmatrix} - P_1\right) L_0 \begin{bmatrix} C_f^T & C_b^T \end{bmatrix}\right] \circ \begin{bmatrix} I & I \end{bmatrix} = 0 \end{cases}$$

$$\vdots$$

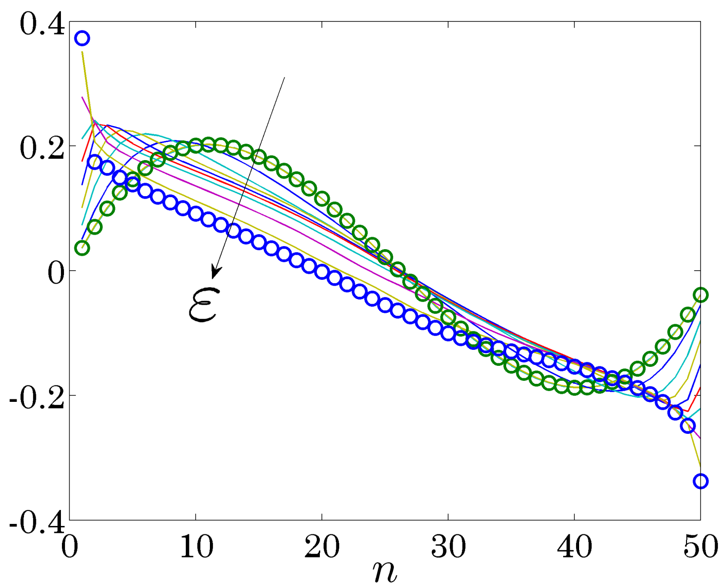
- First-order correction (with $Q_d = I$)

FORWARD/BACKWARD GAINS:



- Homotopy

FORWARD GAINS:



Part I: Summary and concluding remarks

- LOCALIZED CONTROL OF VEHICULAR FORMATIONS
  - ⋆ Identify convex problem (symmetric gains)
  - ⋆ Inverse optimality + perturbation analysis + homotopy (non-symmetric gains)

- ALSO IN THE DISSERTATION:
  - ⋆ Performance of localized controllers: variance per vehicle in large formations
    - ∗ Symmetric gains: $O(N)$
    - ∗ Non-symmetric gains: $O(\sqrt{N})$
  - ⋆ Design of optimal localized controllers for double integrators

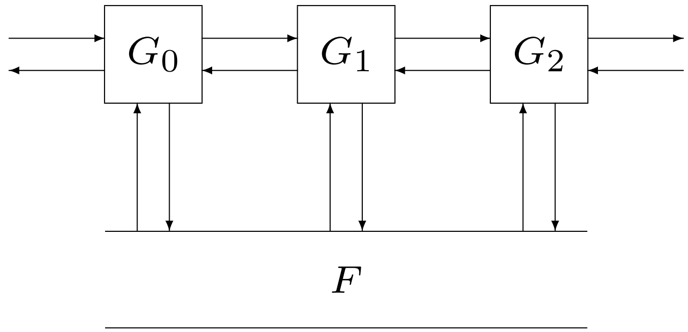
- CONTRIBUTIONS:
  - ⋆ Design of optimal localized controllers
  - ⋆ Optimal gains are non-symmetric and spatially-varying
  - ⋆ Performance limitations of optimal localized controllers

Lin, Fardad, Jovanović, IEEE Trans. Automat. Control, 2012

SPARSITY-PROMOTING OPTIMAL CONTROL

Controller architectures

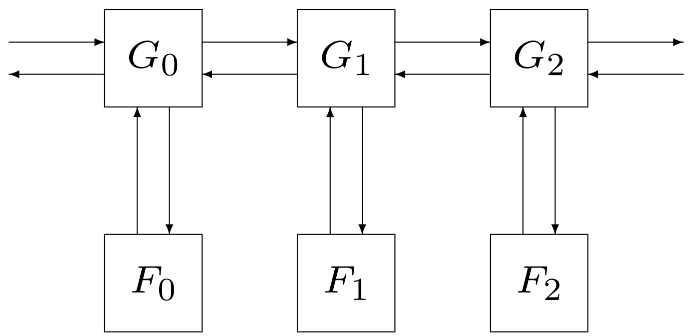
CENTRALIZED



best performance

excessive communication

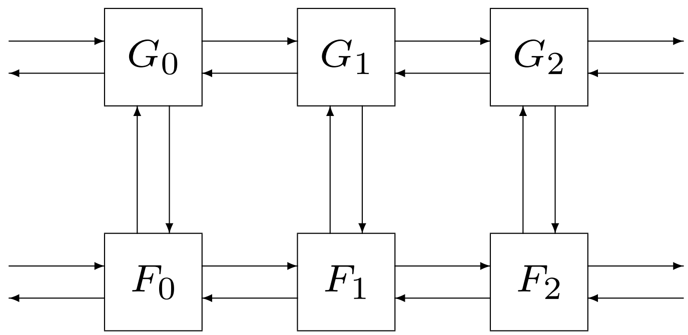
FULLY DECENTRALIZED



worst performance

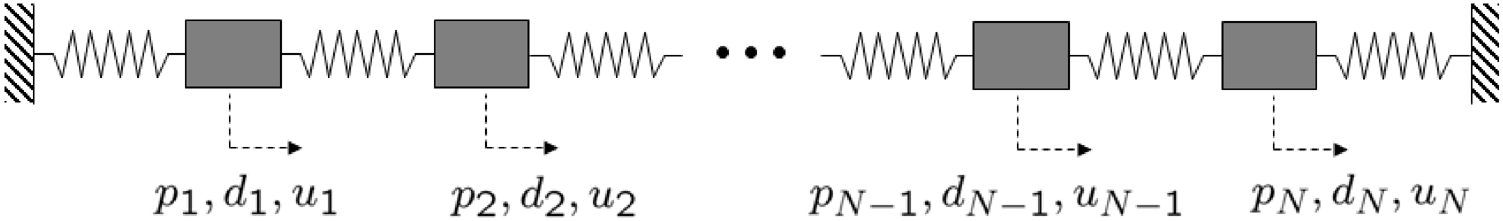
no communication

LOCALIZED



many possible architectures

Example: Mass-spring system



PLANT:
$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ T & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t)$$

CONTROLLER:
$$u(t) = -\begin{bmatrix} F_p & F_v \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix}$$

FEEDBACK GAINS: $F_p, \; F_v$

centralized	localized	fully decentralized
$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & & \\ * & * & * & \\ & * & * & * \\ & & * & * \end{bmatrix}$	$\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix}$

CHALLENGE: Identify controller architectures for complex interconnected systems

Sparse feedback synthesis

OBJECTIVE:

$$\boxed{\text{design sparse } F \text{ that minimizes variance amplification } d \to z}$$

$$\begin{array}{ll} \text{minimize} & J(F) \\ \text{subject to} & \mathbf{card}(F) \leq k \end{array}$$

$$F = \begin{bmatrix} 5.1 & -2.3 & 0 \\ 0 & 0 & 1.6 \end{bmatrix} \Rightarrow \mathbf{card}(F) = 3$$

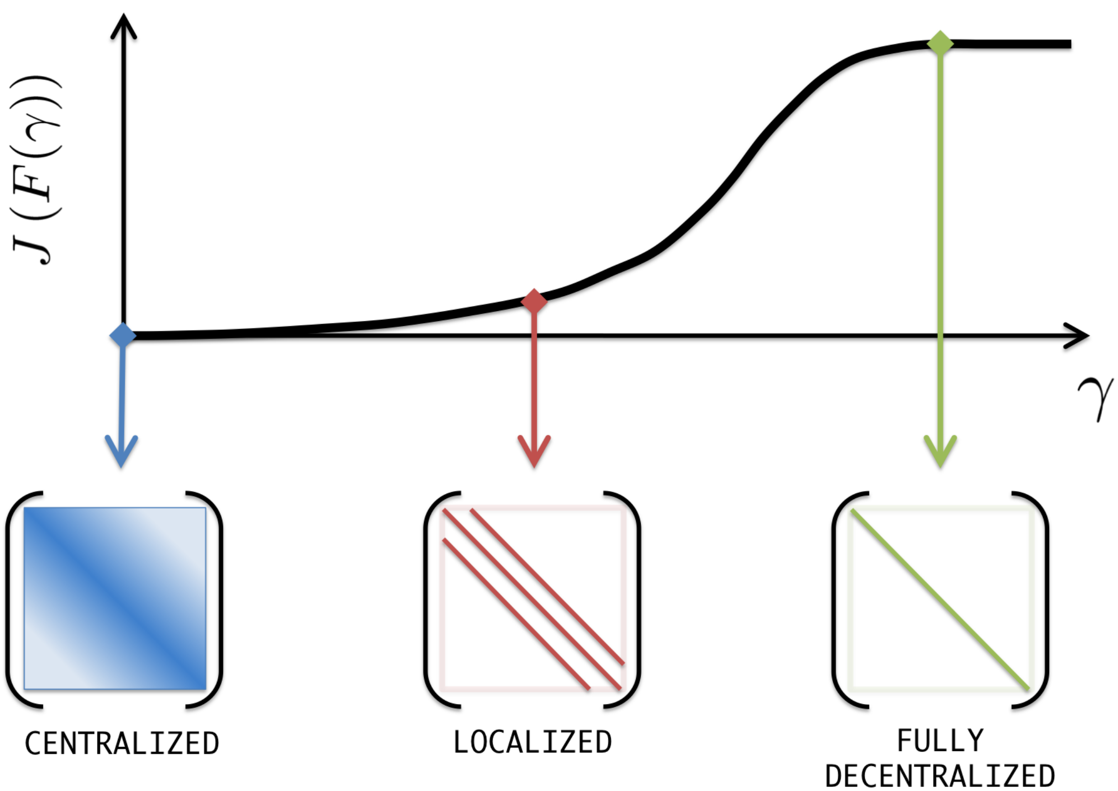
- DIFFICULT COMBINATORIAL PROBLEM

Sparsity-promoting optimal control

$$\text{minimize} \quad J(F) \, + \, \gamma \, \mathbf{card}(F) \qquad \gamma \, \geq \, 0$$

$$\gamma \, = \, 0 \quad \Rightarrow \quad \text{globally optimal controller:}$$

$$\boxed{\begin{aligned} A^T P \, + \, P A \, - \, P B_2 R^{-1} B_2^T P \, + \, Q \, &= \, 0 \\ F_c \, &= \, R^{-1} B_2^T P \end{aligned}}$$



DIFFICULTIES:

⋆ $J(F)$, $\mathbf{card}\,(F)$: nonconvex functions of F

- Convex relaxations of $\mathbf{card}\,(F)$

$$\ell_1 \text{ norm: } \quad \sum_{i,\,j} |F_{ij}|$$

$$\text{weighted } \ell_1 \text{ norm: } \quad \sum_{i,\,j} W_{ij}\,|F_{ij}|, \quad W_{ij} \;\geq\; 0$$

- Identify convex problems: design of undirected consensus networks

Alternating direction method of multipliers

$$\text{minimize} \quad J(F) \,+\, \gamma\, g(F)$$

- Why ADMM?

Observations:

- $\star$ $J(F)$ – nonconvex but smooth

$$J(F) \;=\; \mathrm{trace}\left(\int_0^\infty B_1^T \mathrm{e}^{(A-B_2F)^T t}\left(Q \,+\, F^T R F\right)\mathrm{e}^{(A-B_2F)t} B_1 \,\mathrm{d}t\right)$$

- $\star$ $g(F)$ – convex but nondifferentiable

$$g(F) \;=\; \sum_{i,\,j} |F_{ij}|$$

- $\star$ $J(F) + \gamma\, g(F)$ – nonconvex nondifferentiable

- ADMM splits J and g
  - $\star$ minimization of J – descent methods
  - $\star$ minimization of g – analytical solutions

Introduce additional variable/constraint

$$\begin{array}{ll} \text{minimize} & J(F) \,+\, \gamma\, g(G) \\ \text{subject to} & F \,-\, G \,=\, 0 \end{array}$$

benefit: split J and g

Form augmented Lagrangian

$$\mathcal{L}_\rho(F, G, \Lambda) \;=\; J(F) \,+\, \gamma\, g(G) \,+\, \operatorname{trace}\left(\Lambda^T(F \,-\, G)\right) \,+\, \frac{\rho}{2}\, \|F \,-\, G\|_F^2$$

$$\begin{array}{lll l} F^{k+1} & := & \underset{F}{\arg\min}\; \mathcal{L}_\rho(F, G^k, \Lambda^k) & F\text{-minimization} \\ G^{k+1} & := & \underset{G}{\arg\min}\; \mathcal{L}_\rho(F^{k+1}, G, \Lambda^k) & G\text{-minimization} \\ \Lambda^{k+1} & := & \Lambda^k \,+\, \rho(F^{k+1} \,-\, G^{k+1}) & \Lambda\text{-update} \end{array}$$

- $\star$ F-minimization: descent method
- $\star$ G-minimization: elementwise analytical solutions

Descent method for F-minimization problem

$$\underset{F}{\text{minimize}} \quad J(F) \, + \, \frac{\rho}{2} \, \|F - U\|_F^2$$

$$U \;:=\; G^k \;-\; (1/\rho)\Lambda^k$$

NECESSARY CONDITIONS FOR OPTIMALITY:

$$\boxed{\begin{aligned} (A - B_2 F) L + L (A - B_2 F)^T &= -B_1 B_1^T \\ (A - B_2 F)^T P + P (A - B_2 F) &= -(Q + F^T R F) \\ F L + \rho (2R)^{-1} F &= R^{-1} B_2^T P L + \rho (2R)^{-1} U \end{aligned}}$$

- ITERATIVE SCHEME

$$\boxed{\text{Given} \quad F_0 \quad \text{solve for} \quad \{L_1, P_1\} \;\to\; F_1 \;\to\; \{L_2, P_2\} \;\to\; F_2 \;\cdots}$$

descent direction $+$ **line search** $\Rightarrow$ **convergence**

Polishing: structured optimal design

$$\begin{array}{ll} \text{minimize} & J(F) \\ \text{subject to} & F \in \mathcal{S} \end{array}$$

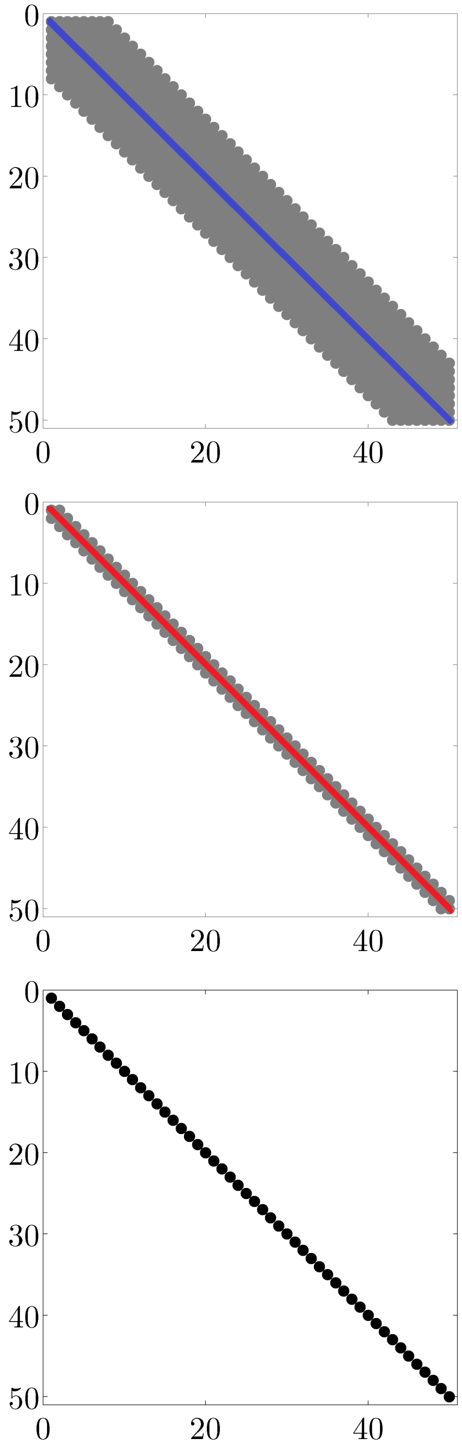
$\star$ ADMM $\left\{ \begin{array}{l} \text{identifies sparsity patterns } \mathcal{S} \\ \text{provides good initial condition for structured design} \end{array} \right.$

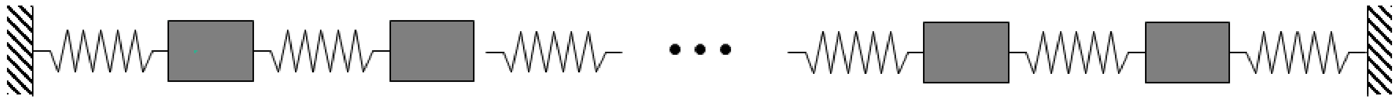
SOFTWARE

$\star$ www.ece.umn.edu/∼mihailo/software/lqrsp/

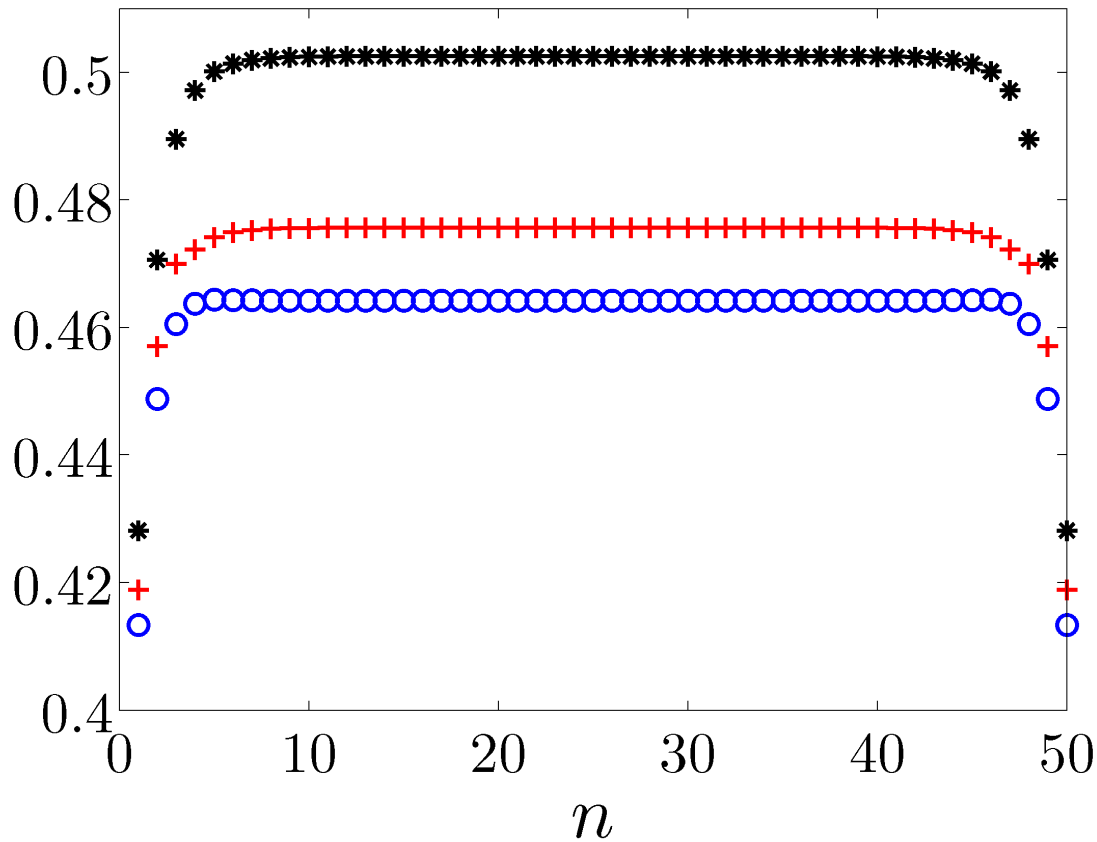
$>> \texttt{solpath} = \texttt{lqrsp}(\texttt{A}, \texttt{B1}, \texttt{B2}, \texttt{Q}, \texttt{R}, \texttt{options});$

Mass-spring system





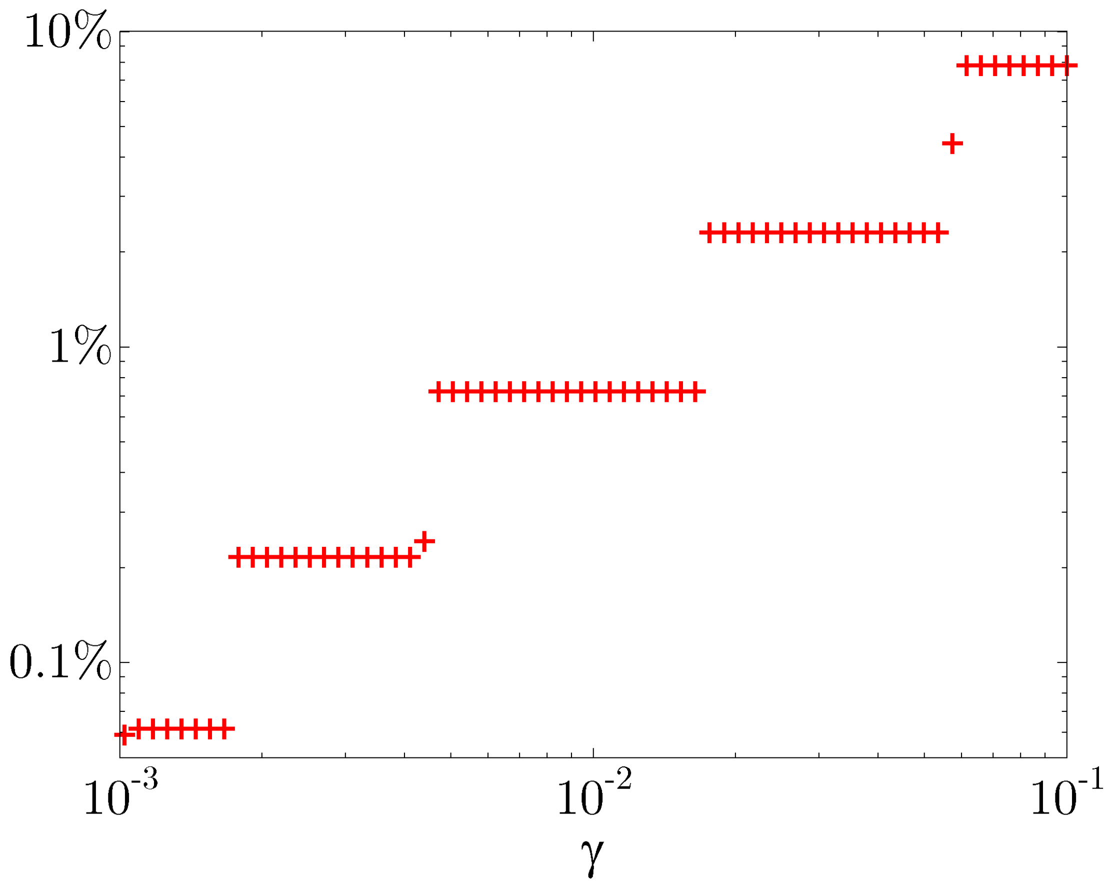
diag (F_v):



$\gamma = 10^{-4}$ $\gamma = 0.03$ $\gamma = 0.1$

- **Performance comparison: sparse vs. centralized**

$(J - J_c)/J_c$:



$\mathbf{card}\,(F)\,/\mathbf{card}\,(F_c)$	$(J - J_c)/J_c$
10%	0.75%
6%	2.4%
2%	7.8%

Part II: Summary and concluding remarks

- SPARSITY-PROMOTING OPTIMAL CONTROL
  - ⋆ Enabling tools from optimization
  - ⋆ Developed framework to identify controller architectures

Lin, Fardad, Jovanović, IEEE Trans. Automat. Control, 2012 (submitted)

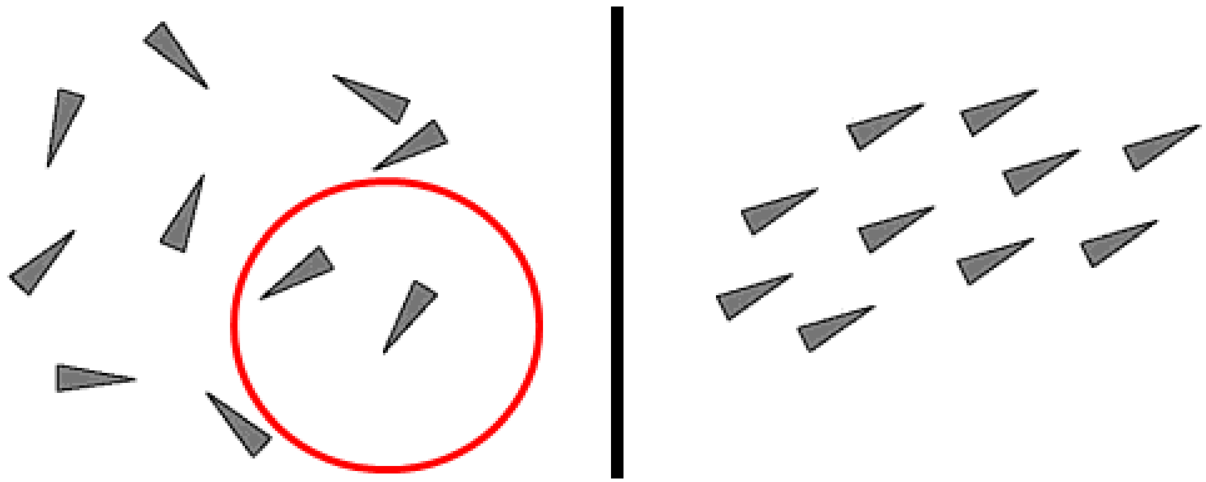
- ALSO IN THE DISSERTATION:
  - ⋆ Augmented Lagrangian approach to structured optimal design

Lin, Fardad, Jovanović, IEEE Trans. Automat. Control, 2011

- CONTRIBUTIONS:
  - ⋆ Identification of controller architectures
  - ⋆ Efficient algorithms for structured controller design
  - ⋆ Sparsity vs. performance

Sparse consensus networks

Consensus



- Update using relative differences with neighbors

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} F_{ij}\,(x_i - x_j)$$

Undirected connected graphs:

$$F = F^T \qquad F + \frac{1}{N}\mathbb{1}\mathbb{1}^T \succ 0 \qquad F\mathbb{1} = 0$$

Reaching consensus: $\lim_{t \to \infty} x_i(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$

Stochastically forced consensus networks

$$\begin{aligned} \dot{x} &= -\,Fx \,+\, d \\ z &= \begin{bmatrix} Q^{1/2} \\ -R^{1/2}F \end{bmatrix} x \end{aligned}$$

$$Q\mathbb{1} \,=\, 0 \qquad\qquad Q \,+\, \frac{1}{N}\mathbb{1}\mathbb{1}^T \,\succ\, 0$$

Variance amplification $d \to z$

$$J(F) \;=\; \frac{1}{2}\,\mathrm{trace}\left(Q^{1/2}(F + \mathbb{1}\mathbb{1}^T/N)^{-1}Q^{1/2} \,+\, RF\right)$$

SDP formulation:

$$\begin{array}{ll} \underset{X,\,F}{\text{minimize}} & \dfrac{1}{2}\,\mathrm{trace}\,(X \,+\, RF) \\ \text{subject to} & \begin{bmatrix} X & Q^{1/2} \\ Q^{1/2} & F + \dfrac{1}{N}\mathbb{1}\mathbb{1}^T \end{bmatrix} \succeq 0 \\ & F\mathbb{1} \,=\, 0 \end{array}$$

Design of sparse consensus networks

$$\begin{array}{ll} \text{minimize} & J(F) \;+\; \gamma \sum_{i,\,j} W_{ij} |F_{ij}| \\ \text{subject to} & F\mathbb{1} \;=\; 0 \qquad F \;+\; \dfrac{1}{N}\mathbb{1}\mathbb{1}^T \;\succ\; 0 \end{array}$$

SDP formulation:

$$\begin{array}{ll} \underset{X,\,Y,\,F}{\text{minimize}} & \dfrac{1}{2}\,\mathrm{trace}\,(X \;+\; RF) \;+\; \gamma \sum_{i,\,j} Y_{ij} \\ \text{subject to} & \begin{bmatrix} X & Q^{1/2} \\ Q^{1/2} & F + \dfrac{1}{N}\mathbb{1}\mathbb{1}^T \end{bmatrix} \succeq 0 \\ & F\mathbb{1} \;=\; 0 \\ & -Y \;\leq\; W \circ F \;\leq\; Y \end{array}$$

Design of structured optimal consensus networks

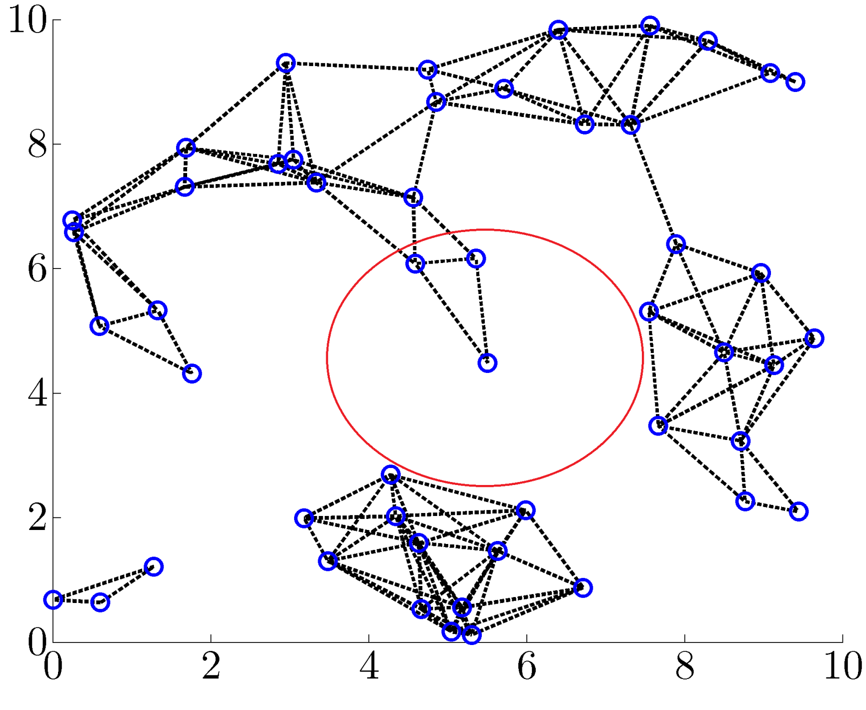
$$\begin{array}{ll} \text{minimize} & J(F) \\ \text{subject to} & F\mathbb{1} \;=\; 0 \quad F \,+\, \dfrac{1}{N}\mathbb{1}\mathbb{1}^T \succ 0 \quad F \,\in\, \mathcal{S} \end{array}$$

SDP formulation:

$$\begin{array}{ll} \underset{X,\,F}{\text{minimize}} & \dfrac{1}{2}\,\mathrm{trace}\,(X \,+\, RF) \\ \text{subject to} & \begin{bmatrix} X & Q^{1/2} \\ Q^{1/2} & F + \dfrac{1}{N}\mathbb{1}\mathbb{1}^T \end{bmatrix} \succeq 0 \\ & F\mathbb{1} \;=\; 0 \\ & F \circ I_{\mathcal{S}}^c \;=\; 0 \end{array}$$

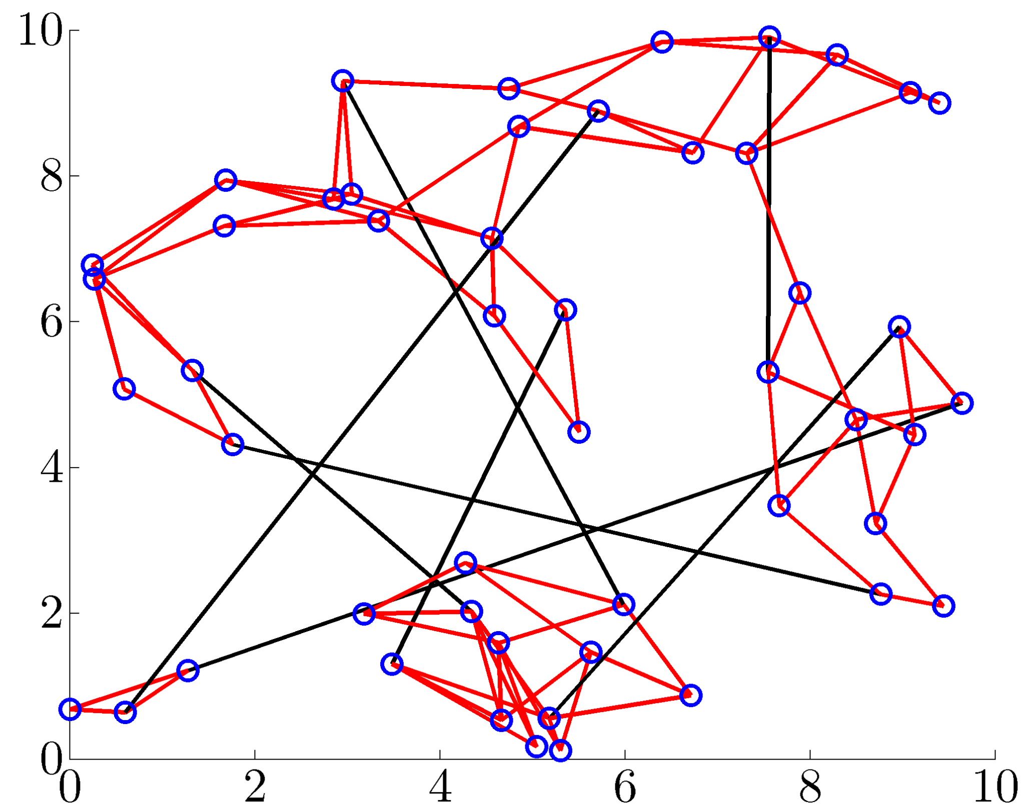
An example

local performance graph:

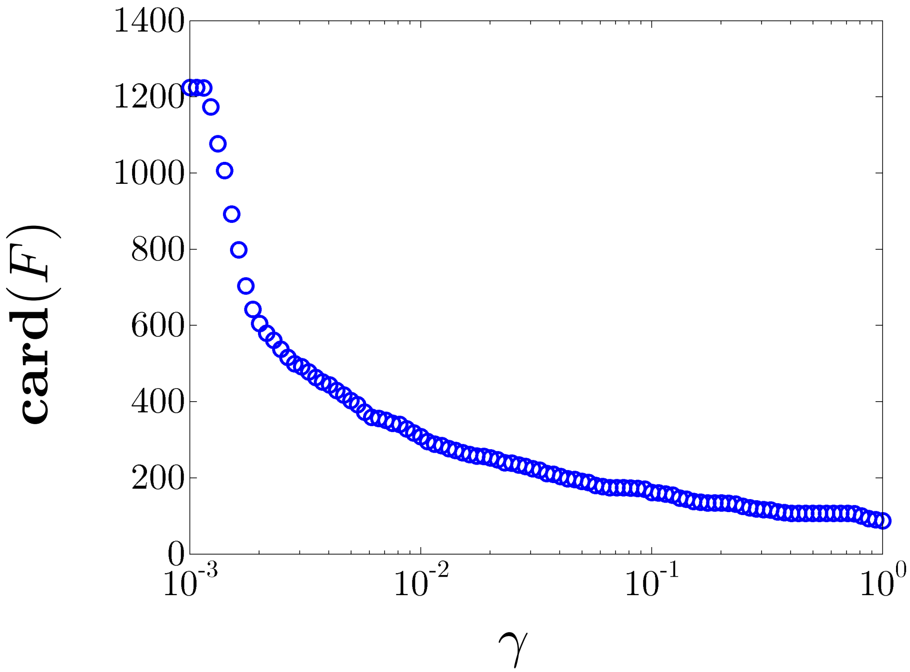


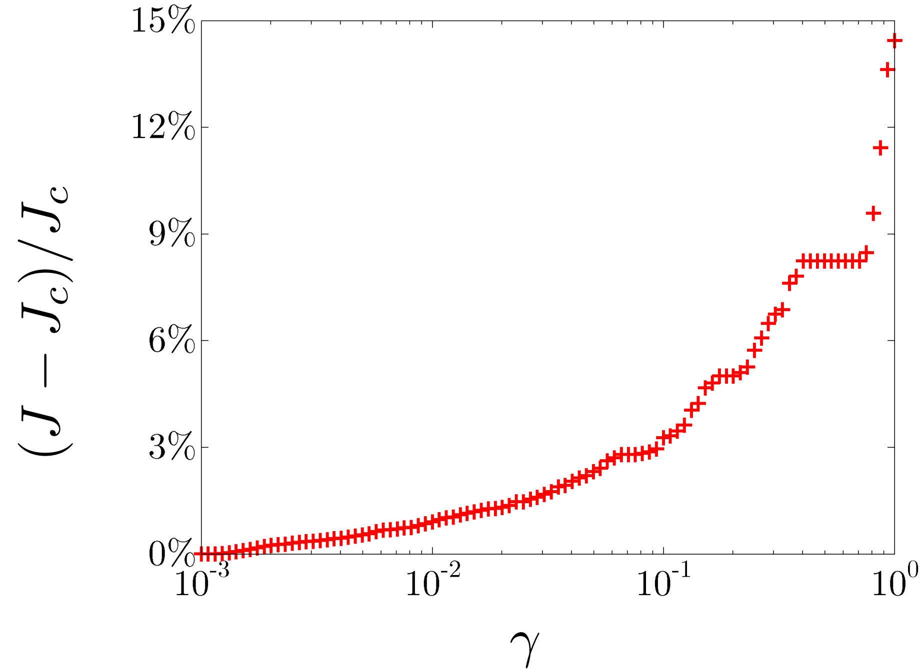
$$z = \begin{bmatrix} Q_{\mathrm{loc}}^{1/2} \\ I - \frac{1}{N}\mathbb{1}\mathbb{1}^T \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} u$$

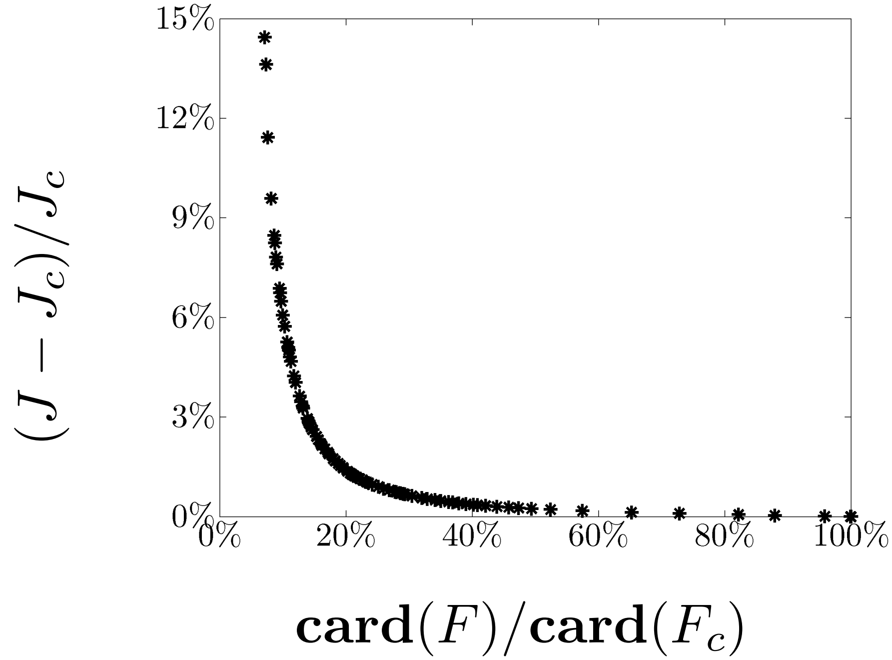
identified communication graph:



$$\begin{aligned} \mathbf{card}\,(F)\,/\,\mathbf{card}\,(F_{\mathrm{c}}) &= 7\% \\ (J - J_{\mathrm{c}})\,/\,J_{\mathrm{c}} &= 14\% \end{aligned}$$


card(F)
γ
1400
1200
1000
800
600
400
200
0
10-3
10-2
10-1
100


(J − Jc)/Jc
γ
15%
12%
9%
6%
3%
0%
10-3
10-2
10-1
100


(J − Jc)/Jc
card(F)/card(Fc)
15%
12%
9%
6%
3%
0%
0%
20%
40%
60%
80%
100%

Part III: Summary and concluding remarks

- SPARSE CONSENSUS NETWORKS
  - ⋆ Building on sparsity-promoting optimal control framework
  - ⋆ Identify convex problems – SDP formulations

Lin, Fardad, Jovanović, Allerton 2012

- ALSO IN THE DISSERTATION:
  - ⋆ Design of structured optimal consensus networks
  - ⋆ Efficient algorithms by exploiting structures of graph Laplacian

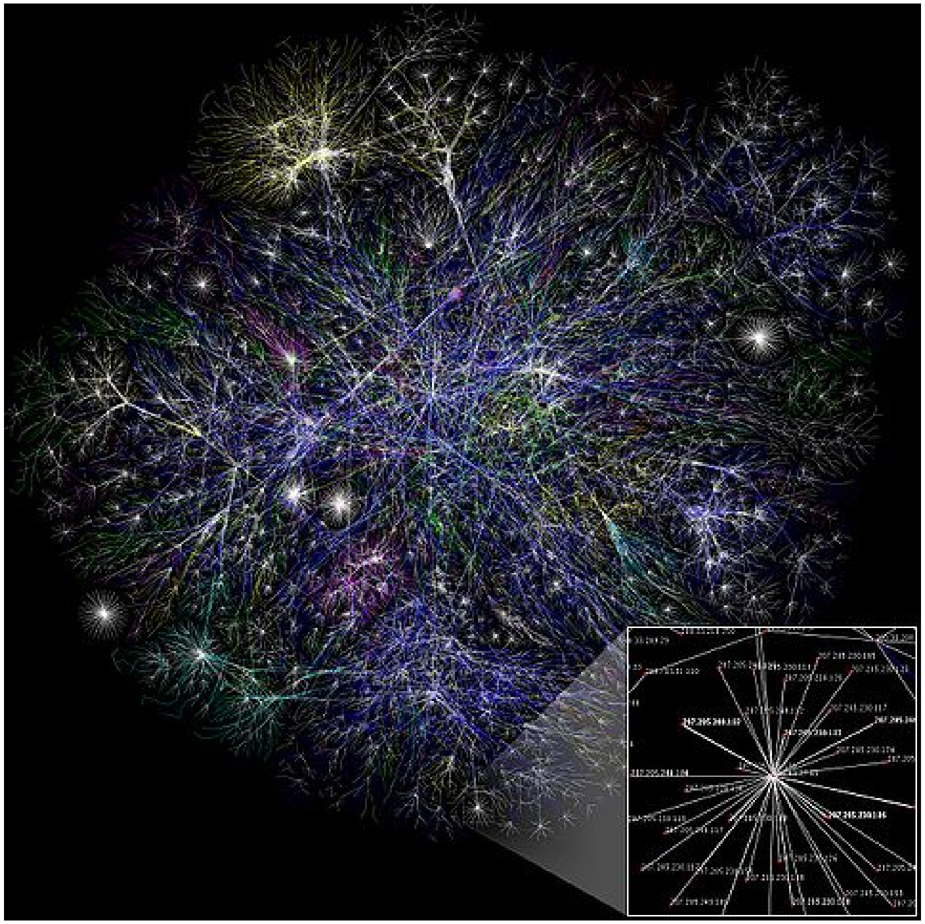
Lin, Fardad, Jovanović, IEEE CDC 2010

- EXTENSIONS:
  - ⋆ Directed trees and lattices – exploiting lower triangular structure

Lin, Fardad, Jovanović, IEEE CDC 2012

ALGORITHMS FOR LEADER SELECTION

Hierarchical structure of Internet



Opte project (www.opte.org)

- Resilient to random failures
- Vulnerable to removal of high degree nodes

Northeast blackout 2003

before:

after:

- Caused by a SINGLE power plant (at Cleveland) going offline

Removal of key nodes can affect performance and survival of networks

Leader selection problem

- CHALLENGE:

  ⋆ Combinatorial optimization problem

- APPROACH:

  ⋆ Convex relaxation $\Rightarrow$ lower bound

  ⋆ Greedy algorithm $\Rightarrow$ upper bound

- CONTRIBUTIONS:

  ⋆ Developing customized algorithm

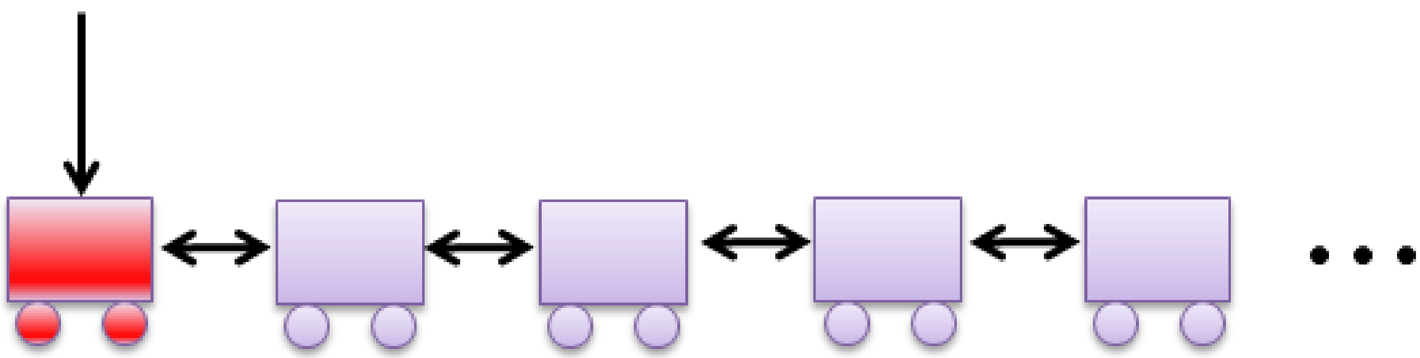
  ⋆ Exploiting structure of low-rank modifications

Leader-follower consensus dynamics

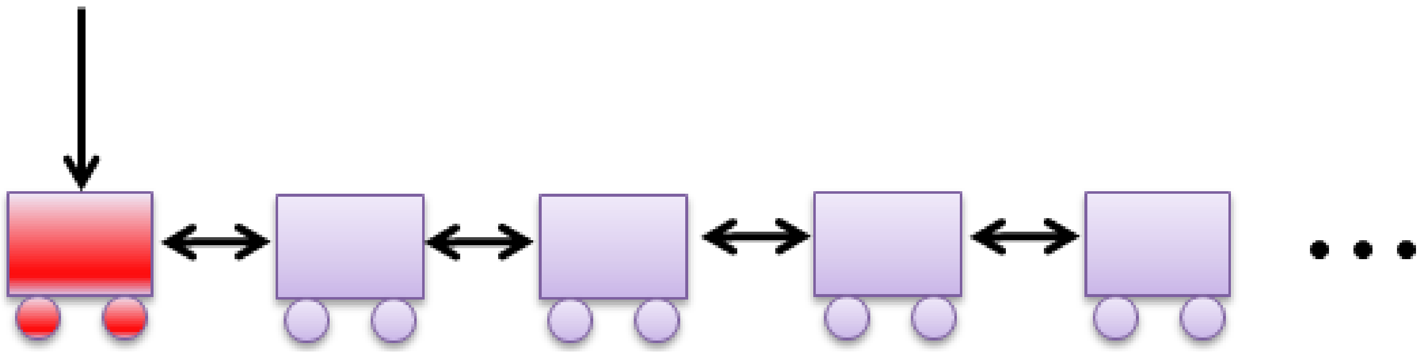
- Undirected connected networks

FOLLOWERS: $$\dot{x}_i = - \sum_{j \in \mathcal{N}_i} F_{ij} (x_i - x_j) + d_i$$

LEADERS: $$\dot{x}_i = - \sum_{j \in \mathcal{N}_i} F_{ij} (x_i - x_j) - x_i + d_i$$



Minimum variance leader selection problem



$$\dot{x} \;=\; -\,(F + \Psi)\,x \;+\; d$$

$$F \;=\; \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad \psi \;=\; \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Psi \;:=\; \mathrm{diag}\,(\psi) \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Assign k leaders to minimize steady-state variance

$$\lim_{t \to \infty} \; \mathrm{trace}\left(\mathcal{E}\{x(t)x^T(t)\}\right)$$

$$\begin{array}{lrcl} \text{minimize} & J(\psi) & = & \operatorname{trace}\left((F + \Psi)^{-1}\right) \\ \text{subject to} & \psi_i & \in & \{0,1\}, \quad i = 1,\ldots,N \\ & \mathbb{1}^T\psi & = & k \end{array}$$

- FEATURES:
  - ⋆ convex objective function, linear constraint, Boolean constraints
- DIFFICULT: **Boolean constraints**
- APPROACH:
  - ⋆ convex relaxation $\Rightarrow$ lower bound
  - ⋆ greedy algorithm $\Rightarrow$ upper bound

Convex relaxation

lower bound on J

$$\begin{array}{lrcl} \text{minimize} & J(\psi) & = & \operatorname{trace}\left((F + \Psi)^{-1}\right) \\ \text{subject to} & \psi_i & \in & [0,1], \quad i = 1,\ldots,N \\ & \mathbb{1}^T\psi & = & k \end{array}$$

SDP formulation:

$$\boxed{\begin{array}{ll} \text{minimize} & \operatorname{trace}(X) \\ \text{subject to} & \begin{bmatrix} X & I \\ I & F + \Psi \end{bmatrix} \geq 0 \\ & \psi_i \in [0,1], \quad i = 1,\ldots,N \\ & \mathbb{1}^T\psi = k \end{array}}$$

without exploiting structure: $O(N^4)$

$n \times n$ matrix and m variables, $O(\max\{mn^3, m^2n^2, m^3\})$, $n = m = N \Rightarrow O(N^4)$

$$\begin{array}{lrcl} \text{minimize} & J(\psi) & = & \operatorname{trace}\left((F + \Psi)^{-1}\right) \\ \text{subject to} & \psi_i & \in & [0,1], \quad i = 1,\ldots,N \\ & \mathbb{1}^T\psi & = & k \end{array}$$

Logarithmic barrier function:

$$\begin{array}{ll} \text{minimize} & q(\psi) = \tau \operatorname{trace}\left((F + \Psi)^{-1}\right) + \sum_{i=1}^{N}\left(-\log(\psi_i) - \log(1-\psi_i)\right) \\ \text{subject to} & \mathbb{1}^T\psi = k \end{array}$$

NEWTON DIRECTION: Form $(F + \Psi)^{-2} \quad \Rightarrow \quad O(N^3)$

Greedy algorithm

- One-leader-at-a-time

$$J_{s+1}^{i} \;=\; \operatorname{trace}\left((F_s \;+\; e_i e_i^T)^{-1}\right)$$

⋆ RANK-1 UPDATE: $\quad \Rightarrow \quad O(N^2)$ per leader

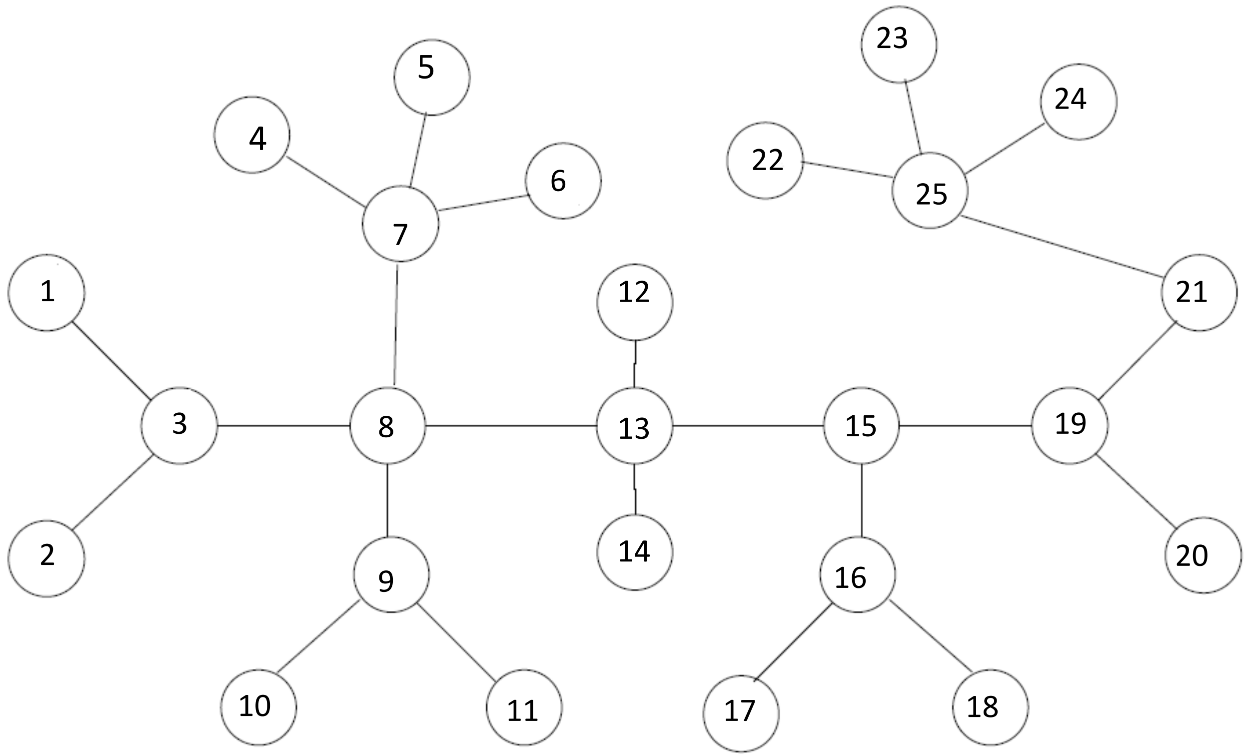
- Swap a leader and a follower

$$J_{ij} \;=\; \operatorname{trace}\left((F_k \;-\; e_i e_i^T \;+\; e_j e_j^T)^{-1}\right)$$

⋆ RANK-2 UPDATE: $\quad \Rightarrow \quad O(N^2)$ per swap

- Total cost: $O(\max\{(k+p)N^2, N^3\}), \quad k, p \ll N \quad \Rightarrow \quad O(N^3)$

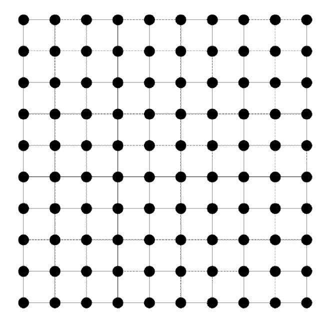
A network with 25 nodes

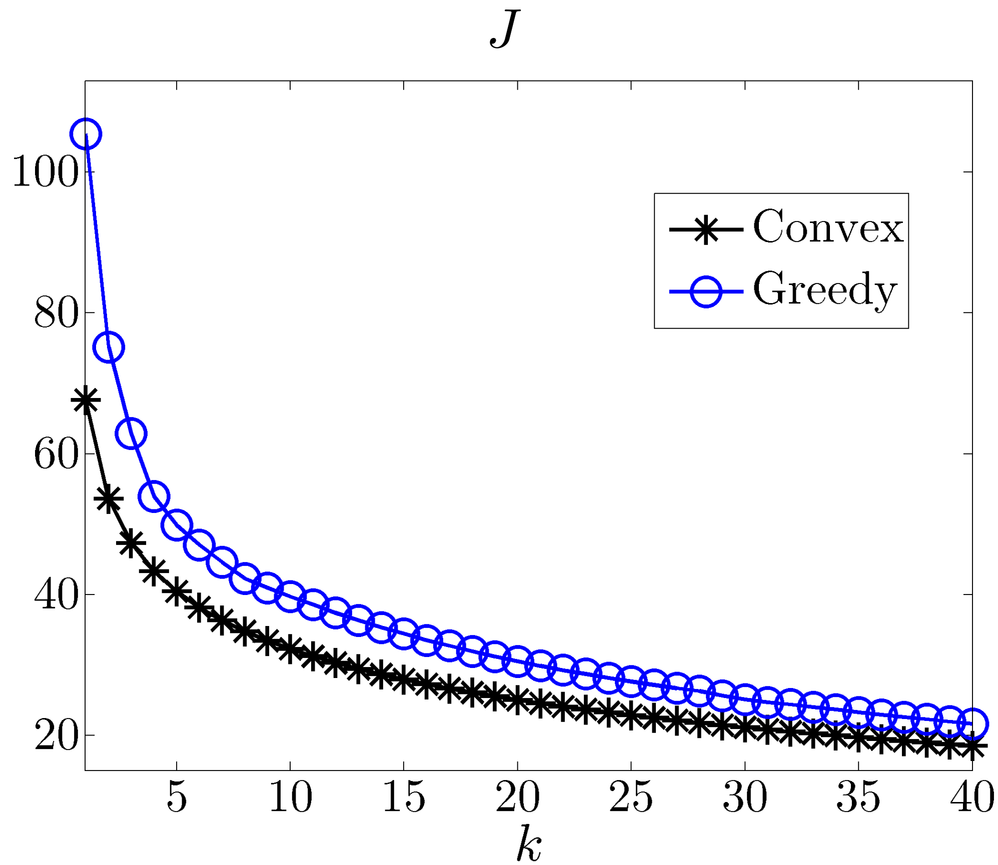


Greedy algorithm finds global solutions

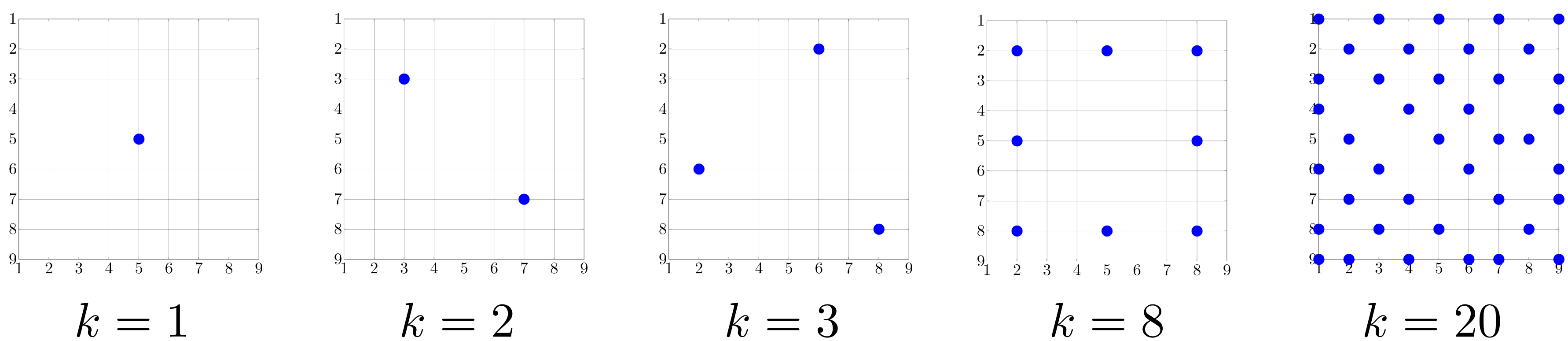
k	leaders
1	13
2	$8, 25$
3	$8, 16, 25$
4	$3, 7, 16, 25$
5	$3, 7, 9, 16, 25$

A 2D lattice

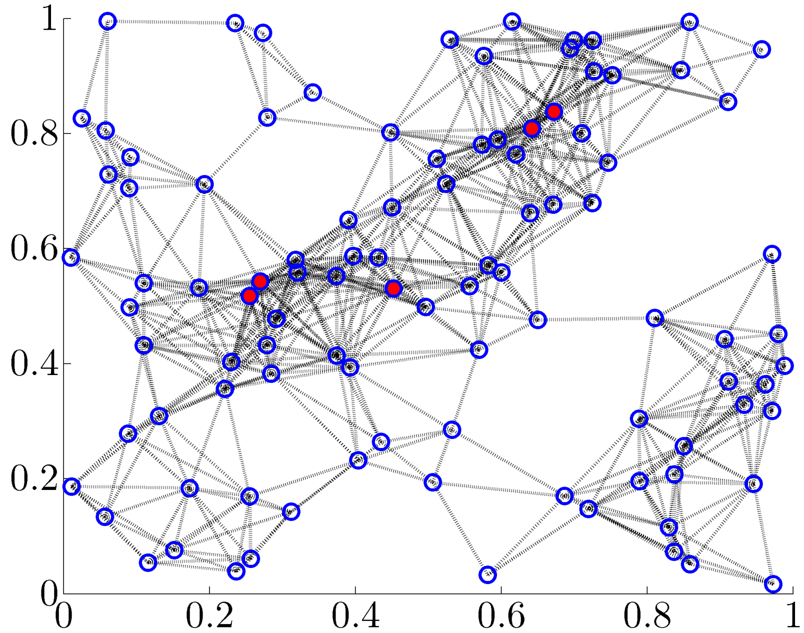




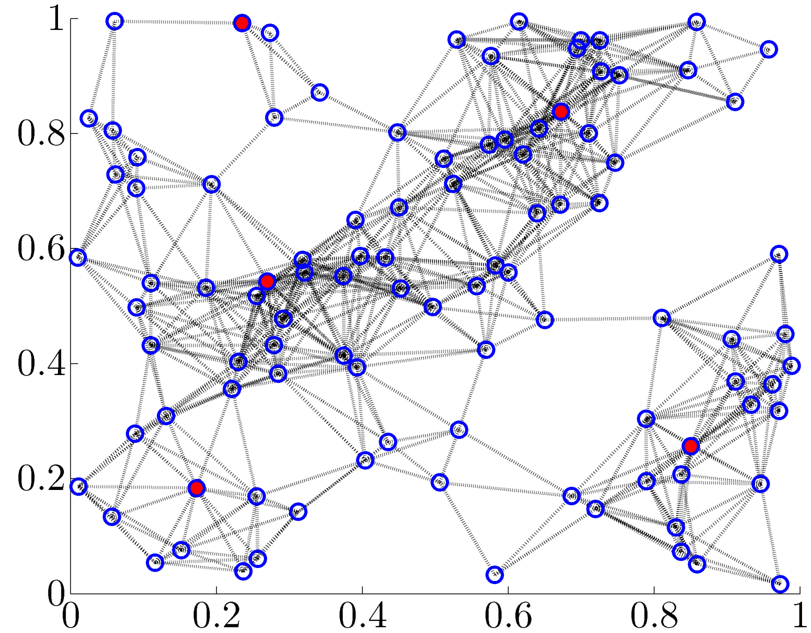
- Leaders: Greedy algorithm



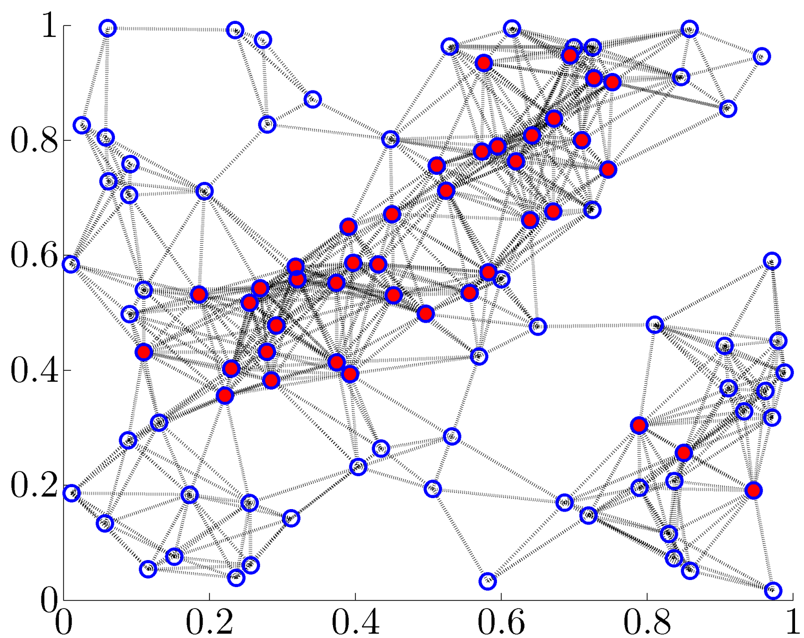
Degree-based-heuristics vs. greedy algorithm



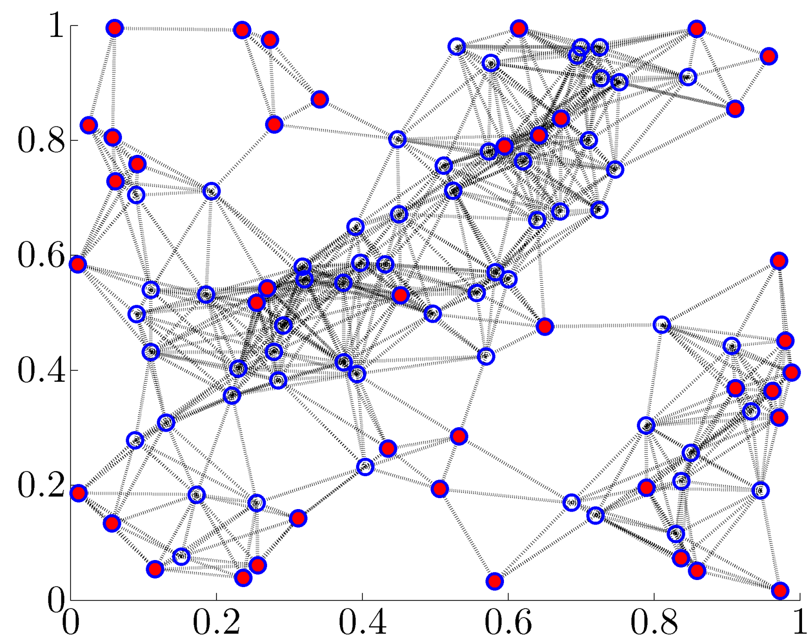
$k = 5 \quad J = 27.7$



$k = 5 \quad J = 15.0$



$k = 40 \quad J = 19.0$



$k = 40 \quad J = 9.5$

Part IV: Summary and concluding remarks

- ALGORITHMS FOR LEADER SELECTION PROBLEM

- CHALLENGE: combinatorial optimization problem

- APPROACH:

  ⋆ Convex relaxation $\Rightarrow$ lower bound

  ⋆ Greedy algorithm $\Rightarrow$ upper bound

- ALSO IN THE DISSERTATION:

  ⋆ Noise-free leader selection problem

    ∗ Identify the source of nonconvexity in objective function

    ∗ Propose LMI-based relaxation

Lin, Fardad, Jovanović, IEEE Trans. Automat. Control, 2012 (submitted)

Acknowledgements