Performance of leader-follower networks in directed trees and lattices

Fu Lin

joint work with:

Makan Fardad

Mihailo Jovanović

UNIVERSITY OF MINNESOTA

51st IEEE CDC, Dec. 2012

Outline

- A motivating example from vehicular formations

- Convexity: undirected vs. directed networks

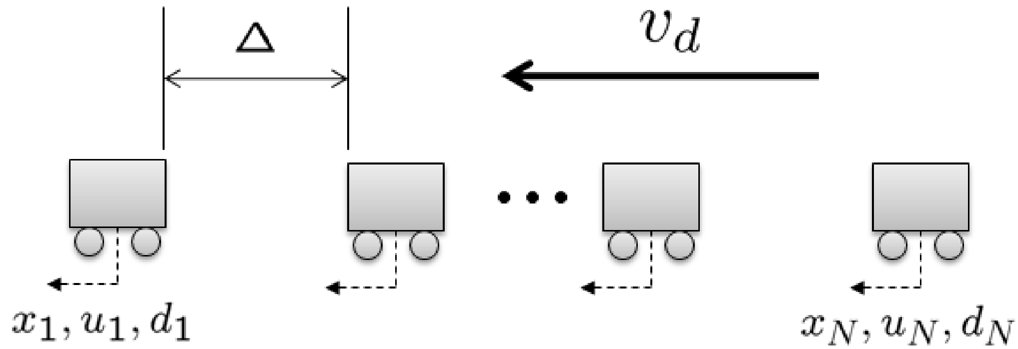
- Lower triangular structure in directed trees and lattices

- A case study: variance distribution in 2D lattices

Vehicular formations

AUTOMATED CONTROL OF EACH VEHICLE

tight spacing at desired speeds



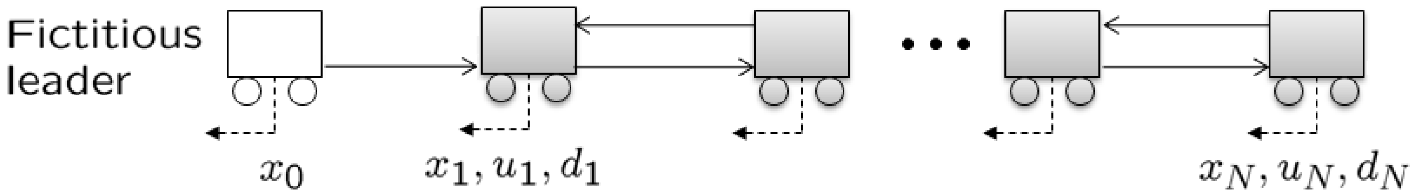
SINGLE INTEGRATOR MODEL

$$\dot{x}_n = \underset{\substack{\uparrow \\ \text{control}}}{u_n} + \underset{\substack{\uparrow \\ \text{disturbance}}}{d_n}$$

- Desired trajectory: $\begin{cases} \bar{x}_n := v_d\, t + n\Delta \\ \text{constant velocity } v_d \end{cases}$

- Deviations: $\left.\begin{array}{lll} \tilde{x}_n & := & x_n - \bar{x}_n \\ \tilde{u}_n & := & u_n - v_d \end{array}\right\} \Rightarrow \dot{\tilde{x}}_n = \tilde{u}_n + d_n$

Nearest neighbor interactions with symmetric gains

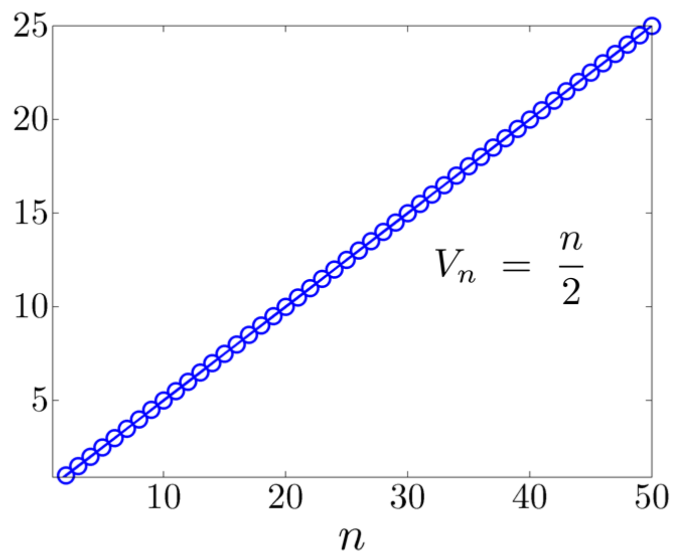


Relative position feedback:

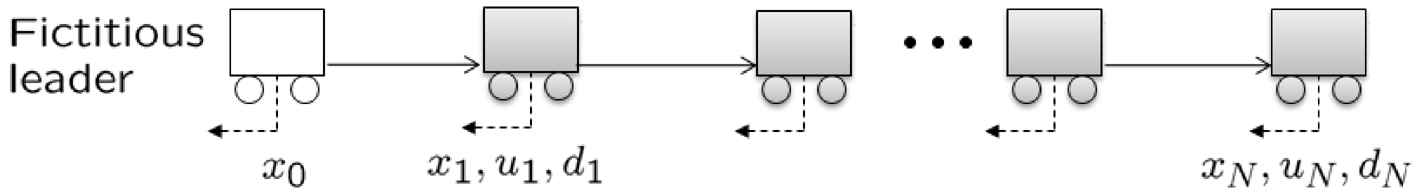
$$\tilde{u}_n \;=\; -\,(\tilde{x}_n - \tilde{x}_{n-1}) \;-\; (\tilde{x}_n - \tilde{x}_{n+1})$$

$$\dot{\tilde{x}} \;=\; -\,K\tilde{x} \;+\; d, \qquad K \;\sim\; \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Steady-state variance: $V_n \;:=\; \lim_{t \to \infty} \mathcal{E}\{\tilde{x}_n^2(t)\}$



Nearest neighbor interactions with non-symmetric gains



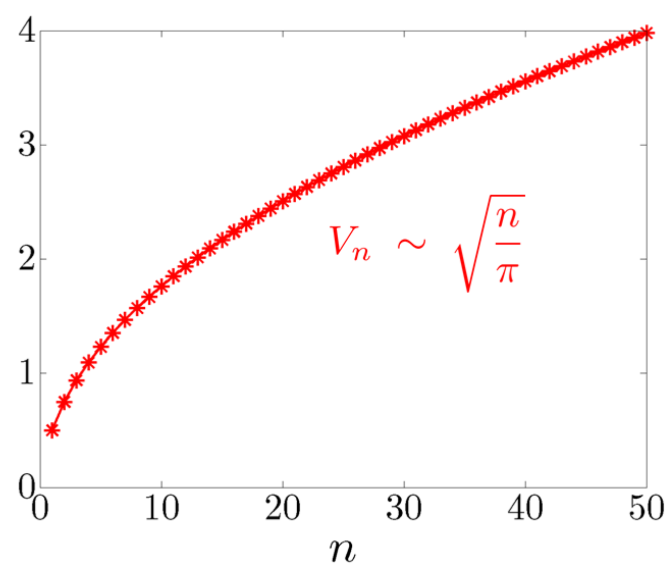
Relative position feedback:

$$\tilde{u}_n = -\left(\tilde{x}_n - \tilde{x}_{n-1}\right)$$

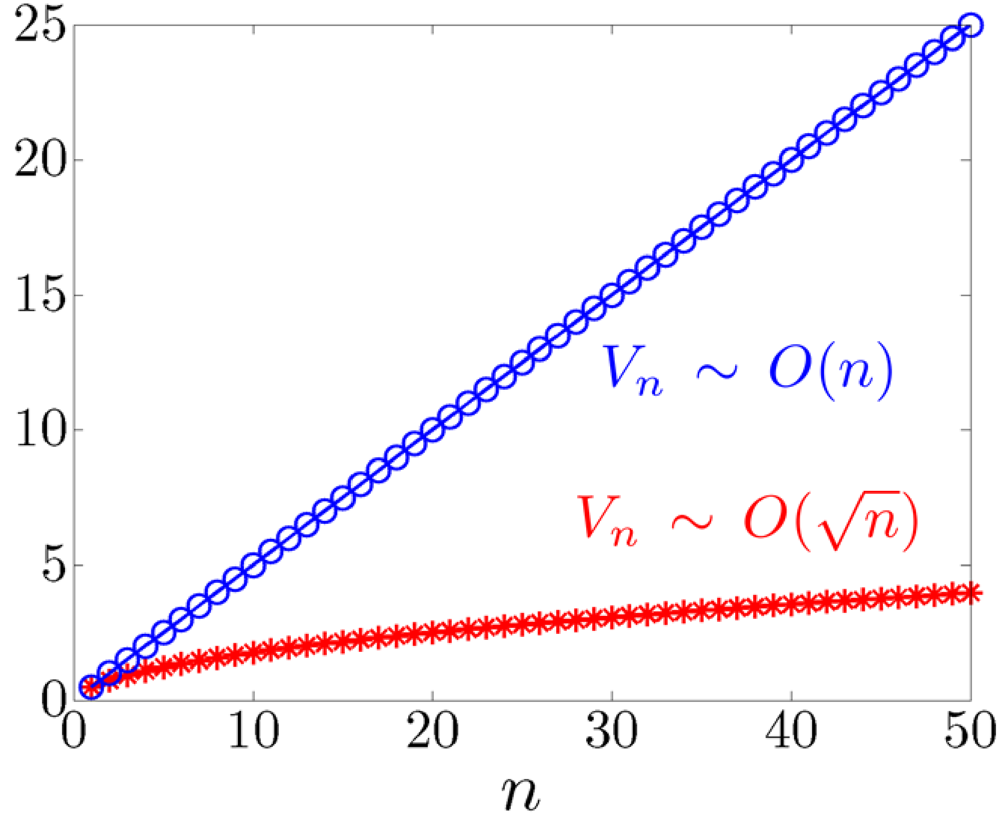
$$\dot{\tilde{x}} = -K\tilde{x} + d, \qquad K \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Steady-state variance:

$$V_n := \lim_{t \to \infty} \mathcal{E}\{\tilde{x}_n^2(t)\}$$



Symmetric gains vs. non-symmetric gains



Lin, Fardad, Jovanović, TAC 2012

- Departure from symmetric controllers improves performance

Barooah, Mehta, Hespanha, TAC 2009

Young, Scardovi, Leonard, ACC 2010

Hao and Barooah, ACC 2012

Leader-follower networks: undirected vs. directed

Variance amplification $d \to \tilde{x}$:

$$\dot{\tilde{x}} \;=\; -\,K\tilde{x} \,+\, d$$

Undirected networks:

$$\boxed{J(K) \;=\; \frac{1}{2}\,\mathrm{trace}\left(K^{-1}\right)}$$

CONVEX FOR $K \;=\; K^T \;\succ\; 0$

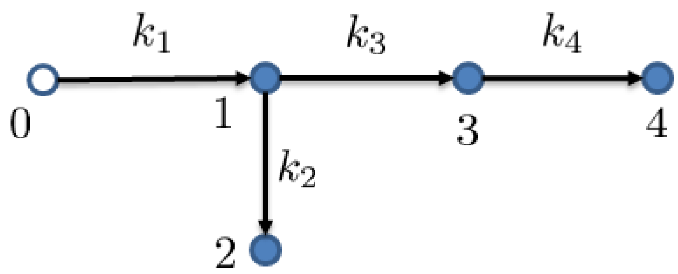
Directed networks:

$$\boxed{J(K) \;=\; \mathrm{trace}\left(\int_0^{\infty} \mathrm{e}^{-\,Kt}\,\mathrm{e}^{-\,K^T t}\,\mathrm{d}t\right)}$$

NONCONVEX FUNCTION

- APPROACH:
  - ⋆ Focus on classes of graphs – rooted trees and lattices
  - ⋆ Exploit lower triangular structure of K

Directed rooted trees



$$K = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ -k_2 & k_2 & 0 & 0 \\ -k_3 & 0 & k_3 & 0 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

Transfer function matrix $d \to \tilde{x}$

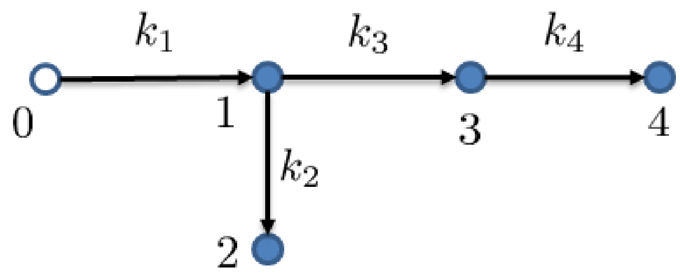
$$H(s) = (sI + K)^{-1}$$ lower triangular matrix

Transfer function $d_j \to \tilde{x}_i$:

$$\boxed{H_{ij}(s) = \frac{1}{k_j}\left(\prod_l \frac{k_l}{s + k_l}\right)}$$

$l \sim$ path $j \to i$

$$d_1 \to \tilde{x}_4 : \qquad H_{41}(s) = \frac{1}{k_1} \times \frac{k_1}{s + k_1} \times \frac{k_3}{s + k_3} \times \frac{k_4}{s + k_4}$$



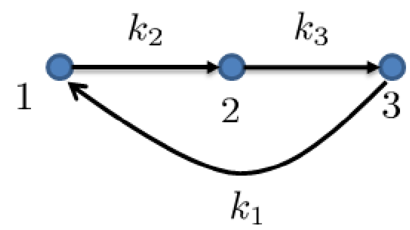
- Elementwise variance amplification $d_j \to \tilde{x}_i$

$$V_{ij} \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_{ij}(\mathrm{j}\omega)|^2 \, \mathrm{d}\omega$$

- Worst-case amplification achieved at $\omega = 0$

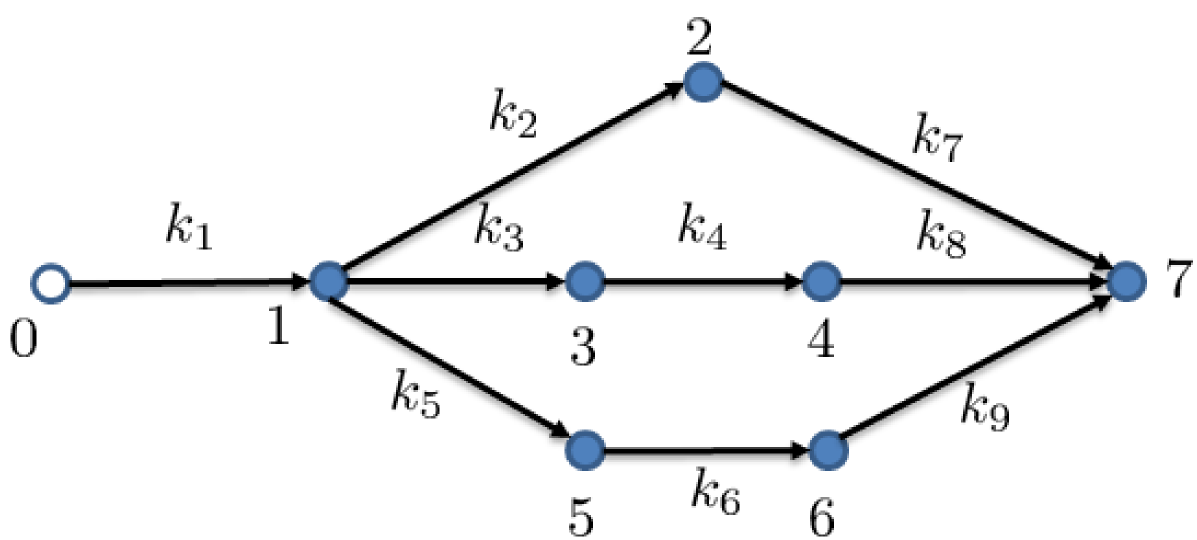
$$\max_{\omega} \; |H_{ij}(\mathrm{j}\omega)|^2 \;=\; \max_{\omega} \; \frac{1}{k_j^2} \left(\prod_{l} \frac{k_l^2}{\omega^2 + k_l^2} \right) \;=\; \frac{1}{k_j^2}$$

Directed networks without cycles



- Networks without cycles $\Rightarrow$ K is still lower triangular

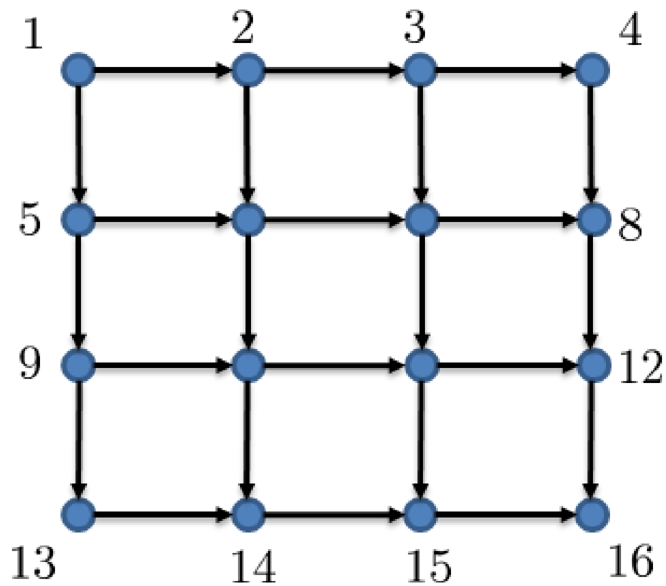
Multiple directed paths



Enumerate all paths $j \to i$

$$H_{ij}(s) = \sum_l H_{ij}^l(s)$$

- Difficulty: number of paths could grow exponentially



$n \times n$ 2D lattice

# of paths from $(1,1) \to (n,n)$

$> \ 2^{n-1}$

- Sparse directed graphs

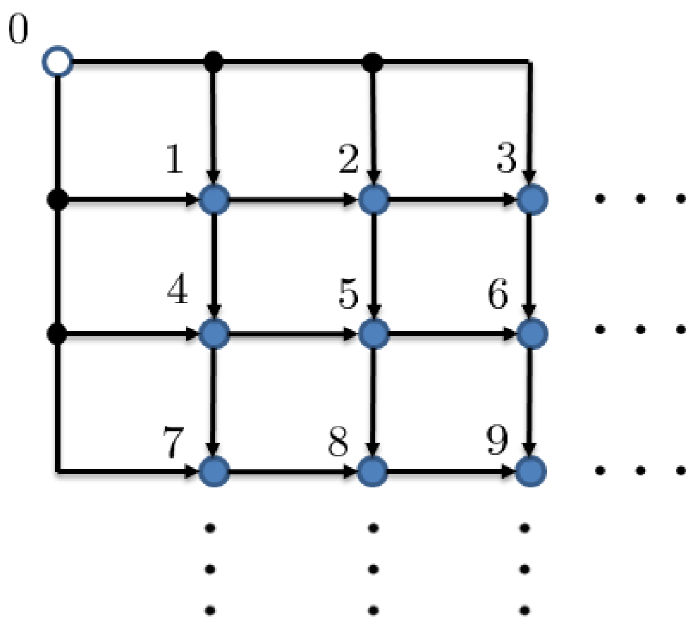
Next: a case study without enumerating all paths

1D and 2D lattices

Toeplitz matrix

$$K \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Preserve lattice structure

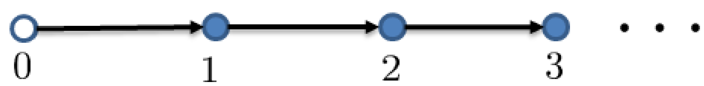


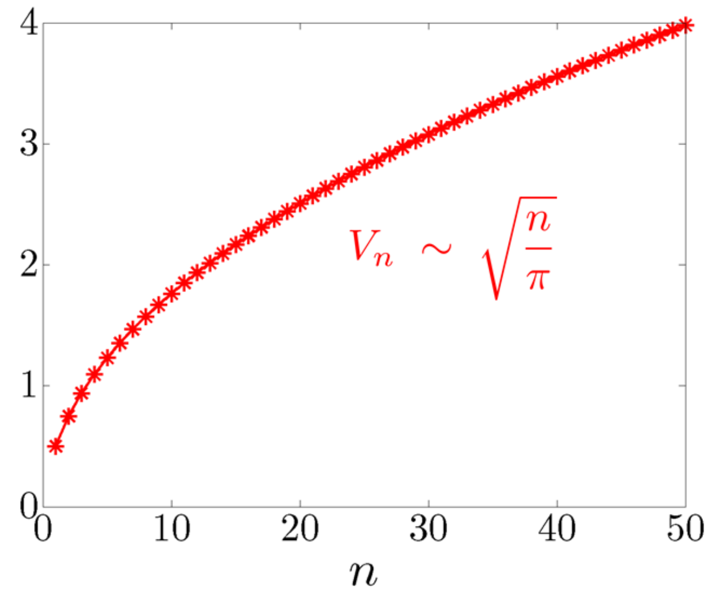
Block Toeplitz matrix

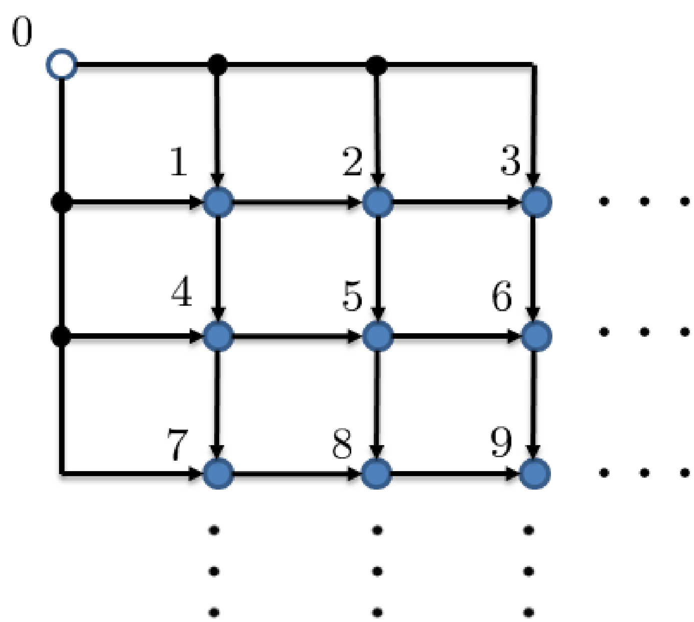
$$K \sim \begin{bmatrix} L & 0 & 0 & 0 \\ -I & L & 0 & 0 \\ 0 & -I & L & 0 \\ 0 & 0 & -I & L \end{bmatrix}$$

$$L \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

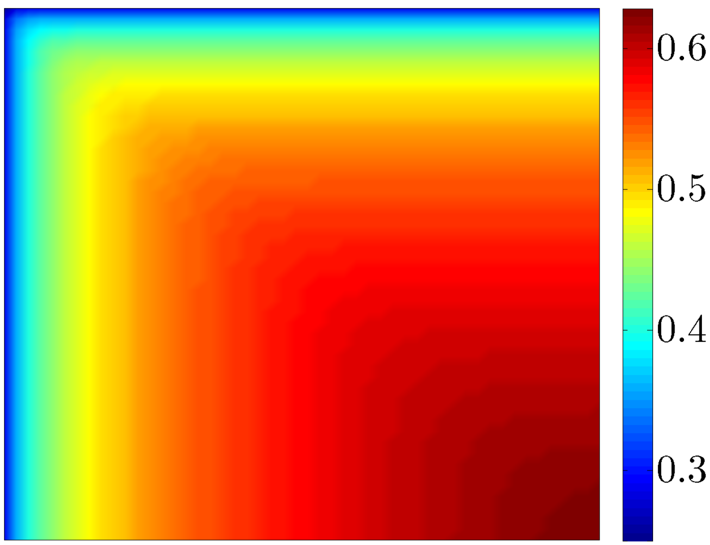
Variance distribution in directed lattices







Variance distribution:



Analytical expressions for variance; diagonal of 2D lattice scales as $O(\log(n))$

Sketch of calculations: 1D case

$$P = \int_0^\infty \mathrm{e}^{-Kt} \mathrm{e}^{-K^T t} \, \mathrm{d}t$$

- Form the lower triangular Toeplitz $(sI + K)^{-1}$

$$(sI + K)^{-1} \sim \begin{bmatrix} (s+1)^{-1} & 0 & 0 \\ (s+1)^{-2} & (s+1)^{-1} & 0 \\ (s+1)^{-3} & (s+1)^{-2} & (s+1)^{-1} \end{bmatrix}$$

- Compute the inverse Laplace transform

$$\mathcal{L}^{-1}\{(s+1)^{-i}\} = \frac{t^{i-1} \mathrm{e}^{-t}}{(i-1)!}$$

- Integrate and compute the summation

$$P_{nn} = \frac{n^2 (2n-1)!}{2^{2n-1} (n!)^2}$$

- Use Stirling's formula to approximate $n!$

$$P_{nn} \approx \sqrt{\frac{n}{\pi}}$$

- Similar but more involved calculations for 2D case
  - ⋆ Use combinatorial formulas in the study of random walks

Doyle and Snell 1984

Concluding remarks

- Directed networks improves performance compared to undirected counterparts
- Focused on directed trees and lattices to exploit lower triangular structure
- Variance distribution in 2D lattices – logarithmic scaling along diagonal

On-going work:

- ⋆ Variance distribution in higher dimension lattices
- ⋆ Connections to social networks