Algorithms for leader selection in consensus networks

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joint work with:

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Overview

Optimal control of dynamical systems on networks

Main topics:

- Localized control of vehicular formations
- Sparsity-promoting optimal control
- Sparse consensus networks
- Algorithms for leader selection in consensus networks
Localized control of vehicular formations

Sparsity-promoting optimal control

Sparse consensus networks

Algorithms for leader selection
• **CHALLENGES:**

  ★ Networks – **combinatorial** objects

  ★ Optimization – **constrained nonconvex** problems

• **APPROACH:**

  ★ Identify classes of **convex** problems

  ★ Exploit **problem structure** to develop efficient algorithms
In this talk

- Leader selection in consensus networks
- Applications in vehicular formations and sensor localization
- Algorithms for lower and upper bounds on global solutions
- Examples from regular lattices and random networks
Leader-follower consensus dynamics

- Time-invariant undirected connected networks

\[ Follower: \quad \dot{\psi}_i(t) = - \sum_{j \in \mathcal{N}_i} (\psi_i(t) - \psi_j(t)) + w_i(t) \]

\[ \text{Leader:} \quad \dot{\psi}_i(t) = 0 \]

\[
\begin{bmatrix}
\dot{\psi}_l(t) \\
\dot{\psi}_f(t)
\end{bmatrix} = - \begin{bmatrix}
0 & 0 \\
L_0 & L_f
\end{bmatrix} \begin{bmatrix}
\psi_l(t) \\
\psi_f(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
w(t)
\end{bmatrix}
\]

Variance of followers depend on network structure and locations of leaders
Leader selection problem

- Select $N_l$ leaders to minimize variance of followers

\[
\begin{align*}
\text{minimize} \quad & J_f(x) = \text{trace}(L_f^{-1}) \\
\text{subject to} \quad & x_i \in \{0, 1\}, \quad i = 1, \ldots, n \\
& 1^T x = N_l \\
& x_i \in \{0, 1\}, \quad 1 \quad \text{LEADER,} \quad 0 \quad \text{FOLLOWER}
\end{align*}
\]

\[
L = \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}, \quad x = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}, \quad L_f = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}
\]
Connections to sensor localization problem

**Goal:** Estimate $n$ sensor positions in 1D

Relative measurements corrupted by noise

$$ y_k = \psi_i - \psi_j + w_k $$

Anchor nodes with known positions $\psi_l$

$$ y = E^T \psi + w $$

$$ = \begin{bmatrix} E_l \\ E_f \end{bmatrix}^T \begin{bmatrix} \psi_l \\ \psi_f \end{bmatrix} + w $$

In this talk: $\mathcal{E}(ww^T) = I$
Laplacian of measurement graph

\[ L = EE^T = \begin{bmatrix} E_lE_l^T & E_lE_f^T \\ E_fE_l^T & E_fE_f^T \end{bmatrix} = \begin{bmatrix} L_l & L_l^T \\ L_0 & L_f \end{bmatrix} \]

Minimum variance estimation

\[ \hat{\psi}_f = (E_fE_f^T)^{-1}E_f(y - E_l^T\psi_l) \]

Covariance of estimation error \( \psi_f - \hat{\psi}_f \)

\[ \Sigma = (E_fE_f^T)^{-1} = L_f^{-1} \]
• Select \( N_l \) anchors to minimize variance of estimation error

\[
\begin{align*}
\text{minimize} & \quad J_f(x) = \text{trace} \left( L_f^{-1} \right) \\
\text{subject to} & \quad x_i \in \{0, 1\}, \quad i = 1, \ldots, n \\
& \quad 1^T x = N_l
\end{align*}
\]

\( x_i \in \{0, 1\}, \quad 1 \text{ – ANCHOR,} \quad 0 \text{ – UNKNOWN SENSOR} \)

• Other applications via the interpretation of effective resistance
Related work

- Greedy algorithms with approximations
  Patterson and Bamieh ’10

- Submodular optimization with performance guarantees
  Clark and Poovendran ’11
  Clark, Bushnell, and Poovendran ’12, ’13, . . .

- Semidefinite programming for related sensor selection problem
  Joshi and Boyd ’09

- A large literature on controllability of leader-follower networks
  Tanner ’04
  Liu, Chu, Wang, and Xie ’08
  Rahmani, Ji, Mesbahi, and Egerstedt ’09
  Clark, Bushnell, and Poovendran ’12
  Kawashima and Egerstedt ’12, . . .
In this talk

- Related **noise-corrupted** leader selection problem

- Efficient algorithms for bounds on global optimal value
  - **Convex relaxations** – lower bounds
  - **Greedy algorithms** – upper bounds (exploiting low-rank structure)

- Examples from regular lattices and random networks
Noise-corrupted leader selection

- Arise in several applications

- Give insights to noise-free leader selection

- Easier to solve ;-)
Noise-corrupted leaders

- Undirected connected networks

FOLLOWERS: $\dot{\psi}_i(t) = - \sum_{j \in \mathcal{N}_i} (\psi_i(t) - \psi_j(t)) + w_i(t)$

LEADERS: $\dot{\psi}_i(t) = - \sum_{j \in \mathcal{N}_i} (\psi_i(t) - \psi_j(t)) - \alpha \psi_i(t) + w_i(t)$

$\alpha > 0$

Leaders have GPS devices
Diagonally strengthened Laplacian matrix

\[
\dot{\psi}(t) = - (L + \alpha \text{diag}(x)) \psi(t) + w(t)
\]

\[x_i \in \{0, 1\}, \quad 1 - \text{LEADER}, \quad 0 - \text{FOLLOVER}\]

\[
L = \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}, \quad \text{diag}(x) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Noise-corrupted leader selection

- Select $N_l$ leaders to minimize variance of the network

$$\begin{align*}
\text{minimize} \quad & J(x) = \text{trace}\left((L + \alpha \text{diag}(x))^{-1}\right) \\
\text{subject to} \quad & x_i \in \{0, 1\}, \quad i = 1, \ldots, n \\
& 1^T x = N_l
\end{align*}$$

Recover the noise-free formulation $\alpha \to \infty$

$$\begin{bmatrix}
\psi_l \\
\psi_f
\end{bmatrix} \cdot \begin{bmatrix}
L_l + \alpha I & L_0^T \\
L_0 & L_f
\end{bmatrix}^{-1} \to \begin{bmatrix}
0 & 0 \\
0 & L_f^{-1}
\end{bmatrix}$$
Connections to sensor localization problem

Goal: Estimate sensor positions $\psi \in \mathbb{R}^n$

- Relative measurements $y_k = \psi_i - \psi_j + w_k$

- Absolute measurements $y_i = \psi_i + \frac{1}{\alpha} w_i$

Select $N_i$ absolute measurements to minimize variance of estimation error
Algorithms for noise-corrupted formulation

\[
\begin{align*}
\text{minimize} & \quad J(x) = \text{trace} \left( (L + \alpha \text{diag}(x))^{-1} \right) \\
\text{subject to} & \quad x_i \in \{0, 1\}, \quad i = 1, \ldots, n \\
& \quad 1^T x = N_l
\end{align*}
\]

- **Feature**: Convex objective function
- **Difficult**: Boolean constraints
- **Approach**:
  - Convex relaxation \(\Rightarrow\) lower bound
  - Greedy algorithm \(\Rightarrow\) upper bound
Convex relaxation

\[
\begin{align*}
\text{minimize} \quad J(x) &= \text{trace} \left( (L + \alpha \text{diag}(x))^{-1} \right) \\
\text{subject to} \quad &x_i \in [0, 1], \quad i = 1, \ldots, n \\
&\mathbb{1}^T x = N_l
\end{align*}
\]

Enlarge feasible set \implies lower bound

- SDP formulation with complexity \( O(n^4) \) — number of nodes
- Customized interior point method \( O(n^3) \)
Greedy algorithm

- One-leader-at-a-time

\[ L + \alpha e_i e_i^T \]

★ **RANK-1 UPDATE:** \( O(n^2) \) per leader

Complexity: \( N_l \ll n \Rightarrow O(n^3) \) one matrix inverse

After selecting \( N_l \) leaders

- Swap a leader and a follower

\[ L - \alpha e_i e_i^T + \alpha e_j e_j^T \]

★ **RANK-2 UPDATE:** \( O(n^2) \) per swap
Recap

\[
\begin{align*}
\text{minimize} \quad J(x) &= \text{trace} \left( (L + \alpha \text{diag}(x))^{-1} \right) \\
\text{subject to} \quad x_i &\in \{0, 1\}, \quad i = 1, \ldots, n \\
1^T x &= N_l
\end{align*}
\]

- Convex relaxation \implies lower bound
  - Semidefinite program $O(n^4)$
  - Customized interior point method $O(n^3)$

- Greedy algorithm \implies upper bound
  - Without exploiting structure $O(n^4 N_l)$
  - Low rank updates $O(n^3)$
A random network with 100 nodes
Number of leaders $N_l$

Gap between bounds

- Upper bounds
- Lower bounds
Degree heuristics vs. greedy algorithm

The graph compares the variance of both the greedy algorithm and degree heuristics as a function of the number of leaders $N_l$. The greedy algorithm shows a faster decrease in variance compared to degree heuristics.
Degree heuristics vs. greedy algorithm

\[ N_l = 5 \quad J = 27.8 \]

\[ N_l = 5 \quad J = 19.0 \]

\[ N_l = 40 \quad J = 15.0 \]

\[ N_l = 40 \quad J = 9.5 \]
Few leaders vs. many leaders

- **Few leaders**: Partition graphs and spread leaders
- **Many leaders**: Boundary with low-degree nodes
A 2D lattice

Gap between bounds

number of leaders $N_l$

lower bounds

upper bounds

number of leaders $N_l$
Leaders spread out from center

$N_l = 31$
So far...

- Leader selection in consensus networks

- Applications in vehicular formations and sensor localization

- Noise-corrupted leaders

- Algorithms for lower and upper bounds on global solutions

- Examples from random networks and 2D lattices
• Alternative formulation for noise-free leader selection

• Algorithms for lower and upper bounds on global solutions

• A flexible framework – amenable to other applications
Alternative formulation

\[ J_f(x) = \text{trace}(L_f^{-1}) \quad \text{NOT EXPLICIT IN } x \]

\[ x_i \in \{0, 1\}, \quad 1 - \text{LEADER,} \quad 0 - \text{FOLLOWER} \]

\[ L = \begin{bmatrix} L_l & L_0^T \\ L_0 & L_f \end{bmatrix}, \quad x = \begin{bmatrix} 1_N_l \\ 0_{N_f} \end{bmatrix} \]

\[ L \circ ((1 - x)(1 - x)^T) = \begin{bmatrix} L_l & L_0^T \\ L_0 & L_f \end{bmatrix} \circ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & L_f \end{bmatrix} \]

\[ (L \circ ((1 - x)(1 - x)^T) + \text{diag}(x))^{-1} = \begin{bmatrix} I_{N_l} & 0 \\ 0 & L_f^{-1} \end{bmatrix} \]

\[ J_f(x) = \text{trace}((L \circ ((1 - x)(1 - x)^T) + \text{diag}(x))^{-1}) - N_l \]
\[ J_f(x) = \text{trace} \left( (L \circ ((1 - x)(1 - x)^T) + \text{diag}(x))^{-1} \right) - N_l \]

\[ y = 1 - x \]

minimize \( J_f(y) = \text{trace} \left( (L \circ yy^T + \text{diag}(1 - y))^{-1} \right) - N_l \)

subject to \( y_i \in \{0, 1\}, \quad i = 1, \ldots, n \)

\[ 1^T y = N_f \]

minimize \( J_f(Y, y) = \text{trace} \left( (L \circ Y + \text{diag}(1 - y))^{-1} \right) - N_l \)

subject to \( Y = yy^T \)

\( y_i \in \{0, 1\}, \quad i = 1, \ldots, n \)

\[ 1^T y = N_f \]
minimize \( J_f(Y, y) = \text{trace} \left( (L \circ Y + \text{diag}(1 - y))^{-1} \right) - N_l \)

subject to

\[
\begin{align*}
Y &= yy^T \\
y_i &\in \{0, 1\}, \quad i = 1, \ldots, n \\
Y_{ij} &\in \{0, 1\}, \quad i, j = 1, \ldots, n \\
1^T y &= N_f \\
1^T Y 1 &= N_f^2
\end{align*}
\]

\[
Y = yy^T \iff \{ Y \succeq 0, \ \text{rank}(Y) = 1 \}
\]

Drop rank constraint + relax Boolean constraints \( \Rightarrow \) convex relaxation

FLEXIBLE FRAMEWORK FOR NODE-SELECTION PROBLEMS
**Convex relaxation**

\[
\begin{align*}
\text{minimize} \quad & J_f(Y, y) = \text{trace} \left( (L \circ Y + \text{diag} (1 - y))^{-1} \right) - N_l \\
\text{subject to} \quad & Y \succeq 0 \\
& y_i \in [0, 1], \quad i = 1, \ldots, n \\
& Y_{ij} \in [0, 1], \quad i, j = 1, \ldots, n \\
& 1^T y = N_f \\
& 1^T Y 1 = N_f^2
\end{align*}
\]

- Semidefinite program formulation \( O(n^6) \)
- Alternating direction method of multipliers
  - Solve a sequence of subproblems
  - Each subproblem costs \( O(n^3) \)
Greedy algorithm

- One-leader-at-a-time
  - RANK-2 UPDATE: $O(n^2)$ per leader

Without exploiting structure $O(n^4 N_l)$

Low rank updates $O(n^3 N_l)$

- Swap a leader and a follower
  - RANK-2 UPDATE: $O(n^2)$ per swap
An example

200 randomly distributed notes in a C-shaped region

Srirangarajan, Tewfik, and Luo ’08
\[ N_l = 3 \]
Both noise-free and noise-corrupted formulations yield similar selection of leaders.
Concluding remarks

- Leader selection in consensus networks
- Applications in vehicular formations and sensor localization
- Algorithms for lower and upper bounds on global solutions
  - Convex relaxations: lower bounds
  - Greedy algorithms: upper bounds

www.umn.edu/~mihailo/software/leaders

Ongoing work:

- Robustness of leader selection w.r.t convergence rate, controllability index, . . .
- Extension to social networks