Algorithms for leader selection in consensus networks

Fu Lin

joint work with:

Makan Fardad

Mihailo Jovanović

UNIVERSITY OF MINNESOTA

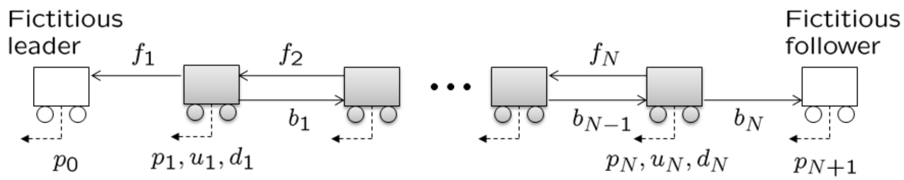
University of Washington, July 2013

Overview

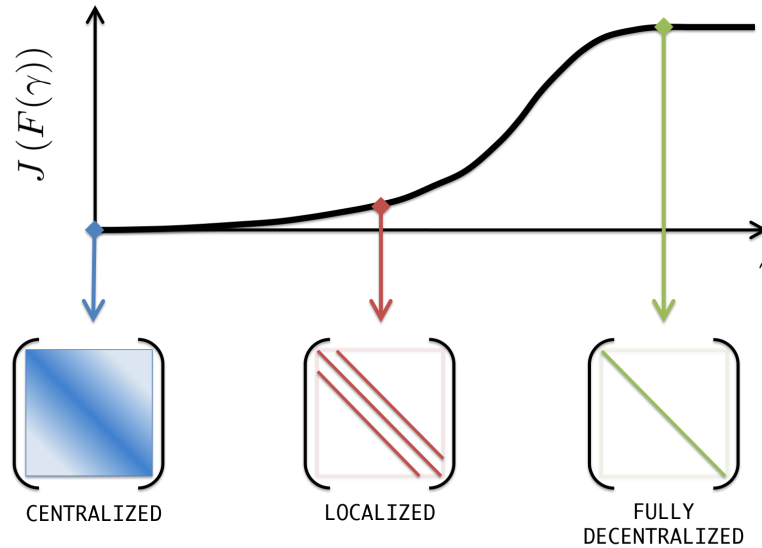
Optimal control of dynamical systems on networks

- MAIN TOPICS:
  - ⋆ Localized control of vehicular formations
  - ⋆ Sparsity-promoting optimal control
  - ⋆ Sparse consensus networks
  - ⋆ Algorithms for leader selection in consensus networks

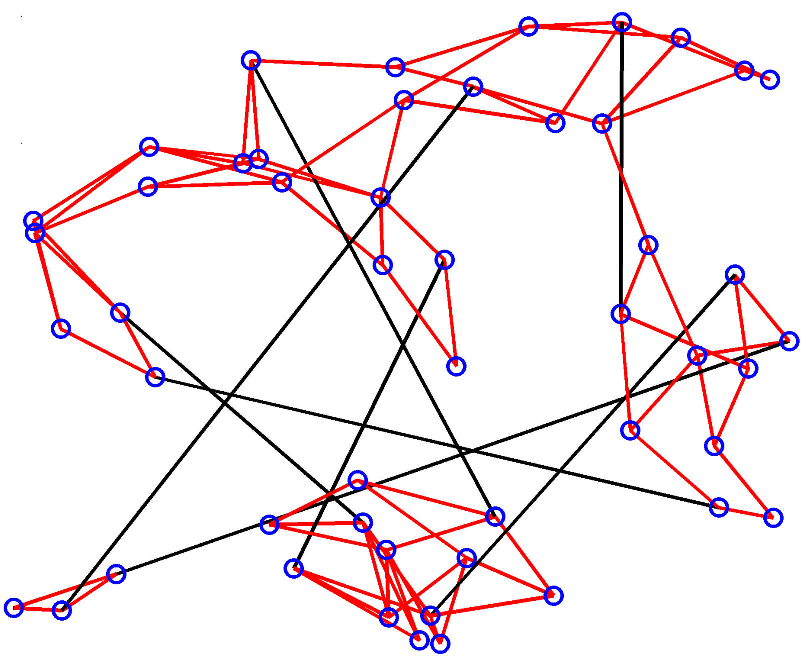
Localized control of vehicular formations



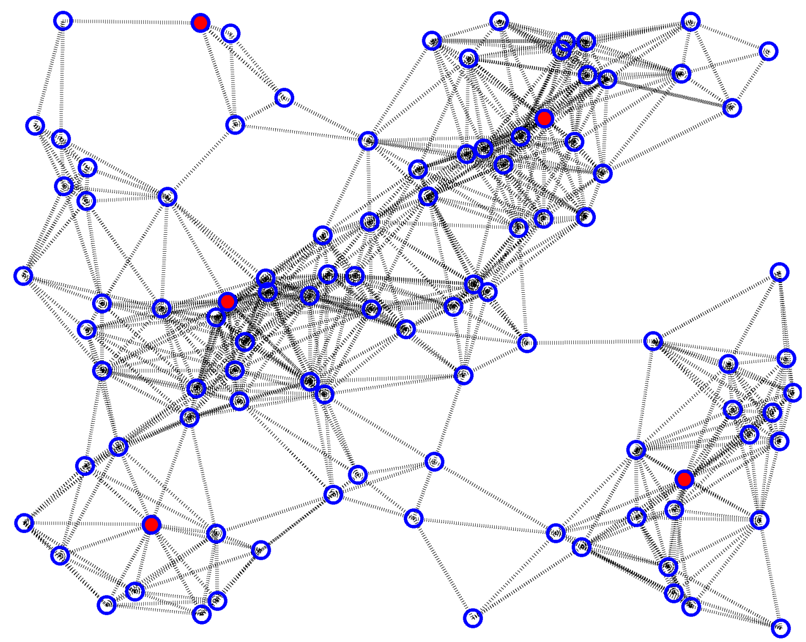
Sparsity-promoting optimal control



Sparse consensus networks



Algorithms for leader selection



- CHALLENGES:
  - ⋆ Networks – combinatorial objects
  - ⋆ Optimization – constrained nonconvex problems

- APPROACH:
  - ⋆ Identify classes of convex problems
  - ⋆ Exploit problem structure to develop efficient algorithms

In this talk

- Leader selection in consensus networks
- Applications in vehicular formations and sensor localization
- Algorithms for lower and upper bounds on global solutions
- Examples from regular lattices and random networks

Leader-follower consensus dynamics

- Time-invariant undirected connected networks

FOLLOWER: $$\dot{\psi}_i(t) \;=\; -\sum_{j \,\in\, \mathcal{N}_i} (\psi_i(t) \;-\; \psi_j(t)) \;+\; w_i(t)$$

↑ disturbance

LEADER: $$\dot{\psi}_i(t) \;=\; 0$$

$$\begin{bmatrix} \dot{\psi}_l(t) \\ \dot{\psi}_f(t) \end{bmatrix} \;=\; -\begin{bmatrix} 0 & 0 \\ L_0 & L_f \end{bmatrix} \begin{bmatrix} \psi_l(t) \\ \psi_f(t) \end{bmatrix} \;+\; \begin{bmatrix} 0 \\ w(t) \end{bmatrix}$$

Variance of followers depend on network structure and locations of leaders

Leader selection problem

- Select N_l leaders to minimize variance of followers

$$
\begin{array}{lrcl}
\underset{x}{\text{minimize}} & J_f(x) & = & \operatorname{trace}\left(L_f^{-1}\right) \\
\text{subject to} & x_i & \in & \{0,1\}, \quad i = 1,\ldots,n \\
 & \mathbb{1}^T x & = & N_l
\end{array}
$$

$$
x_i \in \{0,1\}, \quad 1 - \text{LEADER}, \quad 0 - \text{FOLLOWER}
$$

$$
L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad L_f = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}
$$

Connections to sensor localization problem

GOAL: Estimate n sensor positions in 1D

Relative measurements corrupted by noise

$$y_k = \psi_i - \psi_j + w_k$$

Anchor nodes with known positions ψ_l

$$\begin{aligned} y &= E^T \psi + w \\ &= \begin{bmatrix} E_l \\ E_f \end{bmatrix}^T \begin{bmatrix} \psi_l \\ \psi_f \end{bmatrix} + w \end{aligned}$$

In this talk: $\mathcal{E}(ww^T) = I$

Laplacian of measurement graph

$$L = EE^T = \begin{bmatrix} E_l E_l^T & E_l E_f^T \\ E_f E_l^T & E_f E_f^T \end{bmatrix} = \begin{bmatrix} L_l & L_0^T \\ L_0 & L_f \end{bmatrix}$$

Minimum variance estimation

$$\hat{\psi}_f = (E_f E_f^T)^{-1} E_f \, (y - E_l^T \psi_l)$$

Covariance of estimation error $\psi_f - \hat{\psi}_f$

$$\Sigma = (E_f E_f^T)^{-1} = L_f^{-1}$$

- Select N_l anchors to minimize variance of estimation error

$$
\begin{array}{lrcl}
\underset{x}{\text{minimize}} & J_f(x) & = & \operatorname{trace}\left(L_f^{-1}\right) \\
\text{subject to} & x_i & \in & \{0,1\}, \quad i \;=\; 1,\ldots,n \\
 & \mathbb{1}^T x & = & N_l
\end{array}
$$

$$x_i \in \{0,1\}, \quad 1 - \text{ANCHOR}, \quad 0 - \text{UNKNOWN SENSOR}$$

- Other applications via the interpretation of effective resistance

Related work

- Greedy algorithms with approximations

  Patterson and Bamieh '10

- Submodular optimization with performance guarantees

  Clark and Poovendran '11

  Clark, Bushnell, and Poovendran '12, '13, ...

- Semidefinite programming for related sensor selection problem

  Joshi and Boyd '09

- A large literature on controllability of leader-follower networks

  Tanner '04

  Liu, Chu, Wang, and Xie '08

  Rahmani, Ji, Mesbahi, and Egerstedt '09

  Clark, Bushnell, and Poovendran '12

  Kawashima and Egerstedt '12, ...

In this talk

- Related noise-corrupted leader selection problem

- Efficient algorithms for bounds on global optimal value

  - ⋆ Convex relaxations – lower bounds

  - ⋆ Greedy algorithms – upper bounds (exploiting low-rank structure)

- Examples from regular lattices and random networks

Noise-corrupted leader selection

- Arise in several applications

- Give insights to noise-free leader selection

- Easier to solve ;-)

Noise-corrupted leaders

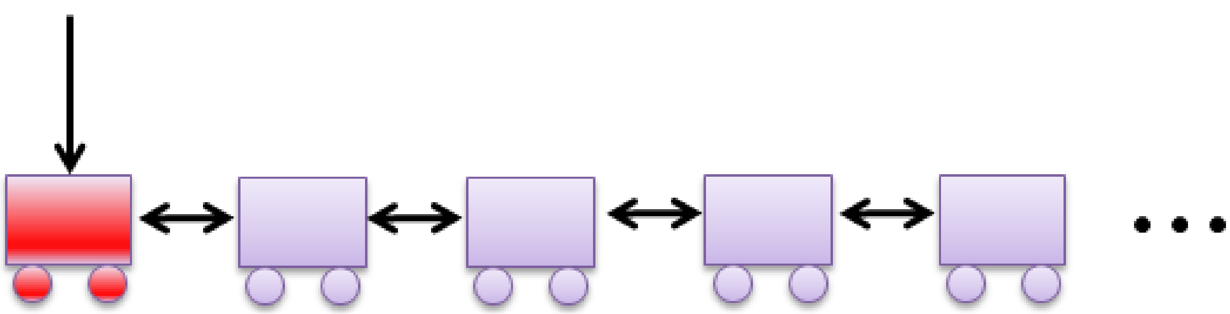
- Undirected connected networks

FOLLOWERS: $$\dot{\psi}_i(t) \;=\; -\sum_{j\,\in\mathcal{N}_i}(\psi_i(t) \;-\; \psi_j(t)) \;+\; w_i(t)$$

LEADERS: $$\dot{\psi}_i(t) \;=\; -\sum_{j\,\in\mathcal{N}_i}(\psi_i(t) \;-\; \psi_j(t)) \;-\; \alpha\,\psi_i(t) \;+\; w_i(t)$$

$$\alpha \;>\; 0$$

Leaders have GPS devices

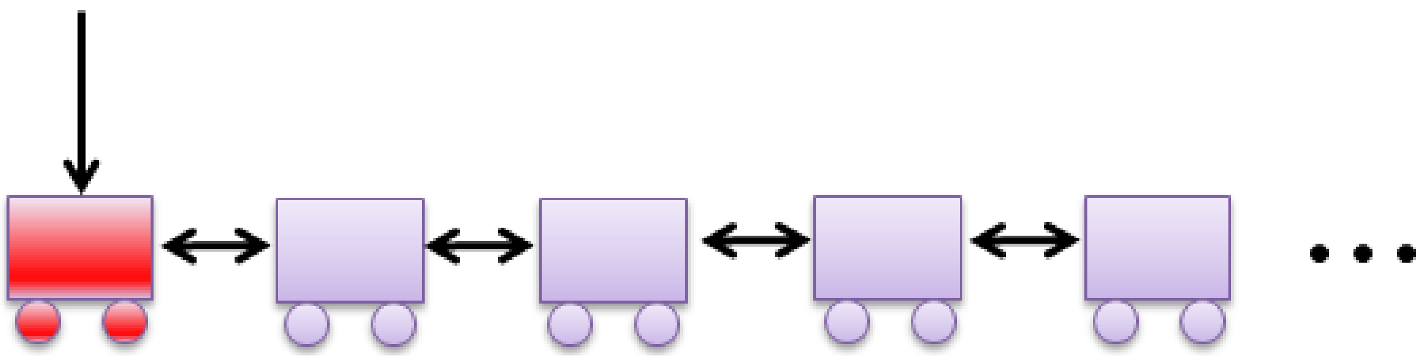


Diagonally strengthened Laplacian matrix

$$\dot{\psi}(t) \;=\; -\;(L \,+\, \alpha \,\mathrm{diag}\,(x))\,\psi(t) \;+\; w(t)$$

$x_i \in \{0, 1\}$, 1 – LEADER, 0 – FOLLOWER

$$L \;=\; \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \qquad \mathrm{diag}\,(x) \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Noise-corrupted leader selection

- Select N_l leaders to minimize variance of the network

$$
\begin{array}{rrcl}
\underset{x}{\text{minimize}} & J(x) & = & \text{trace}\left((L + \alpha\,\text{diag}(x))^{-1}\right) \\
\text{subject to} & x_i & \in & \{0,1\}, \qquad i = 1,\ldots,n \\
& \mathbb{1}^T x & = & N_l
\end{array}
$$

Recover the noise-free formulation $\alpha \to \infty$

$$
\begin{array}{l} \text{LEADERS} \\ \text{FOLLOWERS} \end{array}
\begin{bmatrix} \psi_l \\ \psi_f \end{bmatrix} : \quad
\begin{bmatrix} L_l + \alpha I & L_0^T \\ L_0 & L_f \end{bmatrix}^{-1} \to \begin{bmatrix} 0 & 0 \\ 0 & L_f^{-1} \end{bmatrix}
$$

Connections to sensor localization problem

Goal: Estimate sensor positions $\psi \in \mathbb{R}^n$

- Relative measurements $y_k \;=\; \psi_i \;-\; \psi_j \;+\; w_k$

- Absolute measurements $y_i \;=\; \psi_i \;+\; \dfrac{1}{\alpha} w_i$

Select N_l absolute measurements to minimize variance of estimation error

Algorithms for noise-corrupted formulation

$$
\begin{array}{llll}
\underset{x}{\text{minimize}} & J(x) & = & \text{trace}\left((L + \alpha\,\text{diag}\,(x))^{-1}\right) \\
\text{subject to} & x_i & \in & \{0,1\}, \quad i = 1,\ldots,n \\
& \mathbb{1}^T x & = & N_l
\end{array}
$$

- FEATURE: Convex objective function
- DIFFICULT: Boolean constraints
- APPROACH:
  - ⋆ Convex relaxation $\Rightarrow$ lower bound
  - ⋆ Greedy algorithm $\Rightarrow$ upper bound

Convex relaxation

$$\begin{array}{ll} \underset{x}{\text{minimize}} & J(x) = \operatorname{trace}\left((L + \alpha \operatorname{diag}(x))^{-1}\right) \\ \text{subject to} & x_i \in [0, 1], \quad i = 1, \ldots, n \\ & \mathbb{1}^T x = N_l \end{array}$$

Enlarge feasible set $\Rightarrow$ lower bound

- SDP formulation with complexity $O(n^4)$ – number of nodes
- Customized interior point method $O(n^3)$

Greedy algorithm

- One-leader-at-a-time

$$L \; + \; \alpha \, e_i e_i^T$$

⋆ RANK-1 UPDATE: $O(n^2)$ per leader

Complexity: $N_l \ll n \quad \Rightarrow \quad O(n^3)$ one matrix inverse

After selecting N_l leaders

- Swap a leader and a follower

$$L \; - \; \alpha \, e_i e_i^T \; + \; \alpha \, e_j e_j^T$$

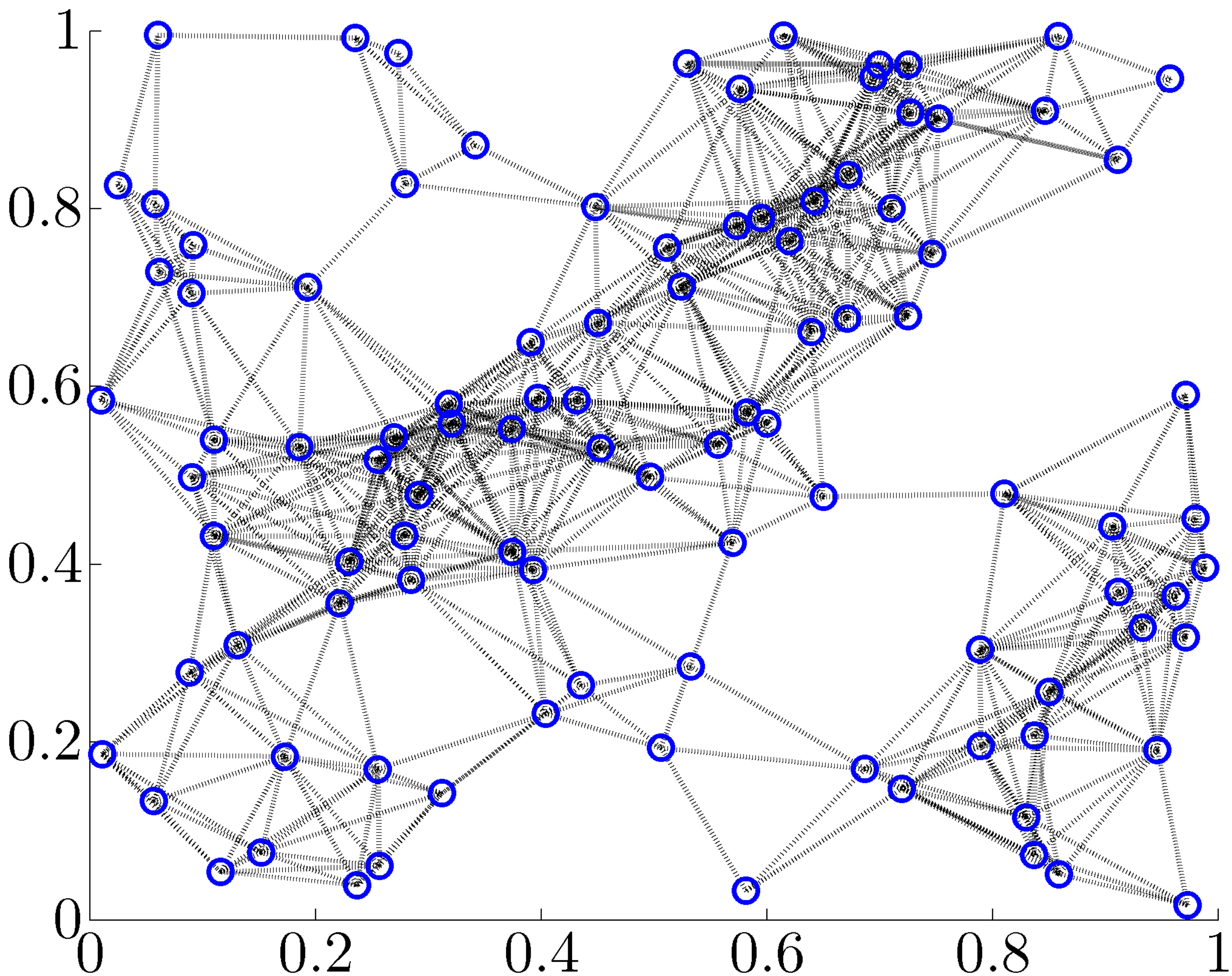
⋆ RANK-2 UPDATE: $O(n^2)$ per swap

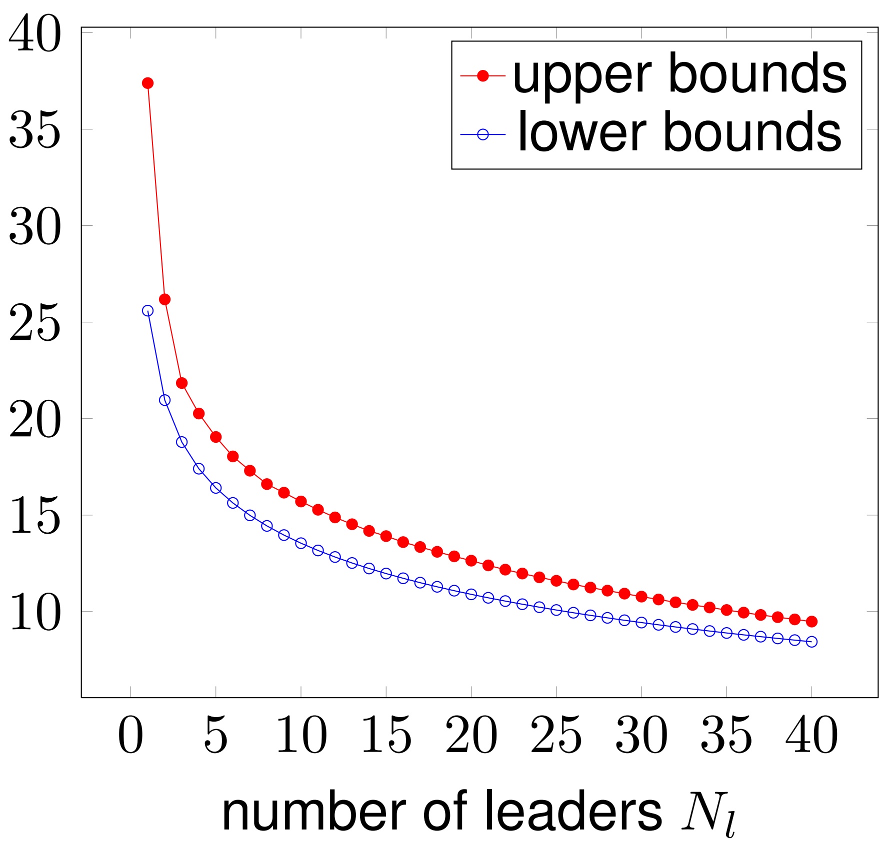
Recap

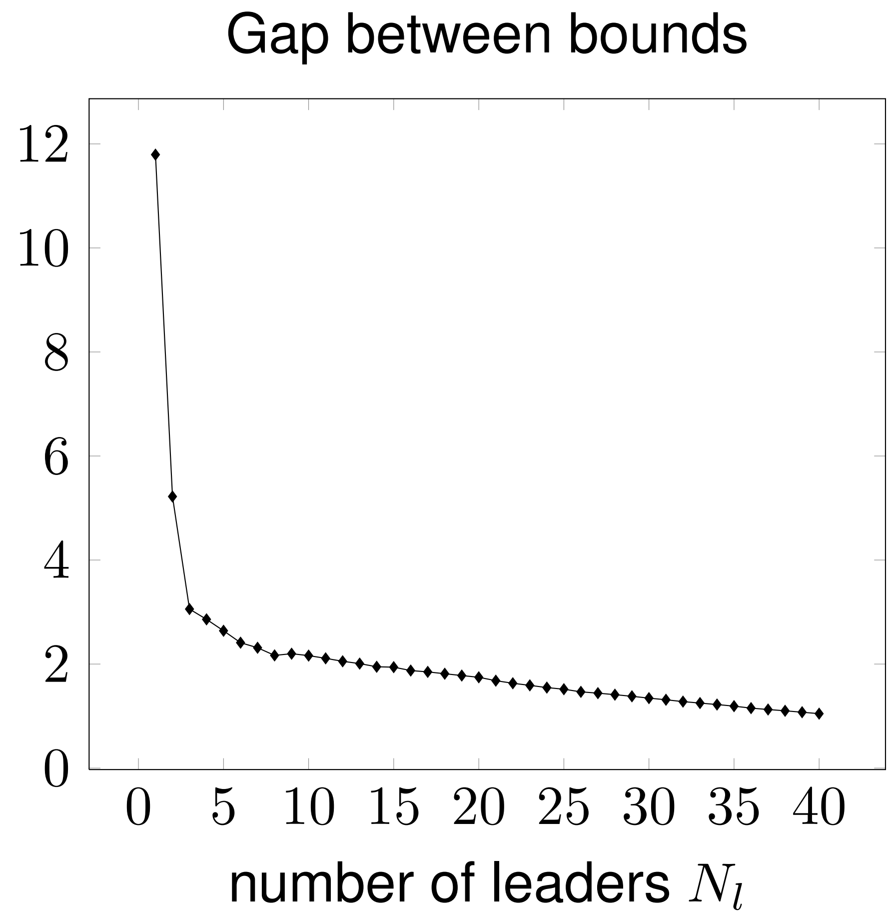
$$
\begin{array}{lrcl}
\underset{x}{\text{minimize}} & J(x) & = & \operatorname{trace}\left((L + \alpha \operatorname{diag}(x))^{-1}\right) \\
\text{subject to} & x_i & \in & \{0,1\}, \qquad i = 1,\ldots,n \\
 & \mathbb{1}^T x & = & N_l
\end{array}
$$

- Convex relaxation $\Rightarrow$ lower bound
  - ⋆ Semidefinite program $O(n^4)$
  - ⋆ Customized interior point method $O(n^3)$
- Greedy algorithm $\Rightarrow$ upper bound
  - ⋆ Without exploiting structure $O(n^4 N_l)$
  - ⋆ Low rank updates $O(n^3)$

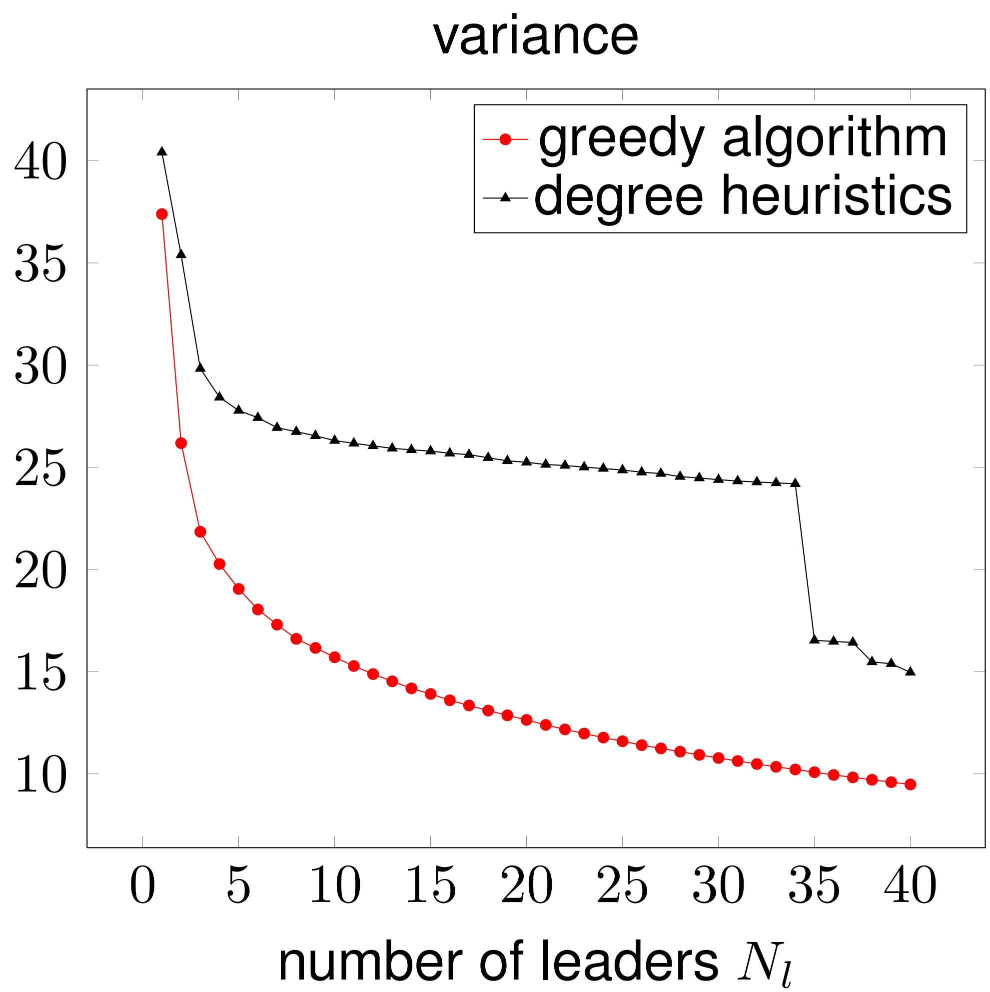
A random network with 100 nodes



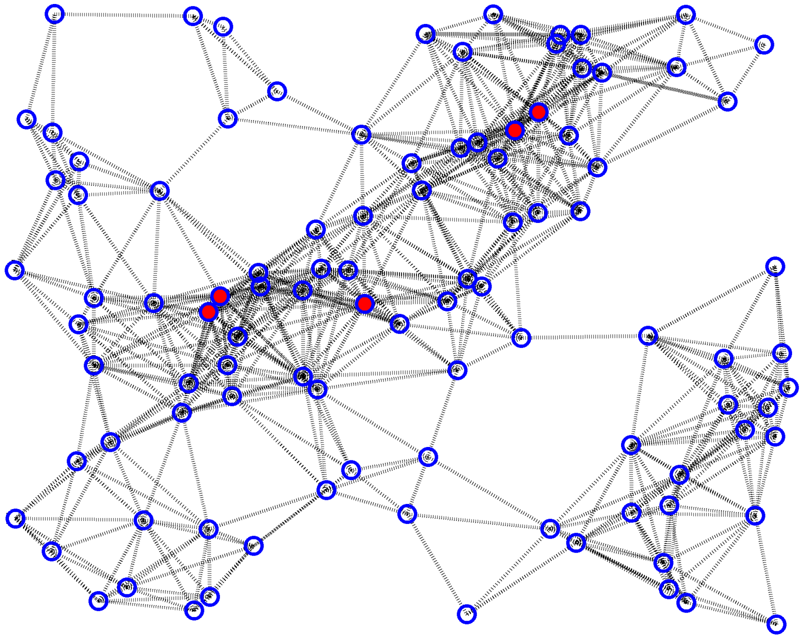

upper bounds
lower bounds
40
35
30
25
20
15
10
0
5
10
15
20
25
30
35
40
number of leaders N_l


Gap between bounds
12
10
8
6
4
2
0
0
5
10
15
20
25
30
35
40
number of leaders N_l

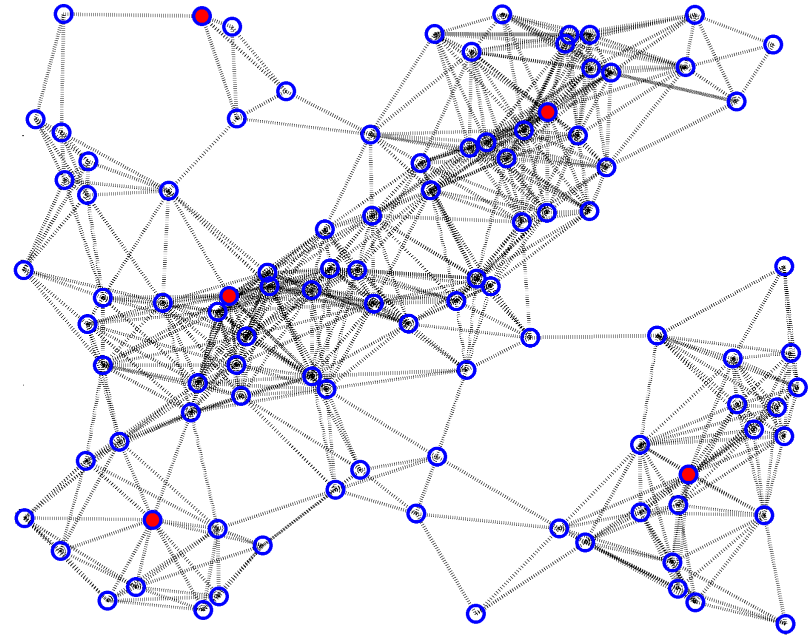
Degree heuristics vs. greedy algorithm



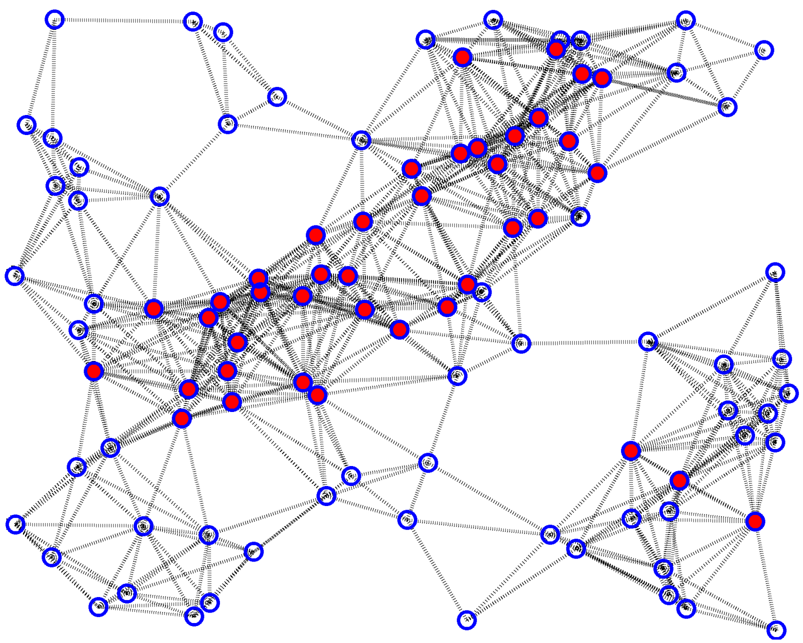
Degree heuristics vs. greedy algorithm



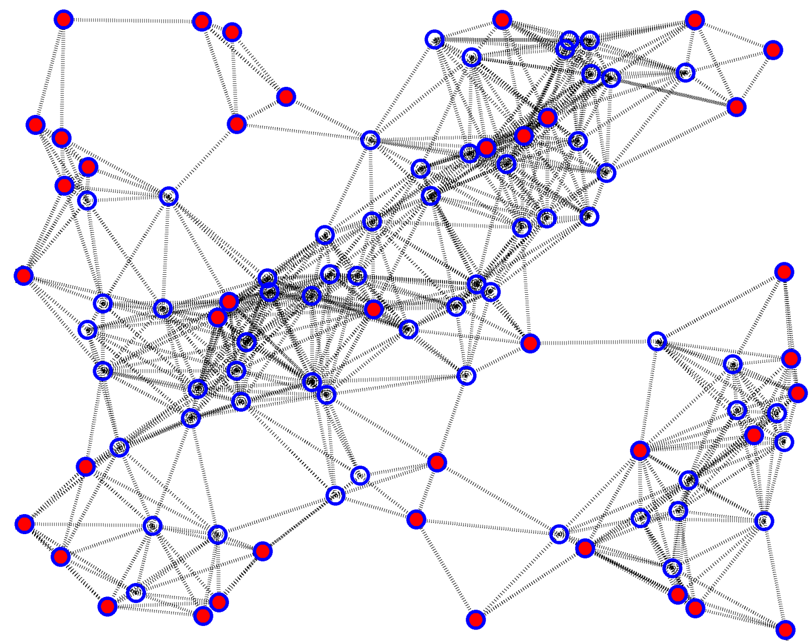
$N_l = 5 \quad J = 27.8$



$N_l = 5 \quad J = 19.0$

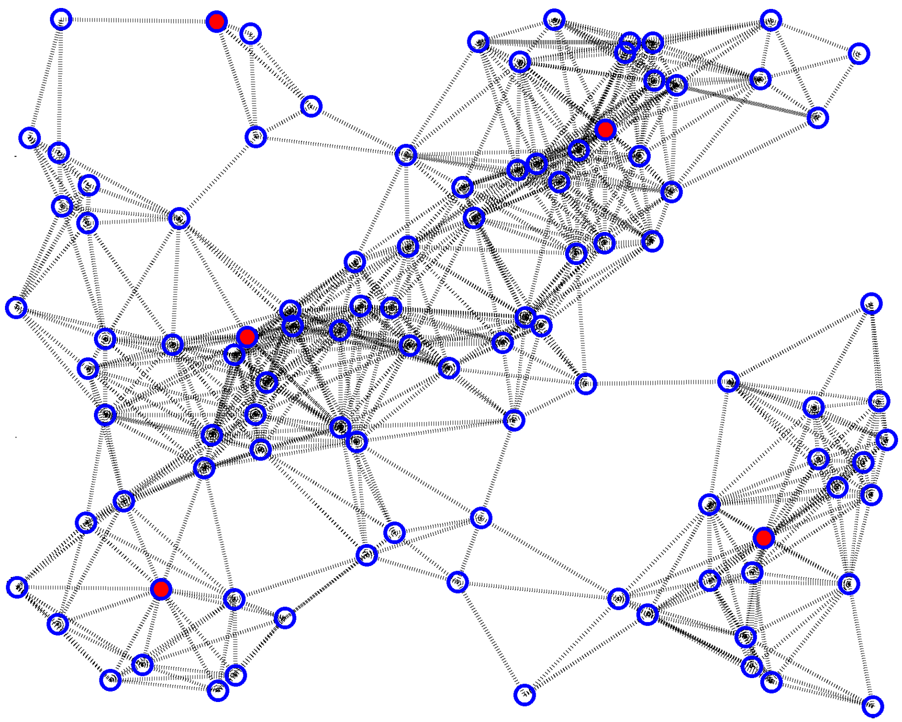


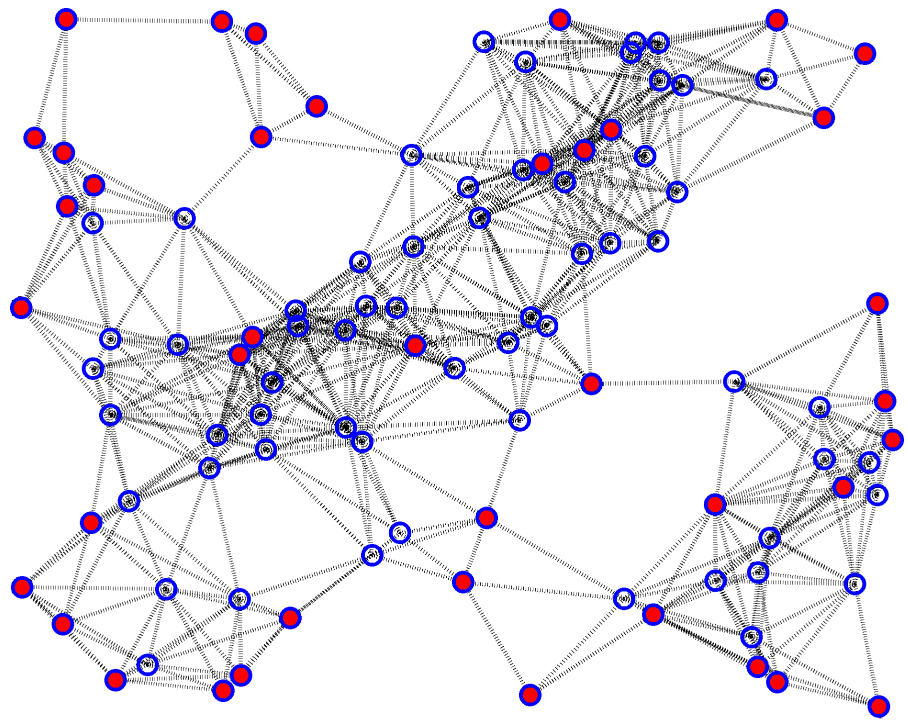
$N_l = 40 \quad J = 15.0$



$N_l = 40 \quad J = 9.5$

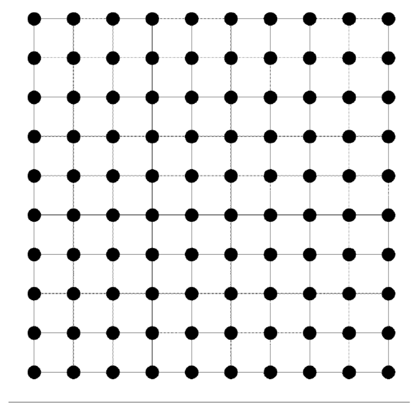
Few leaders vs. many leaders



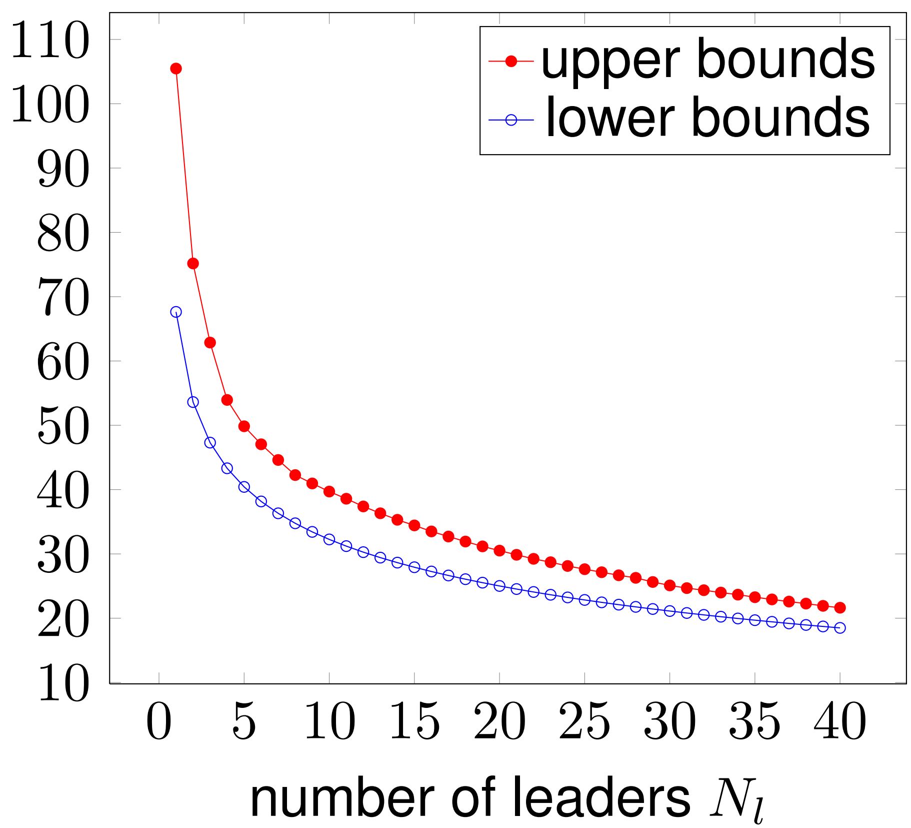


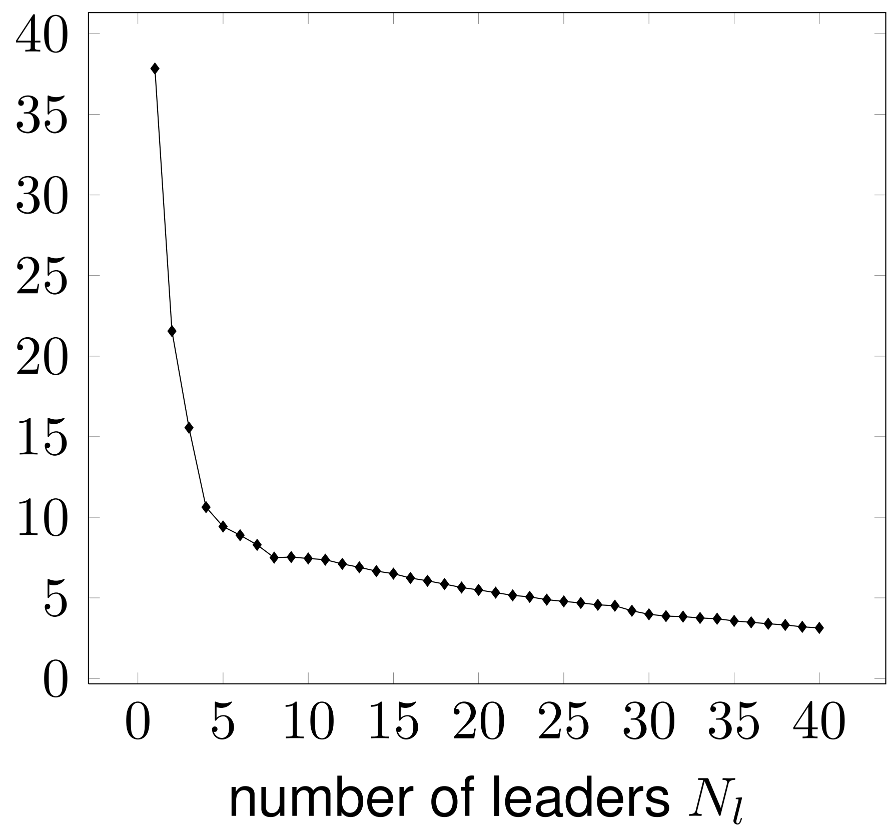
- Few leaders: Partition graphs and spread leaders
- Many leaders: Boundary with low-degree nodes

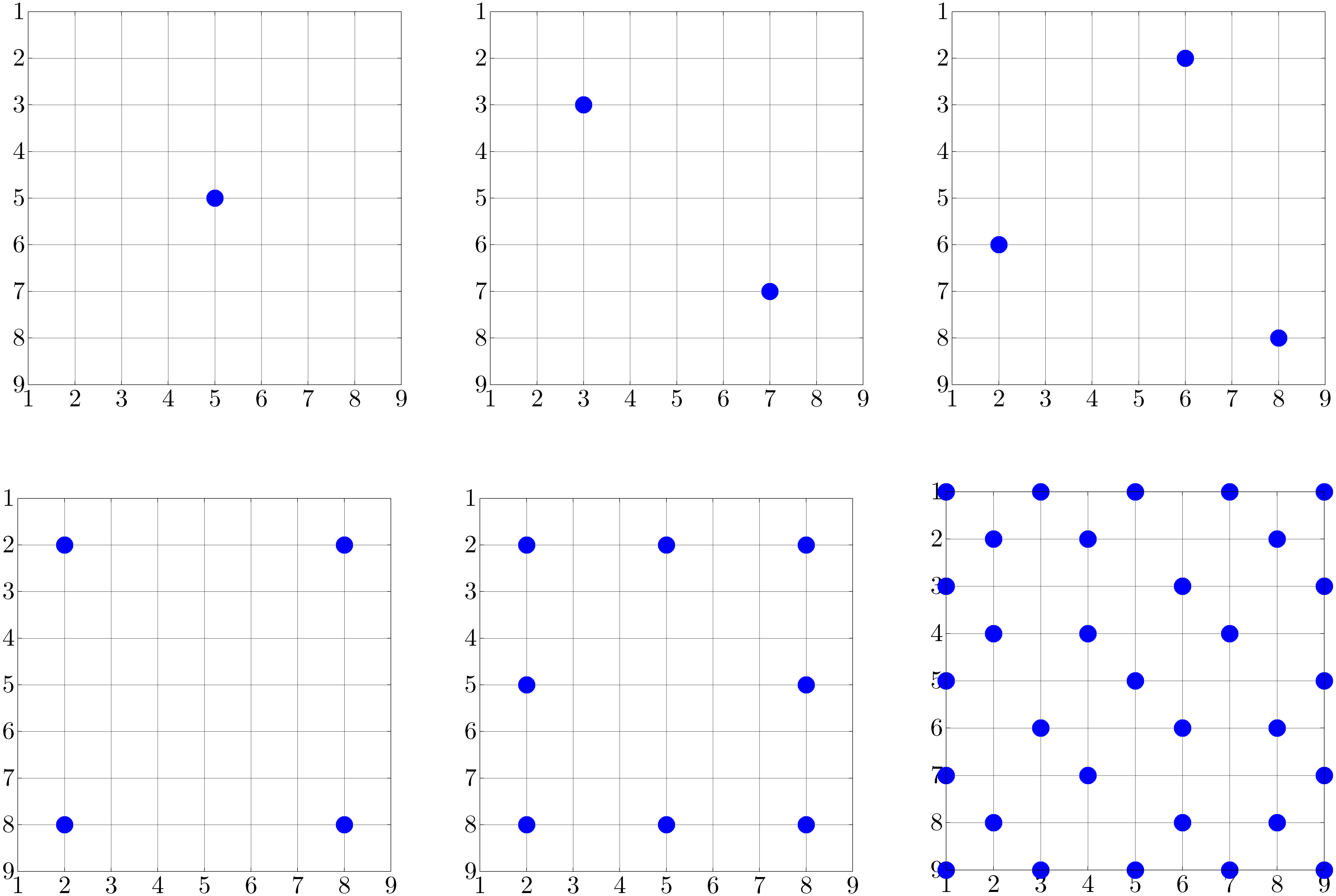
A 2D lattice



Gap between bounds







$N_l = 31$

Leaders spread out from center

So far...

- Leader selection in consensus networks
- Applications in vehicular formations and sensor localization
- Noise-corrupted leaders
- Algorithms for lower and upper bounds on global solutions
- Examples from random networks and 2D lattices

Next...

- Alternative formulation for noise-free leader selection

- Algorithms for lower and upper bounds on global solutions

- A flexible framework – amenable to other applications

Alternative formulation

$$J_f(x) \;=\; \operatorname{trace}\left(L_f^{-1}\right)$$

NOT EXPLICIT IN x

$$x_i \;\in\; \{0, 1\}, \qquad 1 \;-\; \text{LEADER}, \qquad 0 \;-\; \text{FOLLOWER}$$

$$L \;=\; \begin{bmatrix} L_l & L_0^T \\ L_0 & L_f \end{bmatrix}, \qquad x \;=\; \begin{bmatrix} \mathbb{1}_{N_l} \\ 0_{N_f} \end{bmatrix}$$

$$L \circ ((\mathbb{1} - x)(\mathbb{1} - x)^T) \;=\; \begin{bmatrix} L_l & L_0^T \\ L_0 & L_f \end{bmatrix} \circ \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{bmatrix} \;=\; \begin{bmatrix} 0 & 0 \\ 0 & L_f \end{bmatrix}$$

$$(L \circ ((\mathbb{1} - x)(\mathbb{1} - x)^T) \;+\; \operatorname{diag}(x))^{-1} \;=\; \begin{bmatrix} I_{N_l} & 0 \\ 0 & L_f^{-1} \end{bmatrix}$$

$$J_f(x) \;=\; \operatorname{trace}\left((L \circ ((\mathbb{1} - x)(\mathbb{1} - x)^T) \;+\; \operatorname{diag}(x))^{-1}\right) \;-\; N_l$$

$$J_f(x) \;=\; \operatorname{trace}\left(\left(L \circ ((\mathbb{1} - x)(\mathbb{1} - x)^T) \;+\; \operatorname{diag}(x)\right)^{-1}\right) \;-\; N_l$$

$$y \;=\; \mathbb{1} \;-\; x$$

$$\begin{array}{lrcl} \underset{y}{\text{minimize}} & J_f(y) & = & \operatorname{trace}\left(\left(L \circ yy^T \;+\; \operatorname{diag}(\mathbb{1} \;-\; y)\right)^{-1}\right) \;-\; N_l \\ \text{subject to} & y_i & \in & \{0, 1\}, \qquad i \;=\; 1, \ldots, n \\ & \mathbb{1}^T y & = & N_f \end{array}$$

$$\begin{array}{lrcl} \underset{Y,\, y}{\text{minimize}} & J_f(Y, y) & = & \operatorname{trace}\left(\left(L \circ Y \;+\; \operatorname{diag}(\mathbb{1} \;-\; y)\right)^{-1}\right) \;-\; N_l \\ \text{subject to} & Y & = & yy^T \\ & y_i & \in & \{0, 1\}, \qquad i \;=\; 1, \ldots, n \\ & \mathbb{1}^T y & = & N_f \end{array}$$

$$\begin{array}{llcl} \underset{Y,\, y}{\text{minimize}} & J_f(Y,y) & = & \text{trace}\left((L \circ Y \,+\, \text{diag}\,(\mathbb{1} - y))^{-1}\right) \,-\, N_l \\ \text{subject to} & Y & = & y y^T \\ & y_i & \in & \{0,1\}, \quad i \,=\, 1,\ldots,n \\ & Y_{ij} & \in & \{0,1\}, \quad i,j \,=\, 1,\ldots,n \\ & \mathbb{1}^T y & = & N_f \\ & \mathbb{1}^T Y \mathbb{1} & = & N_f^2 \end{array}$$

$$Y \,=\, y y^T \quad \Longleftrightarrow \quad \{\, Y \,\succeq\, 0,\; \mathbf{rank}\,(Y) \,=\, 1 \,\}$$

Drop rank constraint $+$ relax Boolean constraints $\Rightarrow$ convex relaxation

FLEXIBLE FRAMEWORK FOR NODE-SELECTION PROBLEMS

Convex relaxation

$$
\begin{array}{llcl}
\underset{Y,\, y}{\text{minimize}} & J_f(Y,y) & = & \mathrm{trace}\left((L \circ Y + \mathrm{diag}\,(\mathbb{1} - y))^{-1} \right) - N_l \\
\text{subject to} & Y & \succeq & 0 \\
& y_i & \in & [0,1], \qquad i = 1,\ldots,n \\
& Y_{ij} & \in & [0,1], \qquad i,j = 1,\ldots,n \\
& \mathbb{1}^T y & = & N_f \\
& \mathbb{1}^T Y \mathbb{1} & = & N_f^2
\end{array}
$$

- Semidefinite program formulation $O(n^6)$

- Alternating direction method of multipliers

  - ⋆ Solve a sequence of subproblems

  - ⋆ Each subproblem costs $O(n^3)$

Greedy algorithm

- One-leader-at-a-time

  ⋆ RANK-2 UPDATE: $O(n^2)$ per leader

  Without exploiting structure $O(n^4 N_l)$

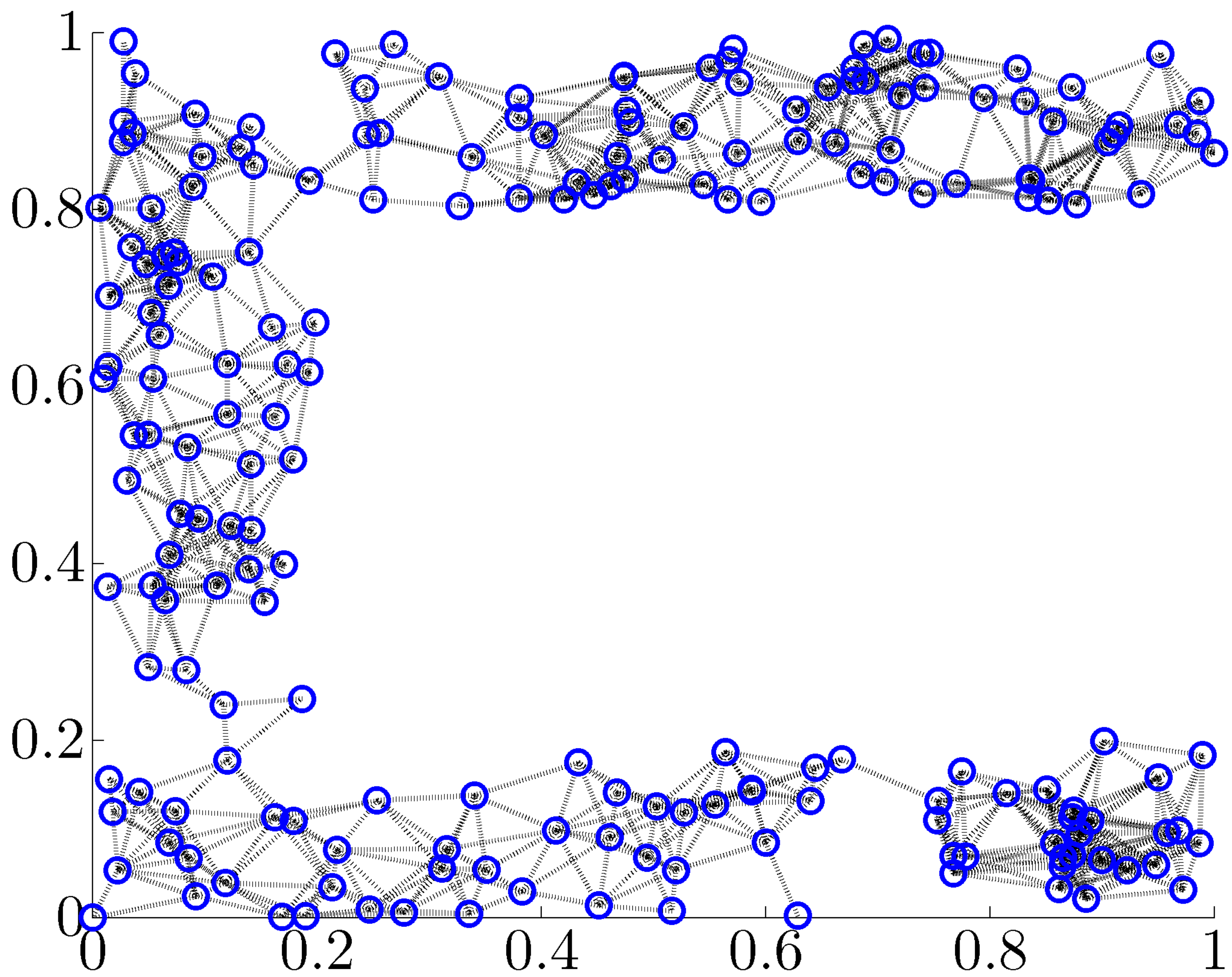
  Low rank updates $O(n^3 N_l)$

- Swap a leader and a follower

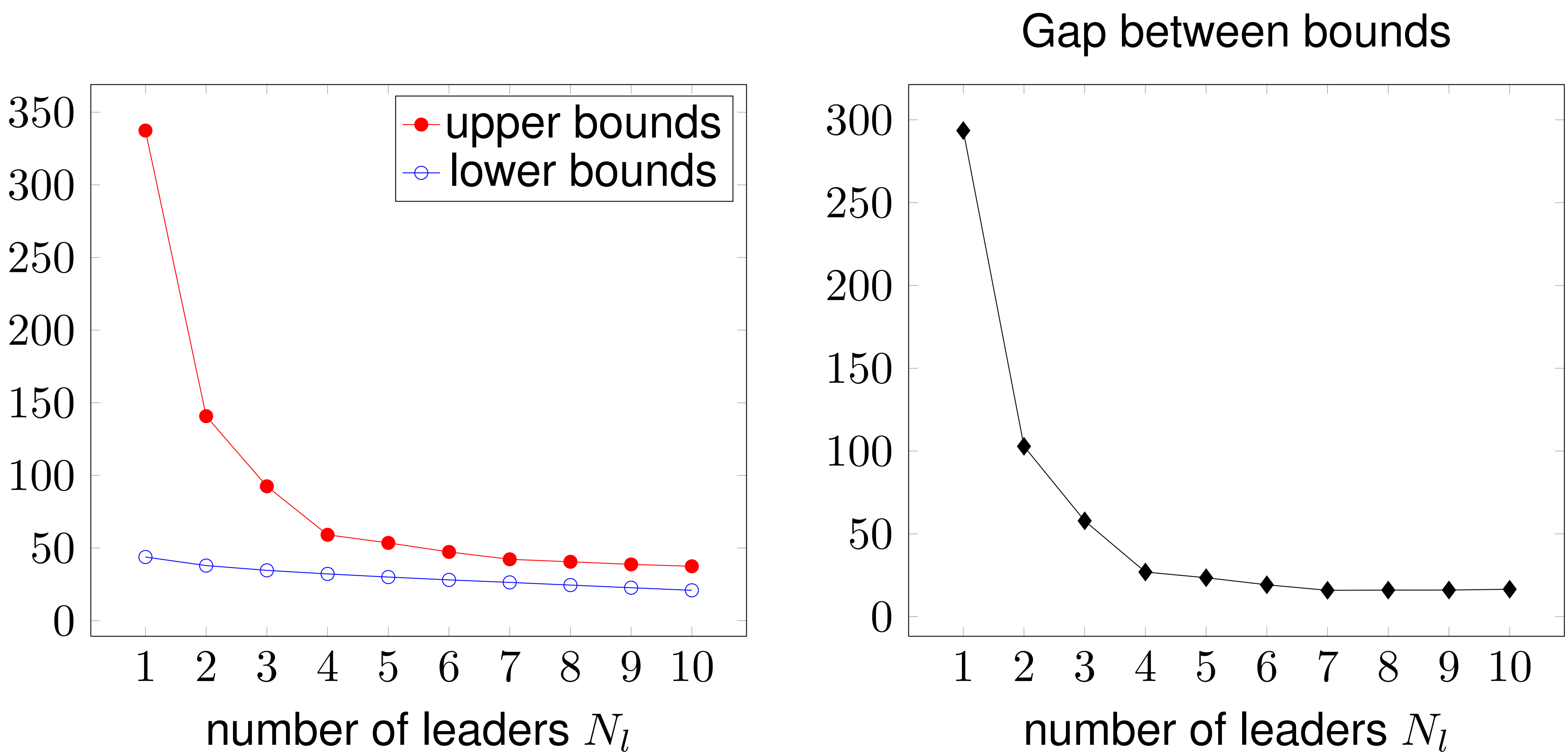
  ⋆ RANK-2 UPDATE: $O(n^2)$ per swap

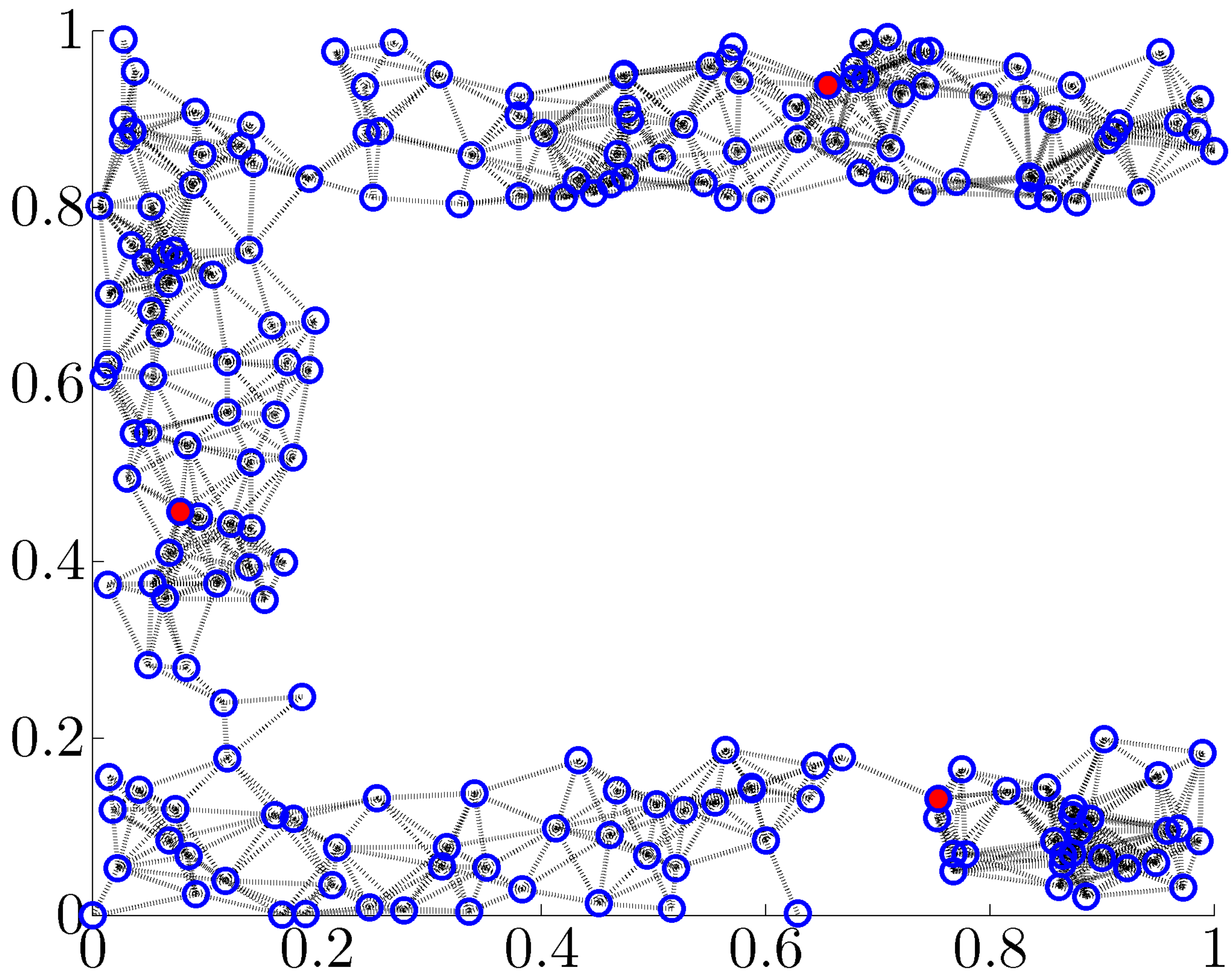
An example

200 randomly distributed notes in a C-shaped region

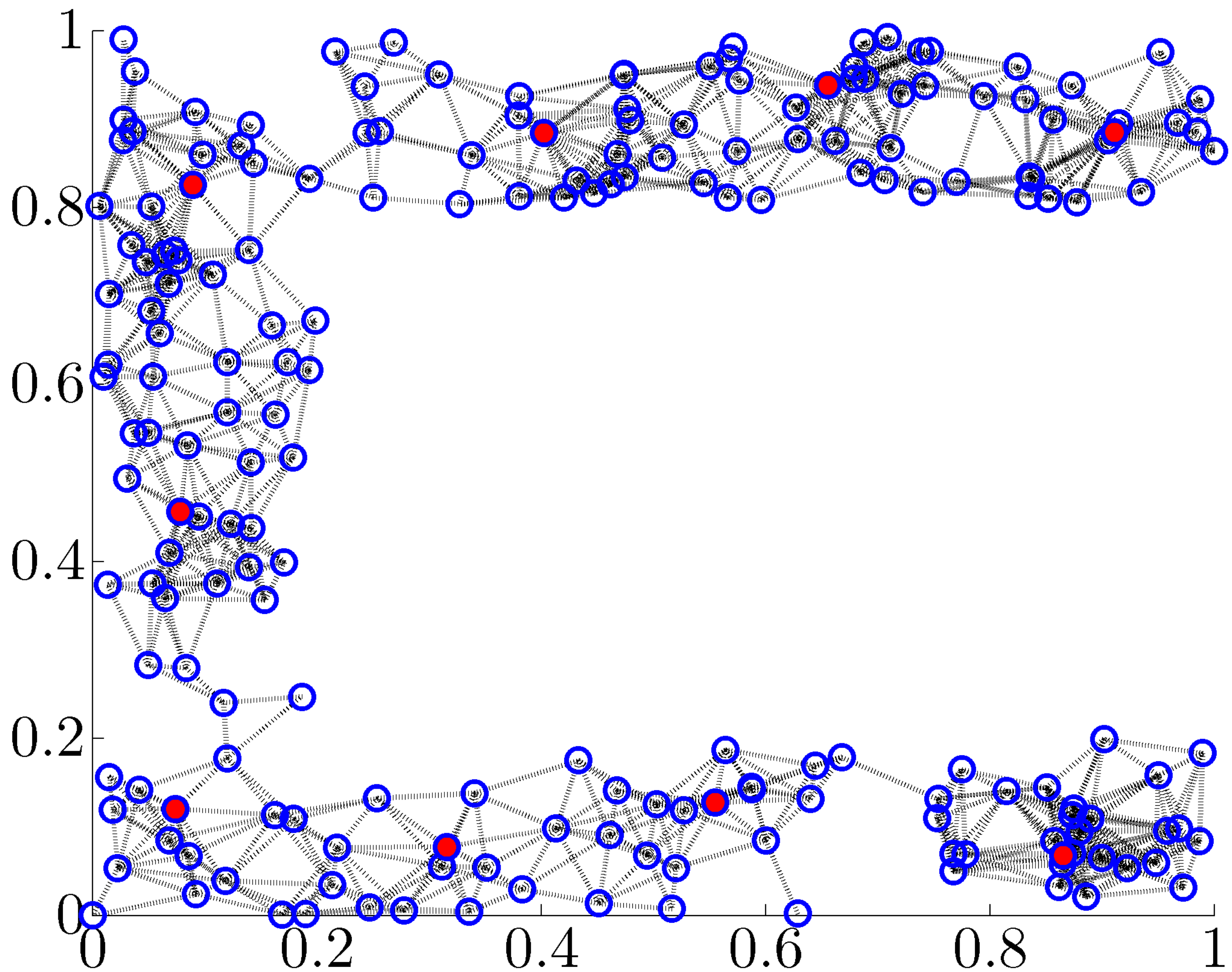


Srirangarajan, Tewfik, and Luo '08


upper bounds
lower bounds
number of leaders N_l
Gap between bounds
number of leaders N_l



$N_l = 3$



$N_l \;=\; 9$

Both noise-free and noise-corrupted formulations yield similar selection of leaders

Concluding remarks

- Leader selection in consensus networks
- Applications in vehicular formations and sensor localization
- Algorithms for lower and upper bounds on global solutions
  - ⋆ Convex relaxations: lower bounds
  - ⋆ Greedy algorithms: upper bounds

www.umn.edu/∼mihailo/software/leaders

Ongoing work:

- Robustness of leader selection w.r.t convergence rate, controllability index, . . .
- Extension to social networks