Optimal localized feedback design for multi-vehicle systems

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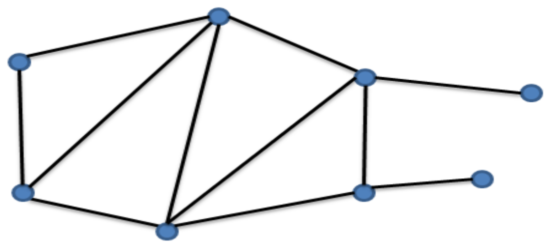
Problem setup

A network (graph) $\mathcal{G}$ of n vehicles (nodes)



Single-integrator dynamics

$$\dot{x}_i \;=\; u_i \,+\, d_i, \qquad d_i \;\text{disturbance}$$

Relative information exchange with neighbors

$$u_i(t) \;=\; -\sum_{j \in \mathcal{N}_i} k_{ij} \Big(x_i(t) \;-\; x_j(t) \Big)$$

Vector form $\quad \dot{x}(t) \;=\; -\,L(k)\,x(t) \;+\; d(t)$

Structured matrix $L(k)$ depends on graph topology

Problem setup

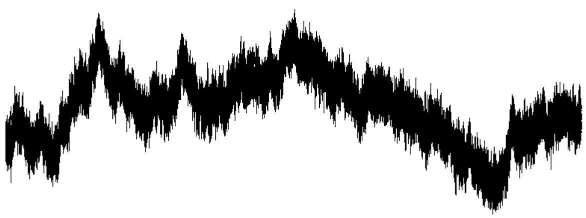
Structured matrix

$$L(k)\mathbf{1} \;=\; 0 \cdot \mathbf{1} \qquad \mathbf{1} \text{ is vector of all } 1$$

Independent of graph topology and feedback gains $\{k_{ij}\}$

$$\dot{x}(t) \;=\; -\,L(k)\,x(t) \;+\; d(t)$$

Average mode $\quad \bar{x}(t) \;=\; (1/n) \sum x_i(t) \quad$ undergoes random walk



If other modes are stable, $x_i(t)$ fluctuates around $\bar{x}(t)$

Deviation from average $\bar{x}(t)$

$$\tilde{x}_i(t) \;=\; x_i(t) \;-\; \bar{x}(t)$$

Steady-state variance $\quad \lim_{t \to \infty} \mathcal{E}\{\tilde{x}^T(t)\tilde{x}(t)\} \quad$ quantifies consensus performance

Optimal control problem

What graph topologies lead to small variance?

How to design feedback gains to minimize variance? (In this talk)

- Choose feedback gains $\{k_{ij}\}$ to minimize H_2 norm $\|d \to z\|_2^2$

$$\dot{x} = -L(k)\,x + d$$

$$z = \begin{bmatrix} \tilde{x} \\ u \end{bmatrix} = \begin{bmatrix} x - \bar{x}\,\mathbf{1} \\ -L(k)\,x \end{bmatrix}$$

Assumptions:

- Bi-directional interaction between vehicles (undirected graphs)
- Symmetric feedback gains $k_{ij} = k_{ji}$ (interpreted as spring constant)

$$L(k) = L^T(k)$$

Need some stuffs to express $L(k)$ explicitly

Incidence matrix

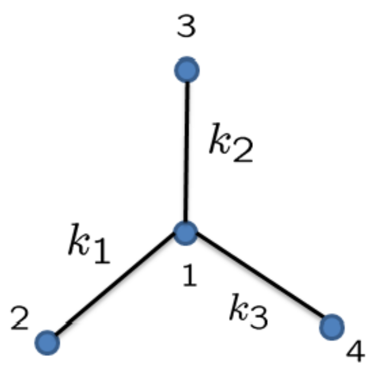
Edge $l \sim (i,j)$ connects nodes i and j

Define $e_l \in \mathbb{R}^n$ with only two nonzero entries

$$(e_l)_i \;=\; 1 \qquad\qquad (e_l)_j \;=\; -1$$

Incidence matrix

$$E \;=\; \left[\, e_1 \;\cdots\; e_m \,\right]$$



$$E \;=\; \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad E^T\mathbf{1} \;=\; 0$$

Edge $l \sim (i,j)$ $\quad k_l \;:=\; k_{ij} \;=\; k_{ji}$

Structured matrix $\qquad L(K) \;=\; EKE^T \qquad K \;=\; \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_m \end{bmatrix}$

Unobservable mode

$$\dot{x} \;=\; -\,EKE^T\,x \;+\; d$$

$$z \;=\; \begin{bmatrix} \tilde{x} \\ u \end{bmatrix} \;=\; \begin{bmatrix} x \;-\; \bar{x}\mathbf{1} \\ -EKE^T x \end{bmatrix}$$

Deviation from average

$$x \;-\; \bar{x}\mathbf{1} \;=\; \big(I \;-\; (1/n)\mathbf{1}\mathbf{1}^T\big)x$$

Can be verified

$$\big(I \;-\; (1/n)\mathbf{1}\mathbf{1}^T\big)\mathbf{1} \;=\; 0$$

and

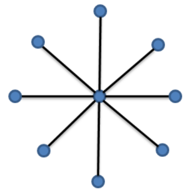
$$-\,EKE^T\mathbf{1} \;=\; 0$$

- Average mode $\bar{x}$ is unobservable, i.e., not minimal representation

Need a change of coordinates to separate $\bar{x}$

Coordinate transformation

Tree: connected graph with no loops

path  star 

Incidence matrix of tree E_t

$$T = \begin{bmatrix} E_t^T \\ (1/n)\mathbf{1}^T \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} E_t\,(E_t^T E_t)^{-1} & \mathbf{1} \end{bmatrix}$$

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$$Tx := \begin{bmatrix} \psi \\ \bar{x} \end{bmatrix}$$

For tree graphs $\quad L = E_t K E_t^T$

$$\begin{bmatrix} E_t^T \\ (1/n)\mathbf{1}^T \end{bmatrix} E_t K {\color{red}E_t^T} \begin{bmatrix} {\color{red}E_t\,(E_t^T E_t)^{-1}} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} E_t^T E_t\, K & 0 \\ 0 & 0 \end{bmatrix}$$

Tree graphs

$$\dot{\psi} \;=\; -\,E_t^T E_t\, K\, \psi \;+\; E_t^T\, d$$

$$z \;=\; \begin{bmatrix} \tilde{x} \\ u \end{bmatrix} \;=\; \begin{bmatrix} E_t\,(E_t^T E_t)^{-1} \\ -\,E_t\,K \end{bmatrix} \psi$$

H_2 norm $\|d \to z\|_2^2$

$$J(K) \;=\; \frac{1}{2}\,\mathrm{trace}\Big((E_t^T E_t)^{-1} K^{-1} \;+\; K\,E_t^T E_t\Big)$$

Diagonal matrix

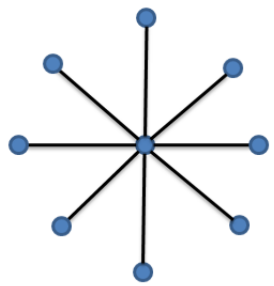
$$K \;=\; \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_{n-1} \end{bmatrix}$$

- Optimal localized feedback gains

$$k_i \;=\; \sqrt{\frac{\big(E_t^T E_t\big)_{ii}^{-1}}{2}} \qquad i \;=\; 1, \ldots, n-1$$

Two simple examples

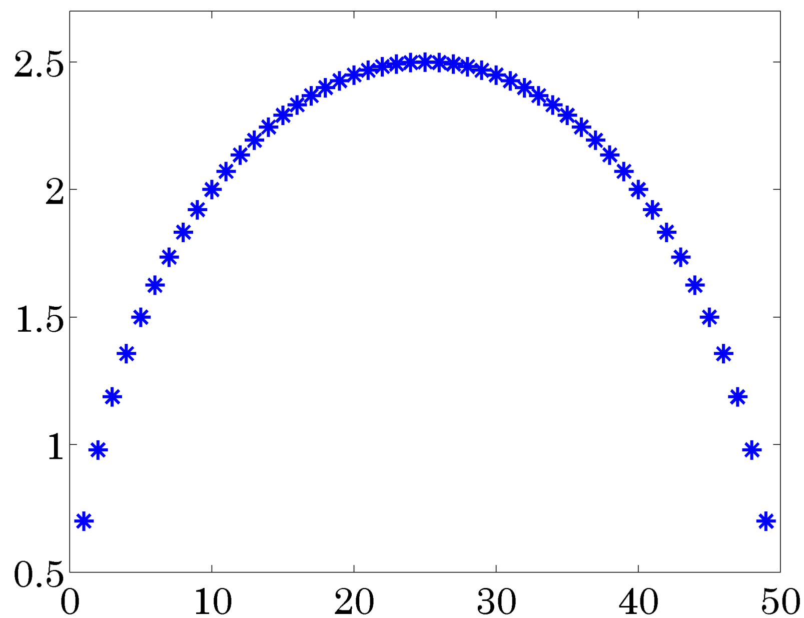
- Star



uniform gain $k = \sqrt{\frac{n-1}{2n}} \approx \frac{1}{\sqrt{2}}$ for large n

- Path:

$$k_i = \sqrt{\frac{i(n-i)}{2n}}$$



larger gains in the center

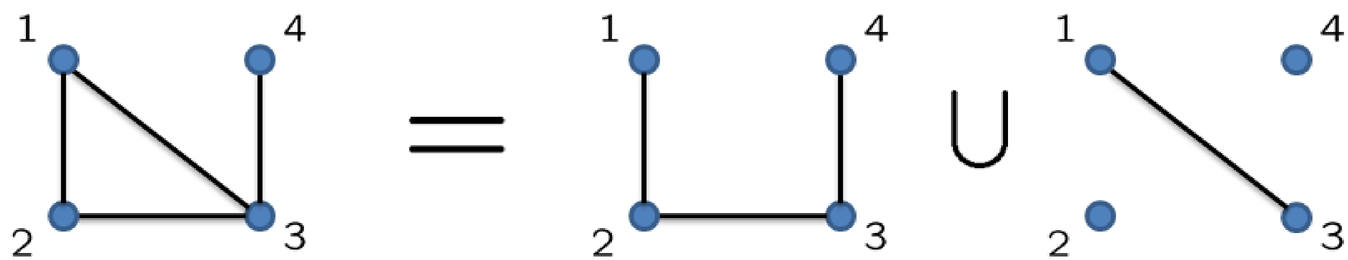
General graphs

Decompose $\mathcal{G}$ into a tree subgraph $\mathcal{T}$ and remaining edges $\mathcal{C}$

(Does not depend on the choice of tree subgraph)

Form corresponding incidence matrix

$$E = \begin{bmatrix} E_t & E_c \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \cup \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$E = \begin{bmatrix} E_t & E_c \end{bmatrix} = E_t \begin{bmatrix} I & (E_t^T E_t)^{-1} E_t^T E_c \end{bmatrix} = E_t \Gamma$$

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General graphs

$$\dot{\psi} \;=\; -\, E_t^T E_t \,\Gamma K \Gamma^T \,\psi \;+\; E_t^T\, d$$

$$z \;=\; \begin{bmatrix} \tilde{x} \\ u \end{bmatrix} \;=\; \begin{bmatrix} E_t (E_t^T E_t)^{-1} \\ -\, E_t \,\Gamma K \Gamma^T \end{bmatrix} \psi$$

$$J(K) \;=\; \frac{1}{2}\,\mathrm{trace}\Big((E_t^T E_t)^{-1} (\Gamma K \Gamma^T)^{-1} \;+\; \Gamma K \Gamma^T E_t^T E_t \Big)$$

Tree graphs $\Gamma \;=\; I$

Main results:

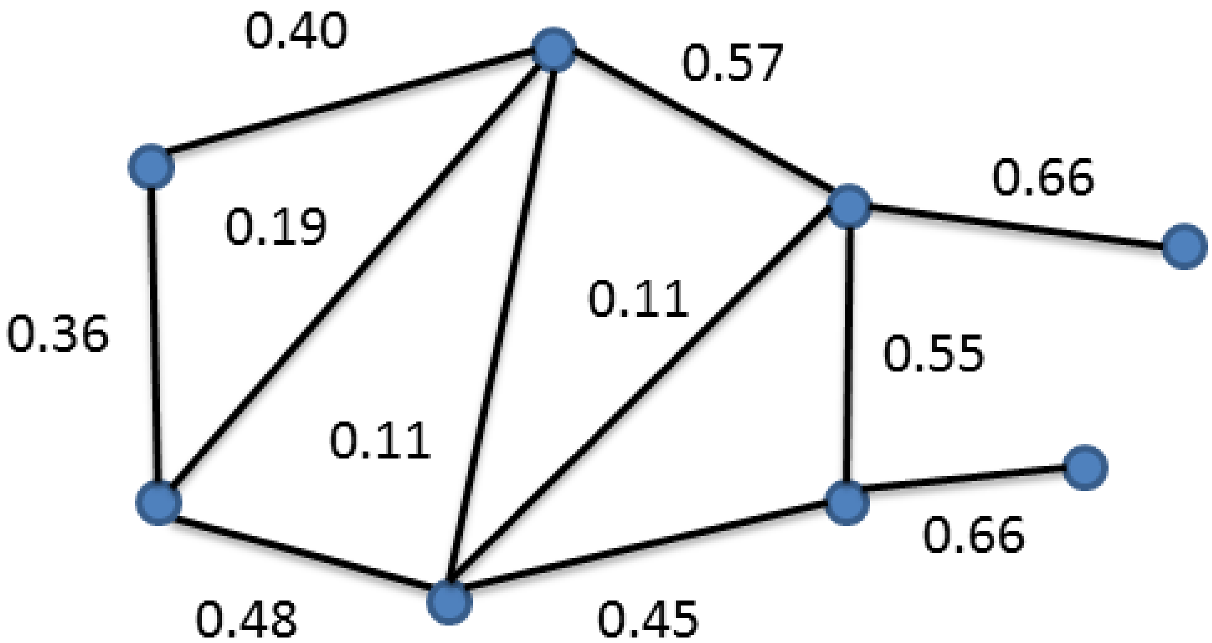
- Closed-loop stability $\iff$ $\Gamma K \Gamma^T \;>\; 0$
- $J(K)$ is convex function if $\Gamma K \Gamma^T \;>\; 0$

  Can be formulated as a semi-definite program

  ⋆ $W_1 \;>\; 0$, $W_2 \;=\; W_2^T$, then $\lambda(-\,W_1 W_2) \;<\; 0$ if and only if $W_2 \;>\; 0$

  ⋆ $\mathrm{trace}\,(W^{-1})$ is convex for $W \;>\; 0$

Example



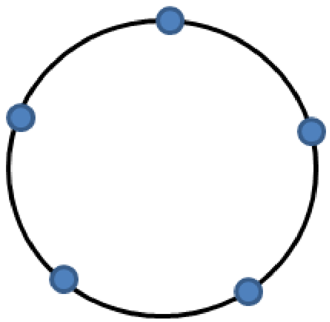
- Compared with performance of uniform gain design

J^*	$J(k=1)$	$(J - J^*)/J^*$
9.1050	13.1929	45%

Two examples

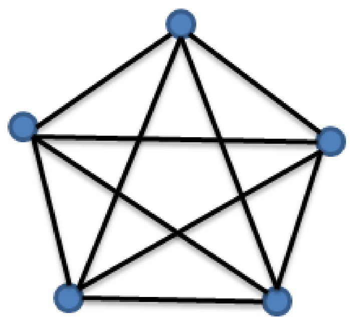
Analytical solutions for some simple graphs

- Circle



uniform gain $k = \sqrt{\frac{n^2 - 1}{24n}}$

- Complete graph

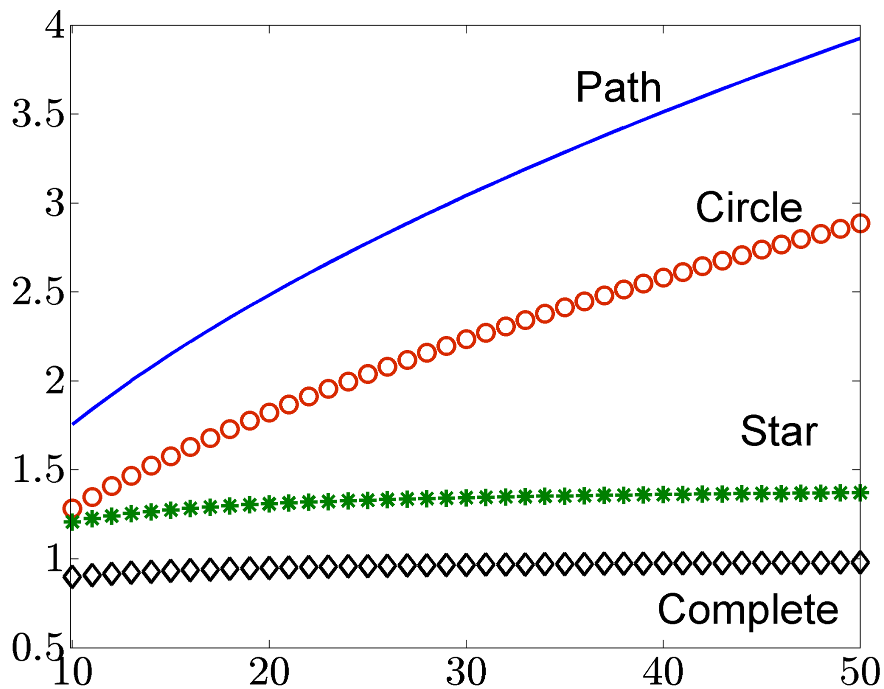


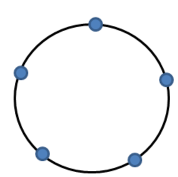
uniform gain $k = \frac{2}{n}$

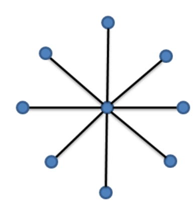
Performance vs. size

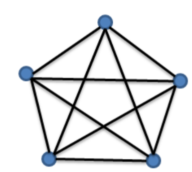
H_2 norm $\|d \to z\|_2^2$ normalized by the number of nodes

	path	circle	star	complete
$\frac{1}{n} J^*(K)$	$\frac{\pi}{8}\sqrt{2n}$	$\sqrt{\frac{n^2-1}{6n}}$	$\sqrt{\frac{2(n-1)^3}{n^3}}$	$\frac{n-1}{n}$









Concluding remarks

Optimal localized feedback gain design

- Analytical solutions for tree graphs
- Characterize stabilizing feedback gains
- Convex problem for general graphs

Ongoing work:

- Feedback gain design of directed graphs

  Hard problem in general, analytical solutions? hidden convexity? etc...

- Is it possible to have sparse feedback gains, i.e., few edges?

  Identify sparsity patterns...

  Fardad, Lin, Jovanović   ACC'11 (submitted)