Hydraulic Transients in Networks

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1 Introduction

The proposed project concerns on the transient simulation of an hydraulic network. That is, the simulation of fluid transients within a closed conduit.

The analysis of hydraulic networks tries to determine maximum pressures and flows along a network for a given disturbance after some disturbance has occurred. This disturbance can be the closure of a valve, the change of water/oil demand, the failure of a pump, etc...

This problem has had much interest over time and historically has been given the name of **Water hammer**. In a simple reservoir-pipe-valve system (see Fig. 2) which is operating in steady state, if there is a sudden closure of the valve, a pressure wave will appear and will propagate along the conduit. Figure 2 shows the periodic nature of the pressure wave.

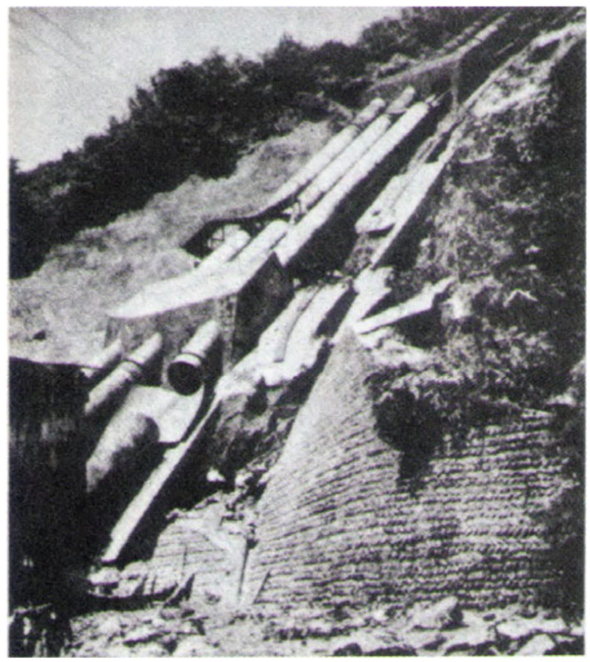

Figure 1: Burst pen stock of Hydroelectric power plant, Japan. 1960. The burst, due to excessive pressure, caused the death of 3 workers and 0.5 million dollars in damage.

Depending on the material and characteristics of the pipes, a sudden surge in pressure can lead to the burst of a pipe or the maloperation of a pump, etc... Figure 1 shows the result of a pipe burst due to water hammer in an hydraulic power plant.

The physical model of the hydraulic network from [1, Chap. 3] can be described with the following PDEs for each pipe:

$$\frac{\partial Q}{\partial t} + gA\frac{\partial H}{\partial x} + RQ\left|Q\right| = 0 \quad (1)$$

$$a^2\frac{\partial Q}{\partial x} + gA\frac{\partial H}{\partial t} = 0 \quad (2)$$

which are the momentum and continuity equations for the flow $Q(x,t)$ and pressure $H(x,t)$, with $R = f/(2DA)$.

Then, the whole set of pipes in the network will be related by the continuity of flows at each node. This set of PDEs related by linear conditions is known as a Differential Algebraic system.

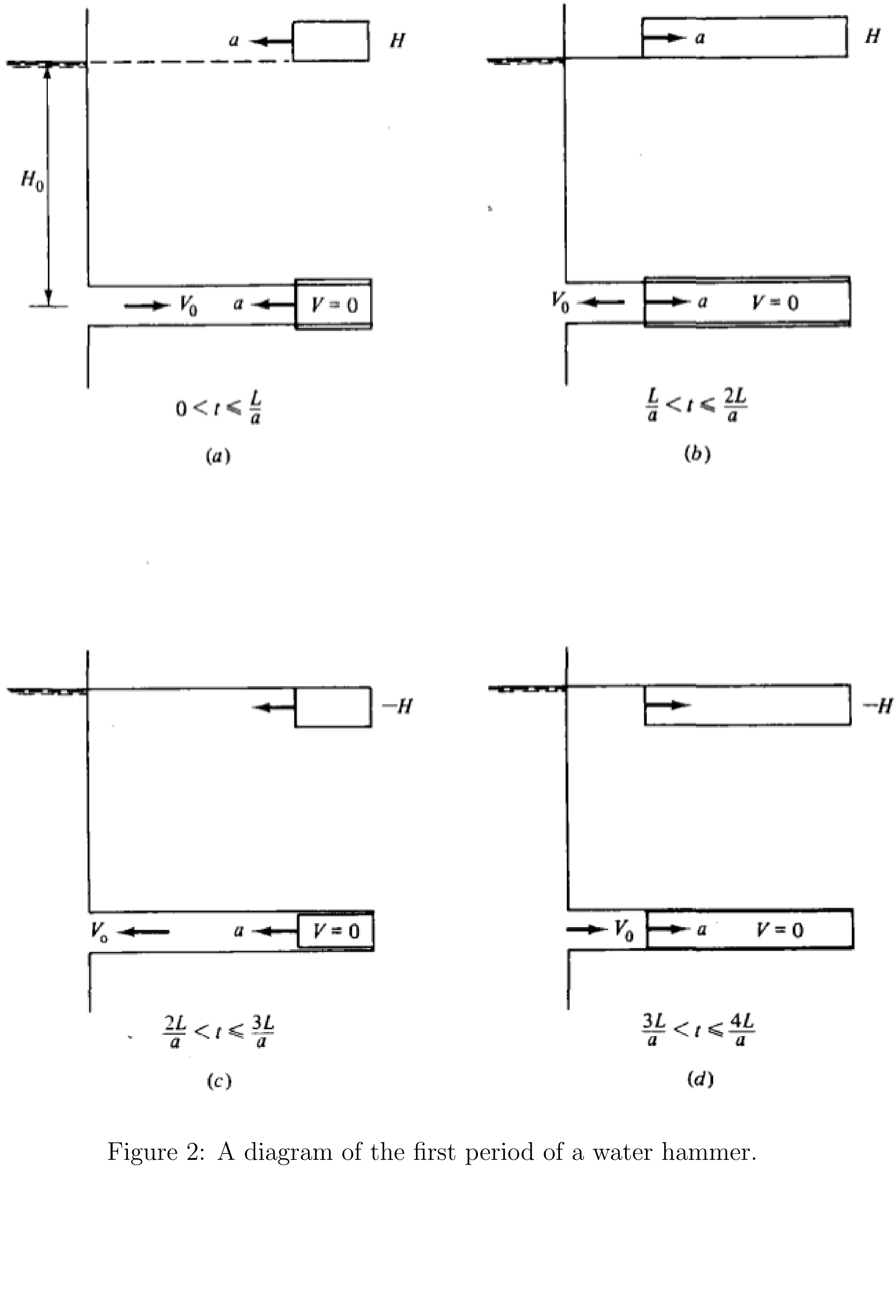

Figure 2: A diagram of the first period of a water hammer.

2 Numerical Computation of Hydraulic Transients

2.1 Method of characteristics

For this type of problem, we will use the characteristic equations method. This method consists on using an unknown multiplier λ and computing (1) + λ(2),

$$\left(\frac{\partial Q}{\partial t} + \lambda a^2 \frac{\partial Q}{\partial x}\right) + \lambda g A \left(\frac{\partial H}{\partial t} + \frac{1}{\lambda}\frac{\partial H}{\partial x}\right) + RQ\,|Q| = 0 \tag{3}$$

Equation (3) is true for all λ although we will only be interested in some particular values of it.

Parallel to that, the total derivatives of the solutions are

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\frac{dx}{dt}$$
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\frac{dx}{dt}$$

In order to plug the total derivatives in equation (3), we only need to impose $\frac{1}{\lambda} = \frac{dx}{dt} = \lambda a^2$. Thus, we need $\lambda = \pm\frac{1}{a}$ and we can rewrite (3) as,

$$\frac{dQ}{dt} + \frac{gA}{a}\frac{dH}{dt} + RQ\,|Q| = 0, \qquad \frac{dx}{dt} = a \tag{4}$$
$$\frac{dQ}{dt} - \frac{gA}{a}\frac{dH}{dt} + RQ\,|Q| = 0, \qquad \frac{dx}{dt} = -a \tag{5}$$

For a graphical explanation of these assumptions, we use Figure 3,

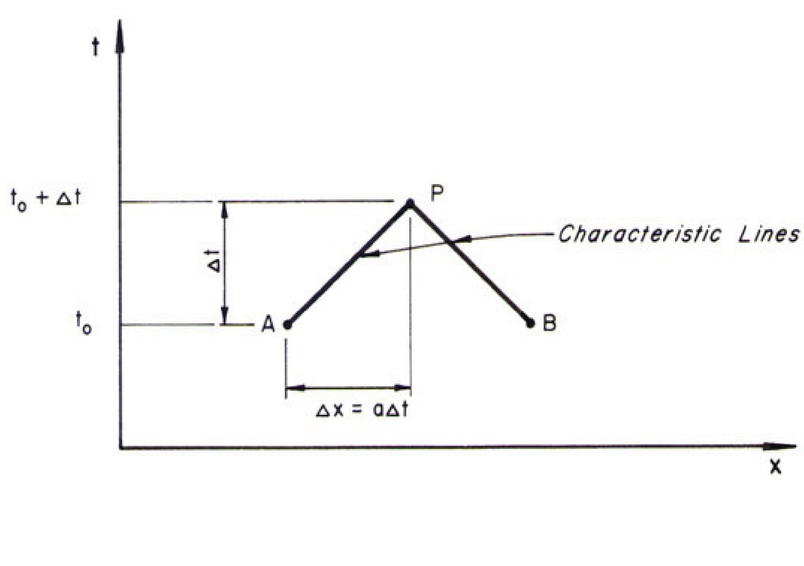

Figure 3: Method of characteristics assumptions scheme.

Slopes AP and BP are paths that a disturbance traverses for Δt in the $x-t$ plane. This is what we will use to integrate over time as follows. If we assume that Q and H are known at A and B, we can find the value at P integrating equations (4) and (5),

$$\int_A^P dQ + \frac{gA}{a}\int_A^P dH + R\int_A^P Q\,|Q|\,dt = 0 \tag{6}$$

$$\int_B^P dQ - \frac{gA}{a}\int_B^P dH + R\int_B^P Q\,|Q|\,dt = 0 \tag{7}$$

The first two terms are easy to evaluate. The third term, the friction term, is not straightforward as we do not know explicitly how Q varies with t. An usual action is to make a first order approximation of the integral:

$$R\int Q\,|Q|\,dt \simeq RQ_A\,|Q_A|\,(t_P - t_A) = RQ_A\,|Q_A|\,\Delta t \tag{8}$$

The problem with this approximation is that it may be unstable for higher friction terms - something characteristic of oil pipelines, for example - and thus a higher order approximation must be used. A common one is the trapezoidal rule:

$$R\int Q\,|Q|\,dt \simeq 0.5R(Q_A\,|Q_A| + Q_P\,|Q_P|)\Delta t \tag{9}$$

In this case, since Q_P is unknown, the resulting equations are a set of nonlinear implicit equations and an iterative solution will be used to find the solution in

$$Q_P - Q_A + \frac{gA}{a}(H_P - H_A) + \frac{1}{2}R\left(Q_A\,|Q_A| + Q_P\,|Q_P|\right)\Delta t = 0 \tag{10}$$

$$Q_P - Q_B - \frac{gA}{a}(H_P - H_B) + \frac{1}{2}R\left(Q_B\,|Q_B| + Q_P\,|Q_P|\right)\Delta t = 0 \tag{11}$$

The characteristic equations have a physical meaning: the state of point P in time $t = t_0 + \Delta t$ is determined by the state on its neighboring points, A and B, at $t = t_0$. Furthermore, the the distance at which this points are, is explicitly determined by the integration time and the speed of a mechanical wave in the water a. In other words, if there is a disturbance in point A at $t = t_0$, this disturbance will reach point P in $t = \frac{\Delta x}{a} = \Delta t$.

An additional algebraic equation is needed to establish continuity along the network, and this is the one of flow continuity:

$$\sum_{p\in\mathcal{P}_n^{\text{in}}} Q_p^{\text{in}} - \sum_{p\in\mathcal{P}_n^{\text{out}}} Q_p^{\text{out}} + \sum_{i\in\mathcal{S}_n} s_i - \sum_{j\in\mathcal{D}_n} d_j = 0 \tag{12}$$

with supplies s_i and demands d_j at all nodes $p \in \mathcal{P}$. This model leads to a Differential Algebraic system (DAE).

2.2 Initial conditions and boundary conditions

The initial condition is supposed to be the steady state of our network, i.e. by assuming the derivative with respect to time to be null in equations (1) and (2). Our base problem is of the form shown in Figure 4

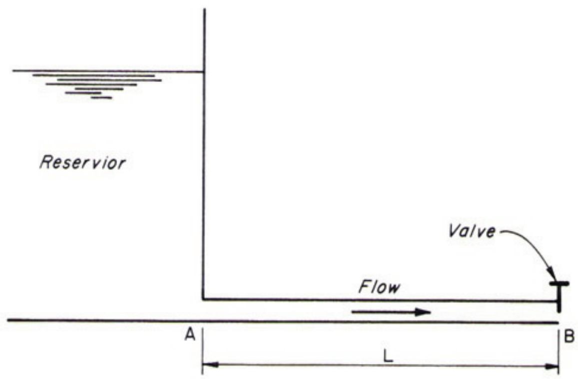

Figure 4: A reservoir and a pipe

Then, we impose upstream boundary conditions for the flow and downstream for the pressure obtaining the initial condition as

$$Q(x,0) = Q^{\text{out}} \tag{13}$$

$$H_p(x,0) = H^{\text{in}} - \frac{R}{gA} Q^{\text{out}} \left| Q^{\text{out}} \right| x \tag{14}$$

For the boundary conditions, following with the same upstream and downstream definitions discussed above, the flow will be modeled as the closure of a valve in time and the pressure will remain constant in the reservoir.

The factor τ representing the opening of the valve, has the following behavior

$$\tau(t) = \begin{cases} \sqrt[3/2]{1 - t/2} & t < 2 \\ 0 & t \geqslant 2 \end{cases}$$

With it, our upstream boundary conditions are

$$Q(L,t) = 0.5 \left(-C_v(t) + \sqrt{C_v(t)^2 + 4C_v(t)C_p} \right)$$
$$H(L,t) = \frac{C_p - Q(L,t)}{C_a}$$

and under the assumption of negligible entrance losses as well as velocity head, the downstream boundary conditions are

$$Q(0,t) = C_n + C_a H_p^{\text{in}}$$
$$H(0,t) = H_p^{\text{in}}$$

where

- $C_a = \frac{gA}{a}$.
- $C_p = Q_A + \frac{gA}{a} H_A - RQ_A |Q_A| \Delta t$, where A is the point at $t - \Delta t$ in Figure 3.
- $C_n = Q_B - \frac{gA}{a} H_B - RQ_B |Q_B| \Delta t$, where B is the point at $t - \Delta t$ in Figure 3.
- $C_v(t) = \frac{(\tau Q_P^2}{C_a H_P}$, where P is the point at t in Figure 3.

Other boundary conditions for dead ends, branchings, junctions, pumps and turbines can be found in [1, Chap. 3.3].

2.3 Convergence of the Finite Differences Method

As shown in [2], to have a convergent multi-step method it is necessary and sufficient to be stable and consistent. It is clear that the characteristic equations is a consistent method because the truncation error vanishes, for Δx and Δt converging to 0. Nonetheless, the stability is more delicate and requires from a condition called the Courant-Friedrichs-Lewy stability condition.

2.3.1 The Courant-Friedrichs-Lewy stability condition and discretization

In order to have a stable computation for the finite-difference scheme, we have to take into account the Courant-Friedrichs-Lewy condition in our discretization scheme. In this case, for the unidimensional flow in pipes:

$$C_N = a \frac{\Delta t}{\Delta x} \leqslant 1 \tag{15}$$

where a is related to the speed of sound in our liquid. However, in our particular case we have $\frac{\Delta x}{\Delta t} = a$. Thus, $C_N = 1$.

This condition leads to 2 new difficulties.

1. It is hard to make $C_N = 1$ since n_{length} is an integer,

$$C_N = a \Delta t \frac{n_{\text{length}}}{L}$$

   However, we can smartly choose Δt such that $C_N \approx 1$ for all pipes.

2. Since the time step must be the same for all pipes, the discretization in space needs to be the same too. Thus, if we have a network with two pipes, one much longer than the other one, the long one will have too many discretization points.

2.4 Solution of the nonlinear equations

The characteristic equations derived in section 2.1 result in a set of nonlinear equations that may be explicit or implicit depending on the approximation of the friction factor.

The solution of these equations is computed using the Newton-Rhapson algorithm implemented in the PETSc library as 'SNES'.

The mathematical characterization of this problem is as follows:

$$\text{For } f : \mathbb{R}^n \to \mathbb{R}^m \text{ find } x^* \in \mathbb{R}^n \text{ such that } f(x^*) = 0$$

This iterative method relies on an initial guess x_0 for the value of x^*, and next elements in the sequence are obtained employing the slope of $f(x)$. The univariate case can be derived using the first order Taylor approximation:

- Approximate $f(x^{k+1})$

$$f(x^{k+1}) \approx f(x^k) + f'(x^k)(x^{k+1} - x^k)$$

- Look for $f(x^{k+1}) = 0$

$$f'(x^k)(x^{k+1} - x^k) = -f(x^k)$$

Analogously, for the multivariate case:

$$J(x^k)(x^{k+1} - x^k) = -f(x^k) \tag{16}$$

where $J(x^k)$ is the Jacobian at x^k. We write $\Delta x^k = x^{k+1} - x^k$ so that our next element x^{k+1} will come from the previous equation as:

$$x^{k+1} = \Delta x^k + x^k = -J^{-1}(x^k)f(x^k) \tag{17}$$

Remember that we look for $f(x^*) = 0$ but an exact solution is usually not possible to reach. Because of this we set a convergence tolerance ϵ and when we find $\|f(x^k)\| \leqslant \epsilon$ we will say that $x^{k+1} \approx x^*$.

2.5 Software architecture

The previous numerical procedures have been coded in C using the PETSc library. Given that the purpose of the project was to build a simulator capable of dealing with large networks, special attention has been given to the software architecture and in particular the data structures that allow to simulate a complex network with various boundary conditions in a systematic way.

Althought our systems are networks (consisting in edges and nodes) they are not lumped networks. In fact, each edge will have a different spatial discretization depending on its **a** term. One of the most difficult planning issues has been the one of relating each pipe Q and H variables - each pipe having a different spatial discretization - to the global X and F vectors. This issue has been solved by doing some 'bookkeeping' and storing the global position in the pipe data structure. An UML diagram describing the architecture of the data structures is shown in Fig. 6. Fig. 2.5 shows the sequence diagram of the overall program.

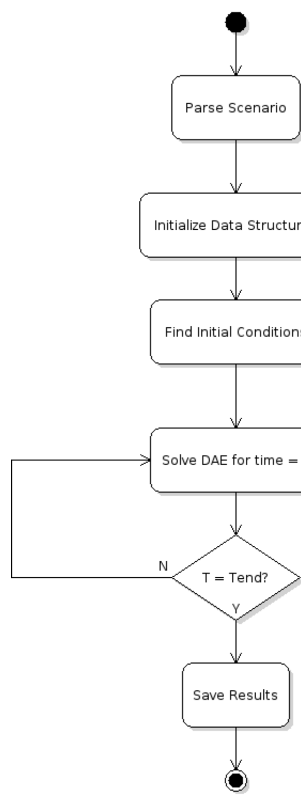

Figure 5: Sequence diagram of program.

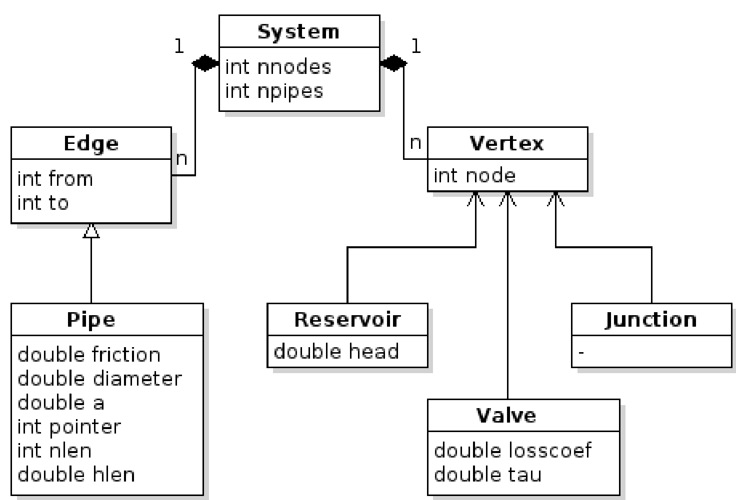

Figure 6: UML diagram of program

3 Results and Analysis

In this section we will run a series of simulations to test:

- The validity of our results.
- The scalability of our program.

3.1 A textbook example

Classical reference [3] is widely used in transient hydraulics literature. The numerical methods employed and the plethora of details concerning boundary conditions, friction approximations, etc... make this book very relevant still after more than 30 years of its publications.

To test the validity and accuracy of our project we have implemented one of the test cases found in this book, which we show in Figure 4. In this test case, a pipe is connected to a reservoir that offers a constant pressure in A. At the end of the pipe, a valve is fully open. The valve starts to close following the equation:

$$\tau = \left(1 - \frac{t}{t_c}\right)^{E_m} \tag{18}$$

where t is the current time step and t_c is the closure time (i.e when the valve will be fully closed).

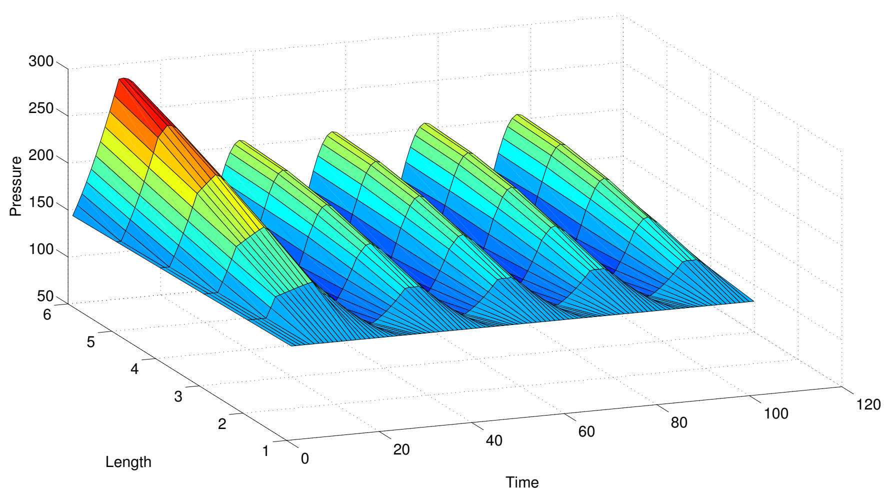

Figure 7: Pressure wave created by closure of valve

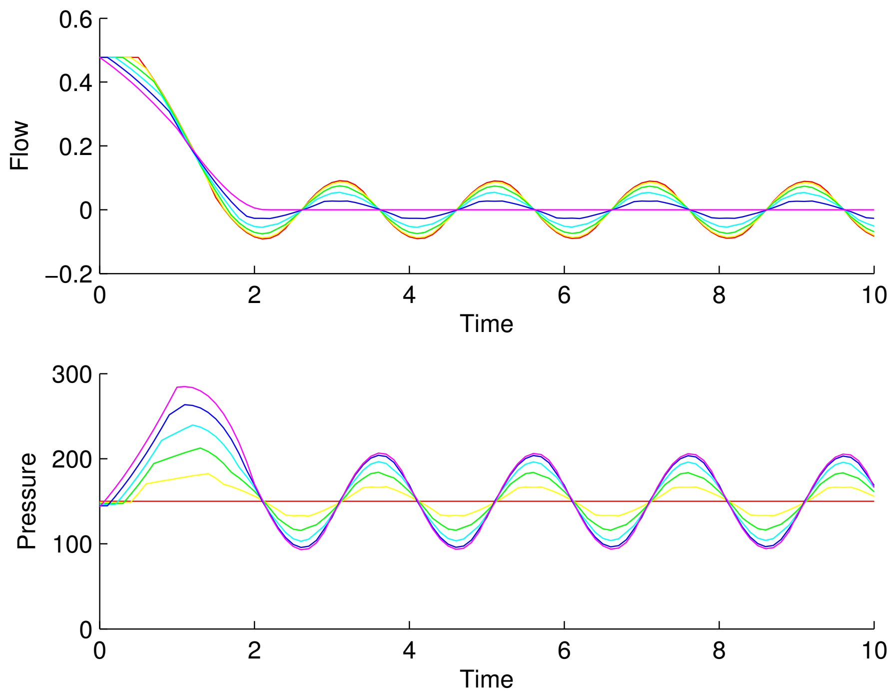

Figure 8: Profile view of pressure and flow. Each color is a different point along the pipe.

When the valve is closed, the pressure will suddenly surge next to the valve. This is very easy to picture, if we have a garden hose and we pinch the exit, the water will flow with higher pressure.

In the water-hammer problem, the closure of the valve will produce a surge in the pressure at the end of the pipe. A pressure wave - a show wave - will propagate from the point B (valve) to the point A (reservoir). When the wave reaches the reservoir - which is a boundary condition that sets the pressure constant at that point - reflect and return to the opposite end.

This behavior can be understood schematically by looking at Figure 2. The actual simulation of this phenomena can be seen in Figure 7 where the pressure disturbance is clearly seen as a wave in the surface plot.

The oscillating nature of the pressure wave can be easily seen in Figure 8. As we discussed before, the period of the pressure wave is proportional to the pipe length and the speed of the sound in the fluid.

A quick calculation of the wave period:

$$T = \frac{4L}{a} = \frac{4 \cdot 600\ m}{1200\ m/s} = 2\ s \tag{19}$$

Which agrees with our simulations.

3.1.1 Increase of friction

One of the main reasons for using a nonlinear solver in our program is the computation of the friction term. As discussed previously, the characteristic equations have an explicit form if the approximation of the friction term is linear.

This a approximation, as discussed before, is unstable for higher friction values and a trapezoidal approximation is used, which results in the characteristic equations being in an implicit form.

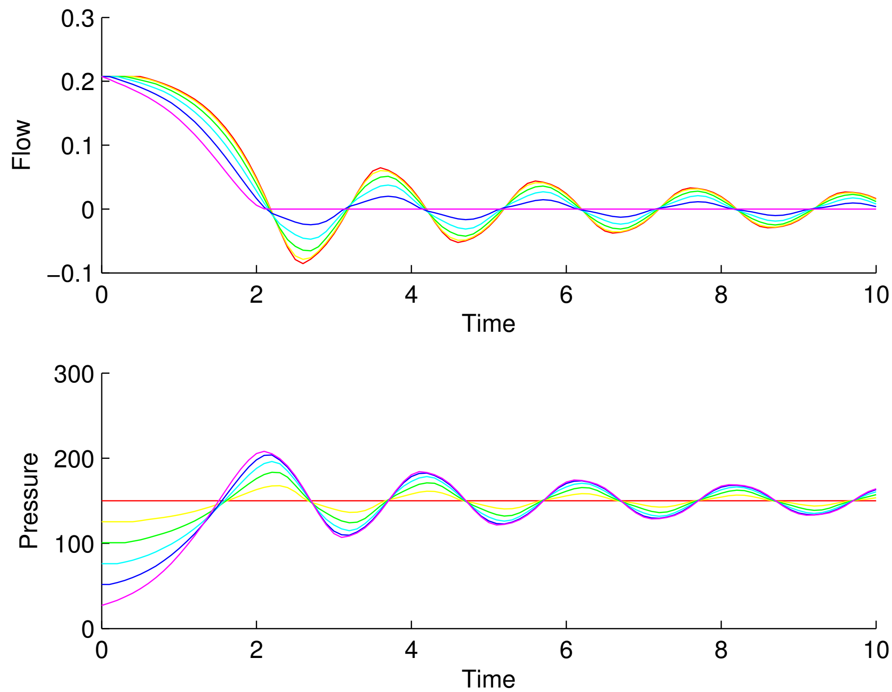

Figure 9: Textbook problem with high friction coefficient.

The importance of the friction term can be seen in Figure 9. A higher friction term in the pipe will lead to more energy dissipation in the transient. We can see how in this case, the oscillation is more damped than in the previous case.

Higher terms of friction are more common in oil pipelines and thus, accurate computation is necessary.

3.1.2 Instantaneous valve closure

One of the harshest scenarios that can occur in a pipe transient is the sudden closure of a valve - a dirac delta - were the output flow at the valve will be set to zero. We can see that the pressure wave reaches much higher values than our previous experiments (Fig, 10) and its profile is much more stiff.

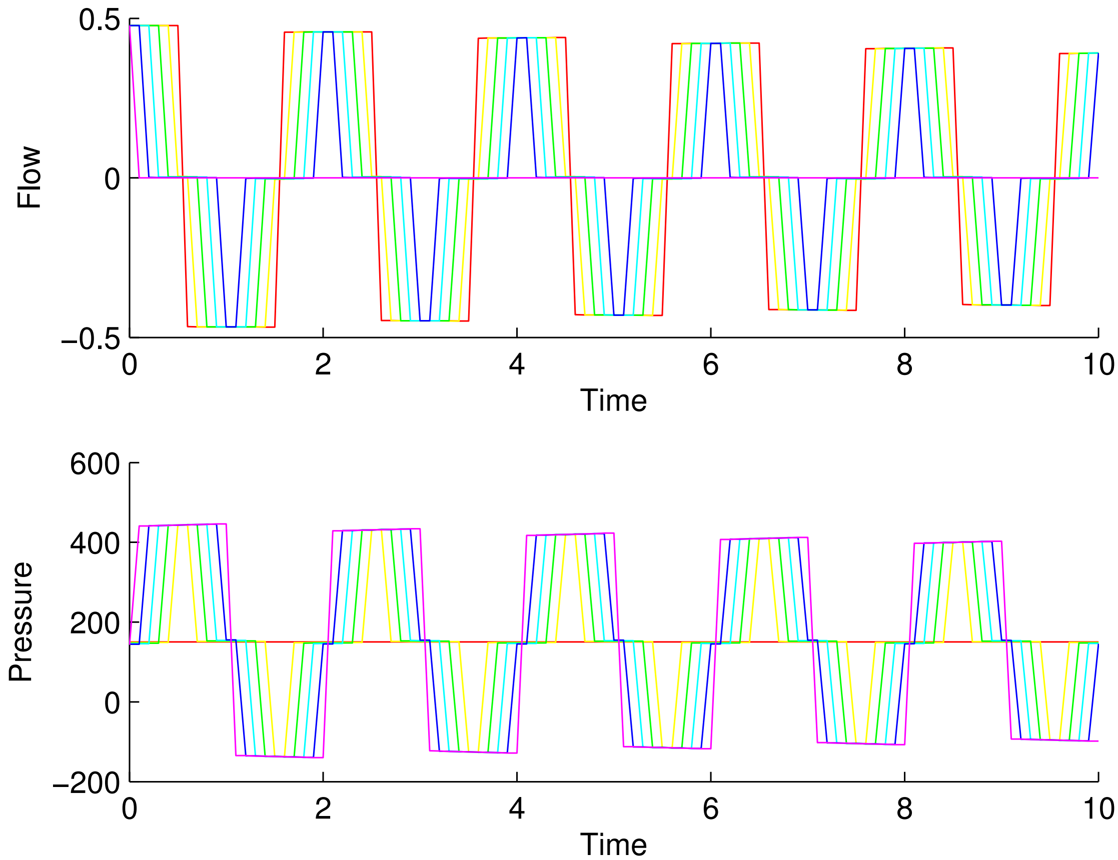

Figure 10: Textbook problem with sudden valve closure.

3.2 Network of contiguous pipes

The necessity of high performance computing arises when the transients are calculated in large and complex networks. Not only the number of variables increases geometrically with the number of pipes, but also the existence of comparatively small pipes will force the usage of small time steps.

Analysis of complex networks is interesting as, by aggregating simple element, the overall results can be quite complex.

In the following figures, we show an example. First, we have created a network just by linking contiguous pipes. The overall length will increase. As calculated previously, the pressure wave period will change depending on the overall length of the transmission network. We can see this, by comparing the period of Figure 11 and Figure 13.

Moreover, as the wave travels more distance between both ends, the friction losses will be more acute. This fact can be observed in the referenced plots.

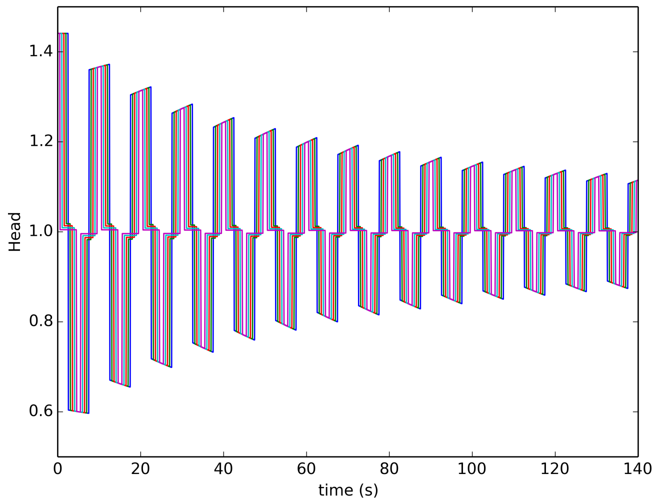

Figure 11: Network with 5 contiguous pipes.

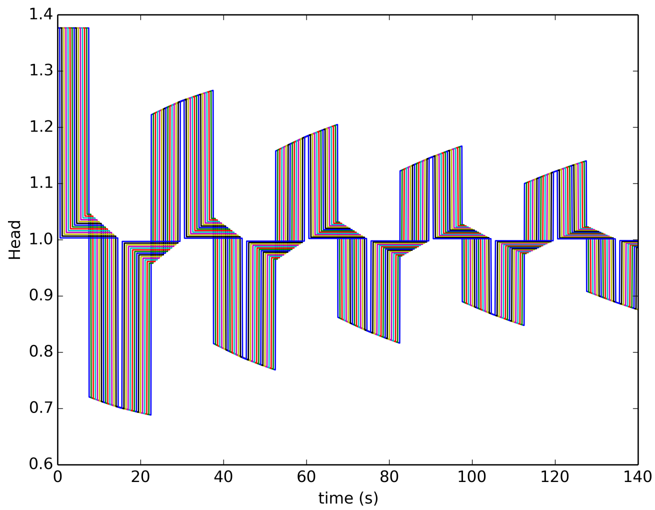

Figure 12: Network with 15 contiguous pipes.

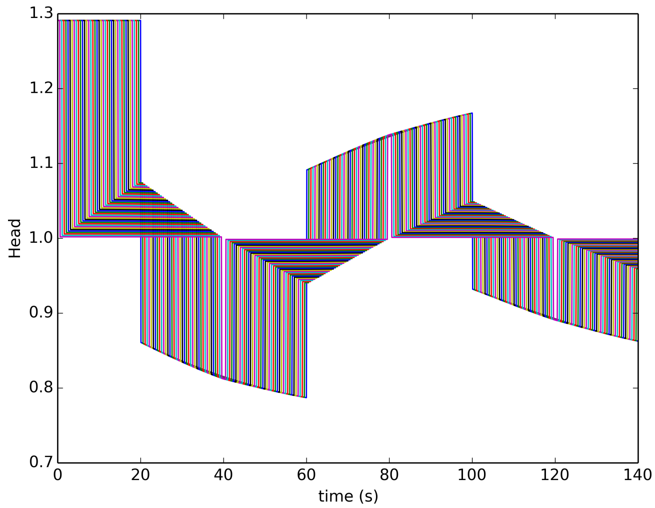

Figure 13: Network with 40 contiguous pipes.

3.3 A more complex network

In the previous network, all the pipes had similar friction and diameters as to being able to observe the resonant phenomena. A modified version of the 5 pipes network where all the pipes have different diameter and friction, and one of the pipes has a very high friction coefficient is shown in Figure 14. The conclusion of this, is that complex networks can show behavior that is difficult to predict without accurate simulations.

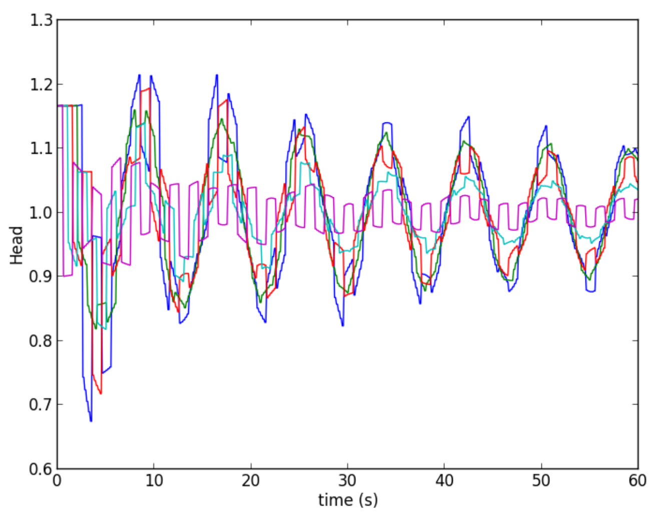

Figure 14: Pressure wave in a complex network.

4 Conclusions and Future Work

In this project we have explored the numerical computation of fluid transients in closed networks. Our solution methodology has been the Characteristic Equations, which is the most common adopted methodology in the literature.

Our focus has been the one of solving large scale networks and developing a platform to be able to handle any-sized and any-configuration network. This, given the particularities of the problem, has not been trivial, and the planning and developing of the main software has been the most time consuming task after the initial literature review. However, this platform will give us the backbone to perform large scale computational experiments.

The nature of the Jacobian matrix of this problem is sparse (if the network is large enough) and is amenable to iterative Krylov methods, which are the ones we are using.

Future work has two lines:

1 **Parallelization** This task has already begun. We are using the DMNetwork module of the PETSc library. In the branch 'parallel' of the repository the initial work of creating data structures in processor 0 and distributing them to each processor can be shown. Some data structures that were suitable for the sequential code are now being rewritten as they are now not amenable to parallelization.

2 **Complex modeling** Current models have few complexities. We need to add pumps and variable aperture valves. We also have to add support to the Characteristic Method with Interpolation to be able to handle small sized pipes accurately. This will allow us to run real large scenarios and compare results and performance with commercial tools.

Being able to accurately model and simulate, with good simulation times and large scale networks, may provide usefulness to researchers in operation and control of hydraulic networks and oil pipelines.

References

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