Restricted Coloring Problems on Restricted Classes of Graphs

Andrew Lyons[1,2]

[1] Computation Institute, University of Chicago
5640 S. Ellis Avenue, RI 405, Chicago, Illinois 60637, USA
[2] Mathematics and Computer Science Division, Argonne National Laboratory
9700 South Cass Avenue, Argonne, Illinois 60439-4844, USA
lyonsam@gmail.com

Keywords. Acyclic coloring, star coloring, forbidden induced subgraph

1 Introduction

An *acyclic coloring* of a graph is a proper coloring such that any two color classes induce a forest. A *star coloring* of a graph is an acyclic coloring with the further restriction that the forest induced by any two color classes is a disjoint collection of stars. We consider the behavior of these problems when restricted to certain classes of graphs. In particular, we give characterizations of the classes of graphs for which two or more of these restricted coloring problems are equivalent, in that they share the same set of solutions. Surprisingly, our characterizations of these classes in terms of forbidden induced subgraphs equate them with classes that are well-studied in the literature. We extend this framework to encompass other restricted coloring problems, both known and new, and outline a method for obtaining results similar to those given here.

We also explore the algorithmic implications of these results in terms of finding optimal acyclic and star colorings on certain classes of graphs. We show that optimal acyclic colorings of certain subclasses of even-hole-free graphs can be found in polynomial time, and that optimal acyclic and star colorings of trivially perfect graphs can be found in linear time.

2 Main Results

We consider the interrelationship between graph coloring problems that arise in the optimal evaluation of sparse Hessian matrices. In particular, we focus on classes of graphs for which different restricted coloring problems share the same set of solutions.

Finding an optimal coloring is **NP**-complete on general graphs for each of these problems; see [1] for a survey of these problems as they relate to the computation of derivative matrices.

Theorem 1 *A graph G is even-hole-free if and only if every coloring of G is also an acyclic coloring.*

Corollary 2 *There is a polynomial time algorithm for finding an optimal acyclic coloring of chordal graphs and (even-hole,diamond)-free graphs.*

Theorem 3 *A graph G is trivially perfect if and only if every coloring of G is also a star coloring.*

Theorem 4 *A graph G is a cograph if and only if every acyclic coloring of G is also star coloring.*

Suppose $\mathcal{C}$ is a class of graphs with the property that the mappings that satisfy the conditions of some coloring problem $\mathcal{P}$ are exactly the mappings that satisfy the conditions of some other coloring problem $\mathcal{Q}$. Then any algorithm for solving $\mathcal{P}$ on class $\mathcal{C}$ can also be used to solve $\mathcal{Q}$. For example, it was shown in [2] that every distance-1 coloring of a chordal graph is also an acyclic coloring. Consequently, the many well-known linear time algorithms for solving distance-1 coloring on chordal graphs can also be used to find optimal acyclic colorings.

We give characterizations for each of the following classes of graphs.

1. The graphs for which every coloring is also an acyclic coloring.
2. The graphs for which every coloring is also a star coloring.
3. The graphs for which every acyclic coloring is a star coloring.

Quasi-threshold graphs can be recognized and optimally distance-1 colored in linear time [3]. The following result implies that an optimal star coloring of a quasi-threshold graph can also be found in linear time.

Theorem 5 *The graphs for which every valid distance-1 coloring is also a star coloring are exactly the quasi-threshold graphs.*

A graph is *even-hole-free* if it contains no induced cycle with an even number of vertices.

Theorem 6 *The graphs for which every valid distance-1 coloring is also a acyclic coloring are exactly the even-hole-free graphs.*

Suppose we are asked to find an optimal acyclic coloring of some graph G. We can determine in polynomial time whether G is even-cycle-free [4] and, if so, we can use one of the many heuristics and approximation algorithms for distance-1 coloring general graphs.

A graph is a *cograph* if it has no induced P_4. Cographs can be recognized in linear time [5].

Theorem 7 *The graphs for which every acyclic coloring is also star coloring are exactly the cographs.*

References

1. Gebremedhin, A.H., Manne, F., Pothen, A.: What color is your Jacobian? Graph coloring for computing derivatives. SIAM Review **47** (2005) 629–705
2. Gebremedhin, A., Pothen, A., Tarafdar, A., Walther, A.: Efficient computation of sparse Hessians using coloring and automatic differentiation. INFORMS J. Computing (2009) to appear.
3. Yan, J.H., Chen, J.J., Chang, G.J.: Quasi-threshold graphs. Discrete Appl. Math. **69** (1996) 247–255
4. Chudnovsky, M., Kawarabayashi, K., Seymour, P.: Detecting even holes. Journal of Graph Theory **48** (2004) 85–111
5. Corneil, D.G., Perl, Y., Stewart, L.K.: A linear recognition algorithm for cographs. SIAM Journal on Computing **14** (1985) 926–934