Restricted Coloring Problems on Restricted Classes of Graphs

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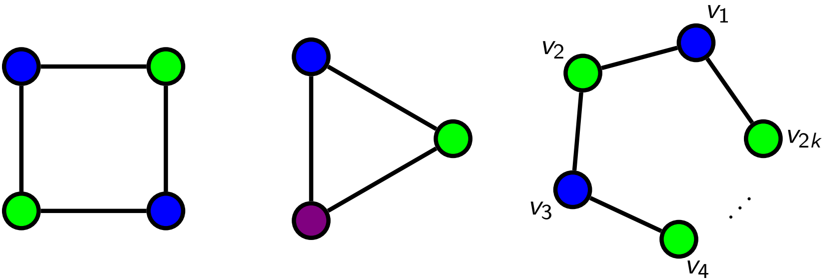
Outline

- Restricted Coloring Problems
  - Acyclic Coloring
  - Star Coloring
- Coloring $\Longleftrightarrow$ Acyclic Coloring
- Coloring $\Longleftrightarrow$ Star Coloring
- Acyclic Coloring $\Longleftrightarrow$ Star Coloring
- Promise Algorithms vs. Robust Algorithms

Coloring

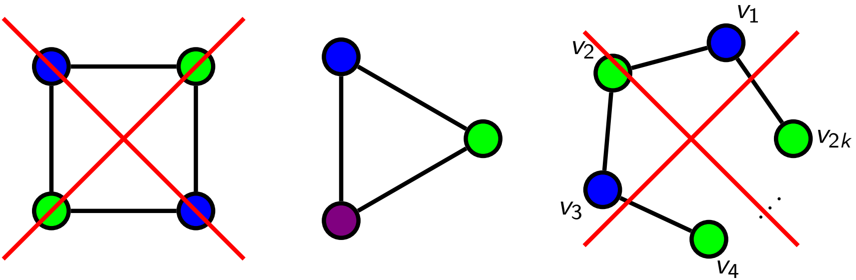
proper vertex coloring



chromatic number $\chi(G)$

Acyclic Coloring

proper vertex coloring without bichromatic cycles



acyclic chromatic number $\chi_a(G) \geq \chi(G)$

Every pair of colors induces a disjoint collection of trees (a forest)

Algorithms for Acyclic Coloring

Algorithms based on maximum degree $\Delta(G)$

- If $\Delta(G) \leq 3$, then G can be acyclically colored using 4 colors of fewer in linear time.
- If $\Delta(G) \leq 5$, then G can be acyclically colored using 9 colors of fewer in linear time.

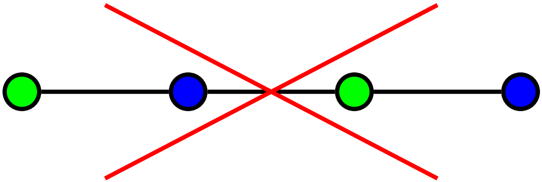
Coloring

proper vertex coloring

chromatic number $\chi(G)$

Star Coloring

proper vertex coloring without bichromatic P_4s



star chromatic number $\chi_s(G) \geq \chi_a(G) \geq \chi(G)$

Every pair of colors induces a disjoint collection of stars

Google Scholar Queries

- graph coloring : $\sim$109,000 results
- "acyclic coloring ": 150 results
- "star coloring ": 95 results

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How can we leverage what is known about the classical coloring problem?

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Acyclic Coloring

Theorem (Gebremedhin et. al., 2008)

If a mapping ϕ is a coloring of a chordal graph G, then ϕ is also an acyclic coloring of G.

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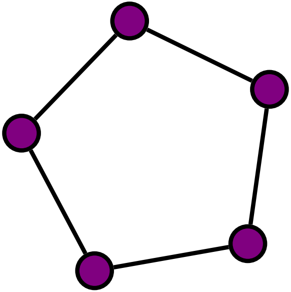

No induced chordless cycles

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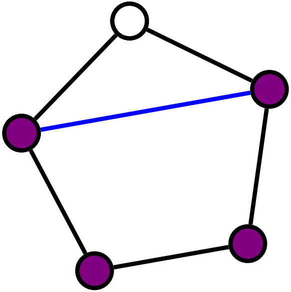

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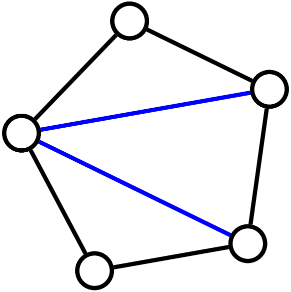

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Application: Band graphs (from banded matrices)

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Application: Band graphs (from banded matrices)

Does G have to be chordal for this to be the case?
For which other graphs is this true?

When is Every Coloring Also an Acyclic Coloring ?

Theorem

Every coloring of G is also an acyclic coloring if and only if G is an even-hole-free graph.

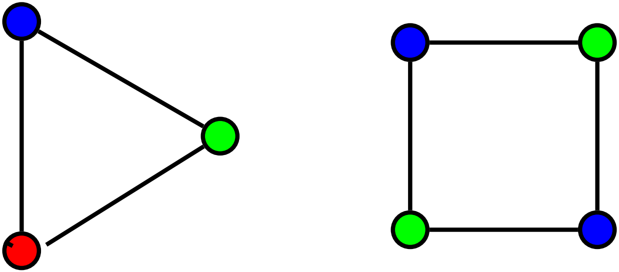
even-hole-free graph

A graph is <u>even-hole-free</u> if it contains no induced even cycle.

Also allows odd chordless cycles

Acyclic Coloring

What about when $\chi(G) = \chi_a(G)$?



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Promise Algorithms vs. Robust Algorithms

When is Every Coloring Also a Star Coloring ?

- every coloring must also be an acyclic coloring
  $\Rightarrow$ must be subclass of the even-hole-free graphs

When is Every Coloring Also a Star Coloring ?

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Definition (trivially perfect graph)

A graph G is trivially perfect if for every induced subgraph G' of G the number of maximal cliques in G' is equal to the size of the largest independent set in G'

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- cannot contain induced P_4

Definition (trivially perfect graph)

A graph G is trivially perfect if for every induced subgraph G' of G the number of maximal cliques in G' is equal to the size of the largest independent set in G'

Theorem (Golumbic 1978)

A graph is trivially perfect if and only if it has no induced C_4 or P_4.

Coloring $\Longleftrightarrow$ Star Coloring

Theorem

The graphs for which every coloring is also a star coloring are exactly the trivially perfect graphs.

Coloring $\Longleftrightarrow$ Star Coloring

Theorem

The graphs for which every coloring is also a star coloring are exactly the trivially perfect graphs.

Proof.

$\square$

Corollary

If G is a trivially perfect graph then $\chi(G) = \chi_a(G) = \chi_s(G)$.

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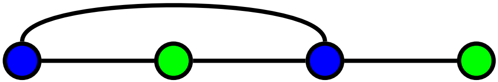

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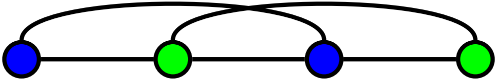

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Acyclic Coloring $\Longleftrightarrow$ Star Coloring

Theorem

The graphs for which every acyclic coloring is also a star coloring are exactly the cographs.

Definition (cograph)

A graph is a cograph if and only if it contains no induced P_4.

Acyclic Coloring $\Longleftrightarrow$ Star Coloring

Theorem

The graphs for which every acyclic coloring is also a star coloring are exactly the cographs.

Definition (cograph)

A graph is a cograph if and only if it contains no induced P_4.

Theorem

cographs can be acyclically colored (and thus star colored) in linear time.

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Promise Algorithms

- Assume input is in the restricted domain
- Behavior undefined when input isn't in the domain
  Example: independent set (**NP**-hard!) in well-covered graphs (**NP**-hard to recognize!)

Robust Algorithms

Solves the problem if the input is in the domain. If not, returns a certificate that says the input isn't in the domain
Example: maximum clique (**NP**-hard!) in unit disk graphs (**NP**-hard to recognize!)

Acyclic Coloring even-hole-free graphs

chordal graphs

Recognizable and colorable in polynomial time

(even-hole,diamond)-free graphs

Recognizable and colorable in polynomial time

$(C_4, 2K_2)$-free graphs (pseudosplit graphs)

Recognizable and colorable in polynomial time

Acyclic Coloring even-hole-free graphs robustly

even-hole-free graphs

- Best known recognition algorithm runs in $O(n^{15})$ time
- No known efficient algorithm for coloring

Example: Suppose we had an algorithm that finds an optimal coloring of any graph.