# Acyclic Colorings and Triangulations of Weakly Chordal Graphs

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Dartmouth CS Theory Reading Group

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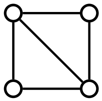
# Outline

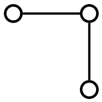
# Subgraphs and Induced Subgraphs

Graph $G$



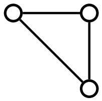
Subgraph $G'$

$V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$



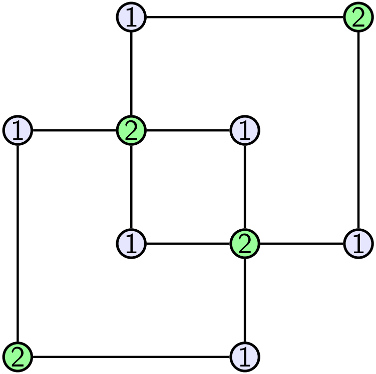
Induced Subgraph $G[S]$

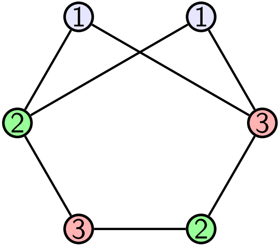
$V(G[S]) = S \subseteq V(G)$ and
$E(G[S])$ consists of all edges with both endpoints in $S$
(vertex-induced subgraph)



# Graph Coloring

An assignment $\phi$ of colors to the vertices $V(G)$
such that $\phi(u) \neq \phi(v)$ for all $\{u, v\} \in E(G)$



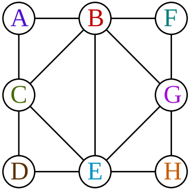




Chromatic number $\chi(G)$

# Chordal (Triangulated) Graphs

## Definition

*A graph $G$ is chordal if it does not contain $C_k$ with $k \geq 4$ as an induced subgraph. (Every $C_k$ with $k \geq 4$ has a chord.)*



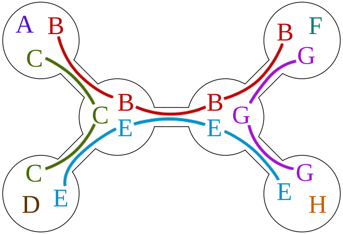




image: David Eppstein (public domain) `http://commons.wikimedia.org/wiki/File:Tree_decomposition.svg`

# The Triangulating Colored Graphs Problem

Triangulating Colored Graphs (TCG)
**Instance**: Graph $G$ and a coloring $\phi$ of $G$.
**Question**: Is there a chordal supergraph $H \supseteq G$ such that $\phi$ is a coloring of $H$?

(Bodlaender, Fellows, Hallett, Wareham, and Warnow 1992). TCG is...

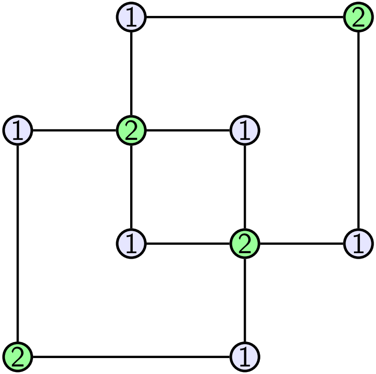
...NP-complete

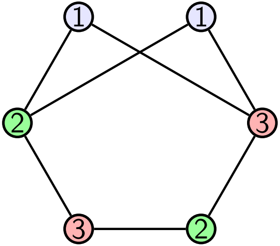
...$W[t]$-hard for all $t \in \mathbb{N}$.

# The Triangulating Colored Graphs Problem

**Lemma**

*If a graph G colored by $\phi$ contains a bichromatic cycle, then G is not $\phi$-triangulatable.*

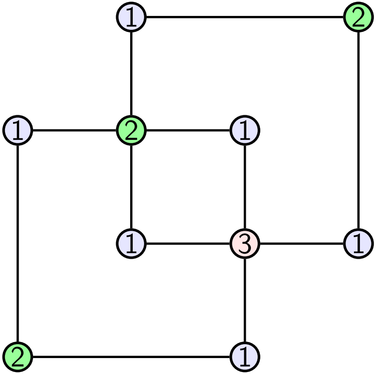


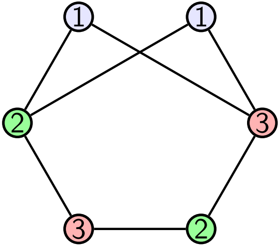



# The Triangulating Colored Graphs Problem

**Lemma**

*If a graph $G$ colored by $\phi$ contains a bichromatic cycle, then $G$ is not $\phi$-triangulatable.*



# Clique Separators

## Definition

*Let $G$ be a connected graph. $S \subset V(G)$ is a...*

separator *if $G - S$ is disconnected.*

clique separator *if $G[S]$ is a clique.*

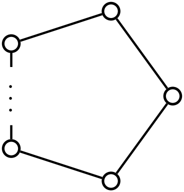
## Lemma

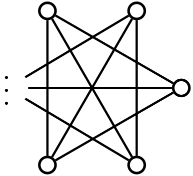
*If $G$ is a graph with coloring $\phi$ and clique separator $S$, then $G$ is $\phi$-triangulatable if and only if $G[S \cup R]$ is $\phi$-triangulatable for every connected component $R$ of $G - S$.*

# Weakly Chordal Graphs

## Definition

*A graph $G$ is weakly chordal if it does not contain $C_k$ or $\overline{C}_k$ with $k \geq 5$ as an induced subgraph.*




Forbidden induced subgraphs for weakly chordal graphs.

# Weakly Chordal Graphs

## Definition

*A graph $G$ is weakly chordal if it does not contain $C_k$ or $\overline{C}_k$ with $k \geq 5$ as an induced subgraph.*

Can be recognized in $O(m^2)$ time

## Subclasses include

- chordal graphs
- permutation graphs
- distance-hereditary graphs
- $P_4$-sparse graphs
- chordal bipartite graphs

# Weakly Chordal Graphs

## Definition

*A graph $G$ is weakly chordal if it does not contain $C_k$ or $\overline{C}_k$ with $k \geq 5$ as an induced subgraph.*

## Lemma

*If $\phi$ is a coloring of a weakly chordal graph $G$, then an edge $uv \in E(G)$ is contained in a bichromatic cycle if and only if $uv$ is contained in a bichromatic $C_4$.*

# Outline

# An Algorithm for Triangulating Colored Graphs

**Theorem**

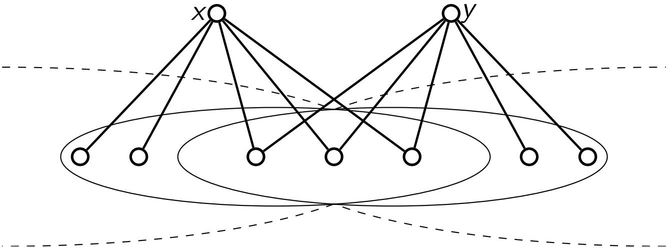
*There exists a polynomial-time algorithm that, given a weakly chordal graph $G$ with a coloring $\phi$, produces a $\phi$-triangulation of $G$ if one exists, and produces a bichromatic cycle in $G$ otherwise.*

This algorithm is certifying: it always returns a certificate

# Two-Pairs

## Definition

*A pair $\{x, y\}$ of distinct, non-adjacent vertices is a two-pair if every induced path from $x$ to $y$ consists of exactly two edges.*



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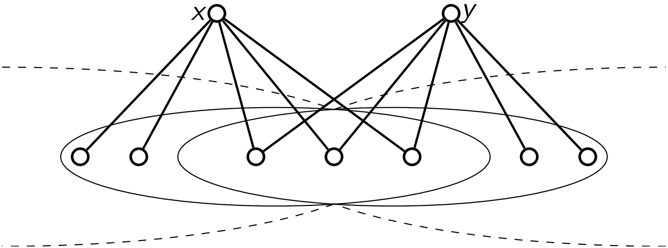


## Theorem (Hayward, Hoàng, and Maffray 1989)

*If $G$ is a weakly triangulated graph, then every induced subgraph of $G$ that is not a clique contains a two-pair.*

# Two-Pairs

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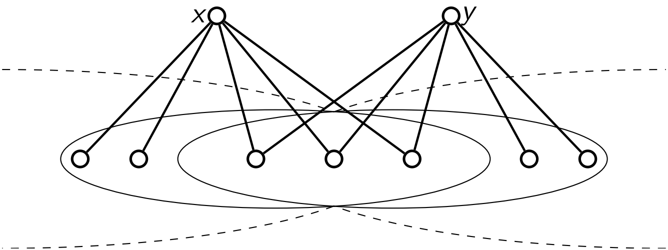
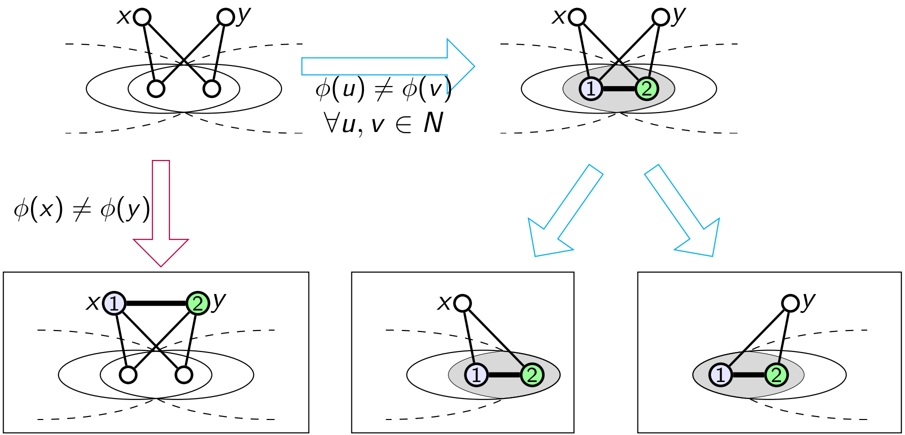



*Key observation: If $G$ is $\phi$-triangulatable, then either $\phi(x) \neq \phi(y)$ or or $\phi(u) \neq \phi(v)$ for all $u, v \in N(x) \cap N(y)$.*

# The Algorithm



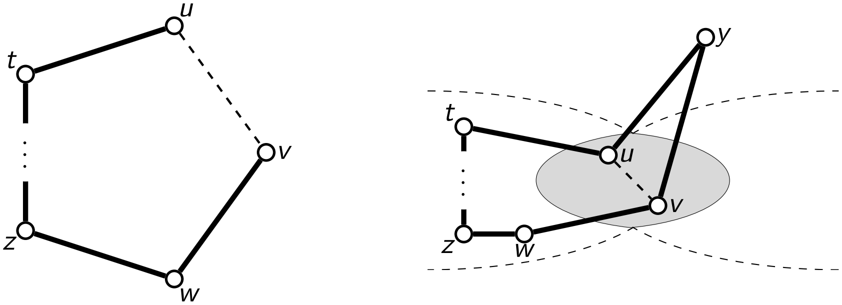

Throughout: $G$ is weakly chordal, $\phi$ is a coloring of $G$,
and we don't create bichromatic cycles

# Completing the Shared Neighborhood of a Two-Pair

**Lemma**

*G weakly chordal* $\Longrightarrow G + \text{clique}(N(x) \cap N(y))$ *weakly chordal*

**Proof.**




The addition of edge $uv$ cannot create a hole.

□

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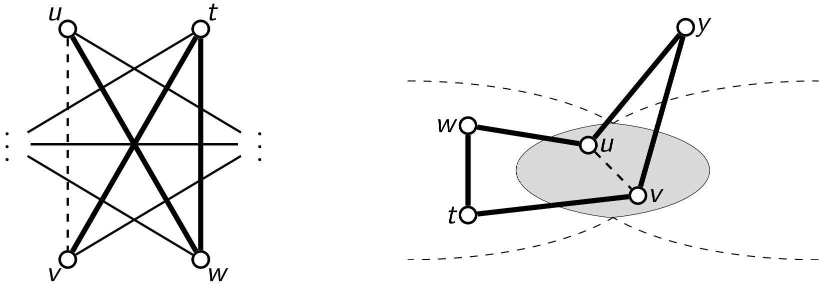



The addition of edge $uv$ cannot create a antihole.

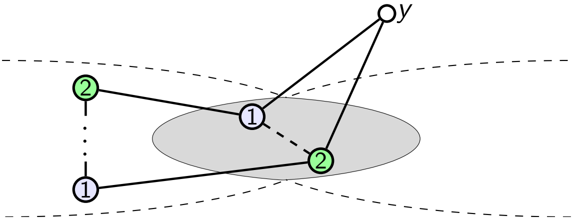
□

# Completing the Shared Neighborhood of a Two-Pair

**Lemma**

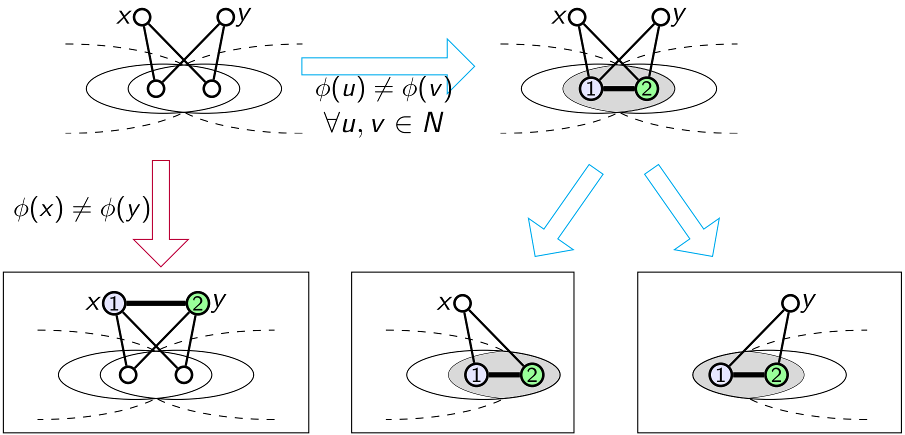
*Completing the shared neighborhood of two-pair $\{x, y\}$ cannot create a bichromatic cycle*

**Proof.**




We've shown that adding edge $uv$ cannot create a hole or an antihole; we still need to show that it cannot create a bichromatic $C_4$. □

# The Algorithm




Throughout: $G$ is weakly chordal, $\phi$ is a coloring of $G$, and we don't create bichromatic cycles

# Connecting a Two-Pair

**Lemma (Spinrad and Sritharan, 1995)**

*If $\{x, y\}$ is a two-pair in a graph $G$, then $G$ is weakly chordal if and only if $G + xy$ is weakly chordal.*

**Lemma**

*Connecting two-pair $\{x, y\}$ cannot create a bichromatic cycle.*

# Connecting a Two-Pair

**Lemma**

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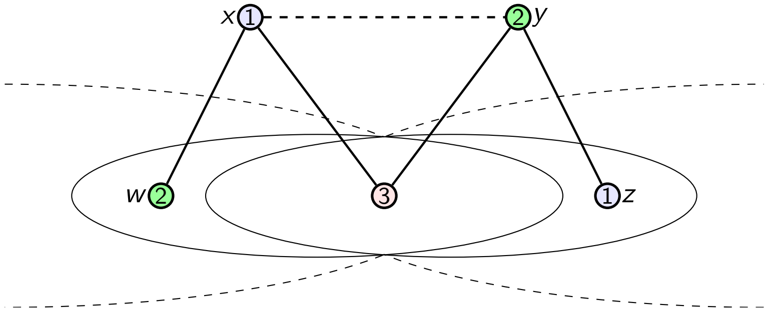
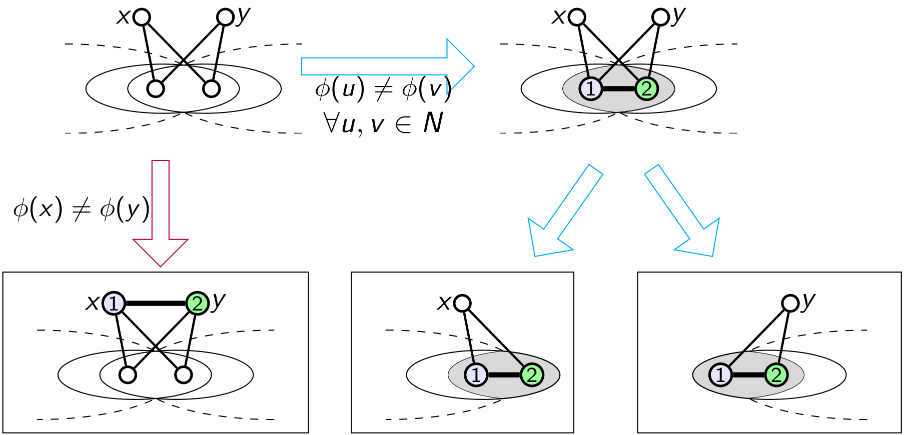
**Proof.**




□

# The Algorithm




Throughout: $G$ is weakly chordal, $\phi$ is a coloring of $G$, and we don't create bichromatic cycles

Algorithm $\texttt{TCWCG}(G, \phi)$

**input** : weakly chordal graph $G$ with coloring $\phi$

**output**: a $\phi$-chordalization of $G$ if one exists, and a bichromatic cycle in $G$ otherwise

---

**if** *$G$ is chordal* **then**
  **return** $G$

find two-pair $\{x, y\}$

**if** $\phi(x) \neq \phi(y)$ **then**
  **return** $\texttt{TCWCG}(G + xy, \phi)$
**else if** $\phi(u) \neq \phi(v)$ *for all distinct* $u, v \in N(x) \cap N(y)$ **then**
  $G \leftarrow G + \text{clique}\big(N(x) \cap N(y)\big)$
  **forall the** *connected components $A$ of $G - S$* **do**
    $G \leftarrow G \cup \texttt{TCWCG}\big(G[A \cup S], \phi\big)$
  **return** $G$
**else**
  **return** *bichromatic cycle uxvy in $G$*

# A Consequence of the Algorithm

**Theorem**

*There exists a polynomial-time algorithm that, given a weakly chordal graph $G$ with a coloring $\phi$, produces a $\phi$-triangulation of $G$ if one exists, and produces a bichromatic cycle in $G$ otherwise.*

**Corollary**

*If $\phi$ is a coloring of a weakly chordal graph $G$, then $\phi$ is a coloring of some triangulation $H \supseteq G$ if and only if $G$ does not contain a bichromatic cycle.*

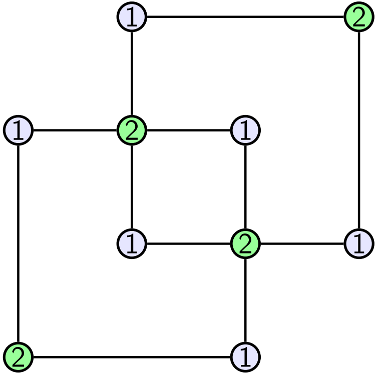
$\Rightarrow$ If all we care about is the decision problem, then we can solve it in linear time: simply check for bichromatic cycles

# Outline

# Acyclic Coloring – No Bichromatic Cycles

## Definition

*A coloring $\phi$ of a graph $G$ is an acyclic coloring if every cycle in $G$ uses at least three colors. (In other words, $G$ does not contain a bichromatic cycle!)*



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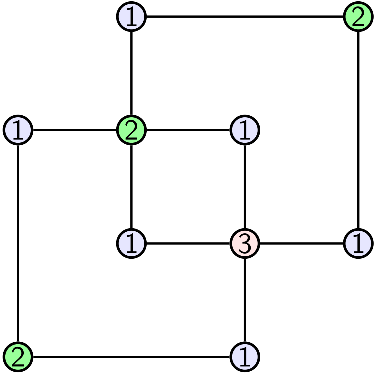
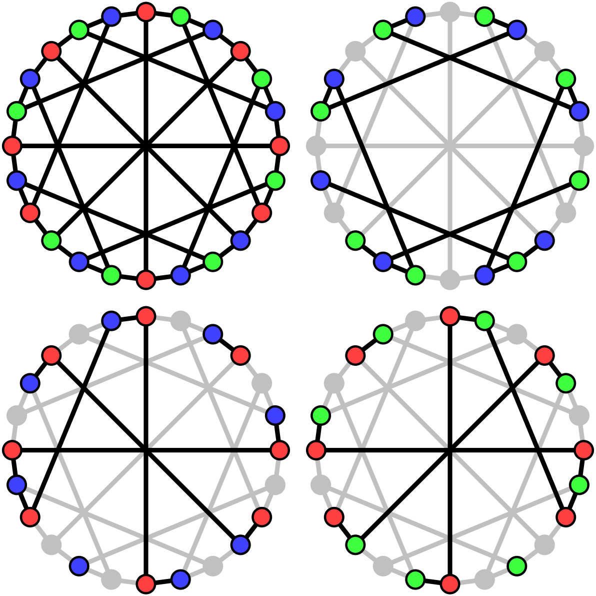



$$\chi(G) < \chi_a(G) = 3$$



credit: Claudio Rocchini (GNU Free Documentation License)

http://commons.wikimedia.org/wiki/File:Acyclic_coloring.svg

# ACYCLIC COLORINGS OF PLANAR GRAPHS[†]

BY

BRANKO GRÜNBAUM

ABSTRACT

A coloring of the vertices of a graph by $k$ colors is called acyclic provided that no circuit is bichromatic. We prove that every planar graph has an acyclic coloring with nine colors, and conjecture that five colors are sufficient. Other results on related types of colorings are also obtained; some of them generalize known facts about "point-arboricity".

## 1. Introduction

Let $G$ denote a graph with vertex set $V$; we shall assume th a $G$ contains no 1- or 2-circuits (that is, loops or multiple edges). A *k-coloring* of $G$ is a partition $V = V_1 \cup \cdots \cup V_k$ of the vertices of $G$ into $k$ pairwise disjoint sets (called *colors*) so that adjacent vertices are in different sets (have different colors). A $k$-coloring of $G$ is called *acyclic* provided that every subgraph of $G$ spanned by vertices of two of the colors is acyclic (in other words, is a forest). If $G$ is the graph of the octahedron then the 4-coloring of $G$ indicated in Fig. 1 by the numerals placed near the vertices is not acyclic (since the colors 1 and 2 span a graph which is not

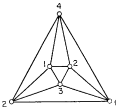



Fig. 1.

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# ON ACYCLIC COLORINGS OF PLANAR GRAPHS

O.V. BORODIN

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Received 14 August 1978

The conjecture of B. Grünbaum on existing of admissible vertex coloring of every planar graph with 5 colors, in which every bichromatic subgraph is acyclic, is proved and some corollaries of this result are discussed in the present paper.

## 1. Introduction and statement of the result

In 1973 Grünbaum has published a large paper [5] on graph colorings, in which various restrictions were given to the type of all 2- and 3-chromatic subgraphs. The main attention in this paper was attached to the planar graphs.

**Definition 1.** An admissible coloring of a graph is called *acyclic* (in narrow sense), if every bichromatic subgraph, induced by this coloring, is a forest (acyclic graph).

The acyclic coloring of a graph should obviously be considered only for loopless graphs without multiple edges, which is assumed below.

The first example of a planar graph, which is not acyclically 4-colorable, has been constructed by Grünbaum [5]. Afterwards Wegner has constructed [12] a planar graph, which possess a cycle in every 2-chromatic subgraph in every admissible 4-coloring.

**Definition 2.** Graph $G$ is called *k-degenerated*, if each subgraph $H$ of $G$ contains a vertex, which induced degree is less than $k$, i.e.

$$W(G) = \max_{G' \subseteq G} \min_{v \in V(G')} s_{G'}(v) + 1 \leq k,$$

where $W(G)$ is known as Vizing-Wilf's number.

In particular, a graph is 1-degenerated, iff it contains no edges, and is 2-degenerated, iff it is a forest.

Kostochka and Melnikov have shown [8] (answering Grünbaum's question), that graphs, acyclically not colorable with 4 colors, can be found even among 3-degenerated bipartite planar graphs.

# The Acyclic Coloring Problem, Algorithmically

Acyclic Coloring (AC)
**Instance**: Graph $G$, positive integer $k$.
**Question**: Is there an acyclic coloring of $G$ that uses $\leq k$ colors?

NP-complete to determine whether $\chi_a(G) \leq 3$ (Kostochka 1978)
Hard to approximate (Gebremedhin, Tarafdar, Manne, Pothen 2007)

If $\Delta(G) \leq 3$, then $G$ can be acyclically colored using 4 colors or fewer in linear time. (Skulrattanakulchai 2004)
If $\Delta(G) \leq 5$, then $G$ can be acyclically colored using 9 colors or fewer in linear time. (Fertin & Raspaud 2008)

# The ACYCLIC COLORING Problem, Algorithmically

SIAM J. ALG. DISC. METH.
Vol. 7, No. 2, April 1986


007

**THE CYCLIC COLORING PROBLEM AND ESTIMATION OF SPARSE HESSIAN MATRICES***

THOMAS F. COLEMAN† AND JIN-YI CAI†

**Abstract.** Numerical optimization algorithms often require the (symmetric) matrix of second derivatives, $\nabla^2 f(x)$. If the Hessian matrix is large and sparse, then estimation by finite differences can be quite attractive since several schemes allow for estimation in much fewer than $n$ gradient evaluations.

The purpose of this paper is to analyze, from a combinatorial point of view, a class of methods known as substitution methods. We present a concise characterization of such methods in graph-theoretic terms. Using this characterization, we develop a complexity analysis of the general problem and derive a roundoff error bound on the Hessian approximation. Moreover, the graph model immediately reveals procedures to effect the substitution process optimally (i.e. using fewest possible substitutions given the differencing directions) in space proportional to the number of nonzeros in the Hessian matrix.

**Key words.** graph coloring, estimation of Hessian matrices, sparsity, differentiation, numerical differences, NP-complete problems, unconstrained minimization

**AMS(MOS) subject classifications.** 65K05, 65K10, 65H10, 68L10

**1. Introduction.** We are concerned with the estimation of a large sparse symmetric matrix of second derivatives $\nabla^2 f(x)$ for some problem function $f: R^n \rightarrow R^1$. In particular, we note that the product $\nabla^2 f(x) \cdot d$ can be estimated, for example, by forward differences

$$\nabla^2 f(x) \cdot d = [\nabla f(x+d) - \nabla f(x)] + o(\|d\|). \tag{1.1}$$

When the structure of $\nabla^2 f(x)$ is known, then usually a few well chosen differencing directions $d_1, \cdots, d_p$ affords the recovery of estimates of all nonzeros of $\nabla^2 f(x)$. Let us denote our estimate by $H$. We will assume that the sparsity pattern of $H$ is known; the diagonal elements are specified as nonzero; $H$ is symmetric. (Restricting the diagonal to be zero-free is reasonable in many contexts: In particular, a minimizer of $f$ usually possesses a positive definite Hessian matrix.) We will be concerned with

↑ NP-complete even when restricted to bipartite graphs

# The Acyclic Coloring Problem, Algorithmically

## Chordal Graphs

- Every proper coloring is also an acyclic coloring (in particular, $\chi_a(G) = \chi(G) = \omega(G)$). (Bodlaender et al. 2000, Gebremedhin et al. 2009)
- Chordal graphs can be colored in $O(n+m)$ time.

# The ACYCLIC COLORING Problem, Algorithmically

## Chordal Graphs

- Every proper coloring is also an acyclic coloring (in particular, $\chi_a(G) = \chi(G) = \omega(G)$). (Bodlaender et al. 2000, Gebremedhin et al. 2009)
- Chordal graphs can be colored in $O(n + m)$ time.

## Cographs (L. 2009)

Also known as the $P_4$-free graphs

- The cographs are *exactly* the graphs for which every acyclic coloring is also a star coloring.
- An optimal acyclic (and star) coloring of a cograph can be found in $O(n)$ time (if a cotree is given as part of the input).

# TREEWIDTH

**Definition**

*The* treewidth $\mathrm{tw}(G)$ *of a graph* $G$ *is*

$$\min\{\omega(H) \mid H \text{ is a triangulation of } G\} - 1.$$

**Theorem (Bouchitté and Todinca 1999)**

TREEWIDTH *can be solved in* $O(n^6)$ *time on weakly chordal graphs.*

# Treewidth

### Definition

*The treewidth* tw$(G)$ *of a graph* $G$ *is*

$$\min\{\omega(H) \mid H \text{ is a triangulation of } G\} - 1.$$

The Key: chordal graphs are perfect!

### Theorem (Bouchitté and Todinca 1999)

Treewidth *can be solved in* $O(n^6)$ *time on weakly chordal graphs.*

### Corollary

*Every weakly chordal graph* $G$ *satisfies* $\chi_a(G) = \mathrm{tw}(G) + 1$.

### Corollary

*There exists a polynomial-time algorithm for* Acyclic Coloring *on weakly chordal graphs.*

# Acyclic Coloring and Treewidth

## Algorithms for subclasses of weakly chordal graphs

- chordal graphs – $O(n + m)$
- permutation graphs – $O(n + m)$
- distance-hereditary graphs – $O(n + m)$
- $P_4$-sparse graphs – $O(n + m)$
- chordal bipartite graphs – $o(n^6)$ (and $O(\min(m \log n, n^2))$ to recognize)

# Constructive Algorithms

We consider an algorithm for treewidth to be constructive if it produces a triangulation $H \supseteq G$.

Note: given tw-optimal triangulation $H$ we can find an optimal acyclic coloring in $O(n + |E(H)|)$ time (this is coloring chordal graphs).

**Theorem**

*If $\mathcal{C}$ is a subclass of the weakly chordal graphs for which* Treewidth *can be solved constructively (producing $H \supseteq G$) in $f_{\mathcal{C}}(n, m)$ time for every $G \in \mathcal{C}$, then an optimal acyclic coloring can be constructed in $O(f_{\mathcal{C}}(n, m) + |E(H)|)$ time for every $G \in \mathcal{C}$.*

# Outline

**Given**: Set $S$ of species, characteristic matrix $C$ of size $(k \times |S|)$ for each species, describing character traits

**Goal**: Determine whether there is a phylogeny $T$ for $S$ consistent with $C$: all species with value $i$ for characteristic $j$ form a connected subtree.

(Assume no trait develops independently in two or more places)

**Given**: Set $S$ of species, characteristic matrix $C$ of size $(k \times |S|)$ for each species, describing character traits

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(Assume no trait develops independently in two or more places)

Question: what is the meaning of a "NO" instance??

# Outline

# Conclusions: Open Problems

## Open Problems

- Characterize the graphs for which $\phi$ acyclic $\Leftrightarrow$ $G$ is $\phi$-triangulatable.
- Can we beat the best known algorithm for treewidth (and thus acyclic coloring) on weakly chordal graphs?

# Thank You!

Questions?

# Summary

### Theorem

*There exists a polynomial-time algorithm that, given a weakly chordal graph $G$ with a coloring $\phi$, produces a $\phi$-triangulation of $G$ if one exists, and produces a bichromatic cycle in $G$ otherwise.*

### Corollary

*Every weakly chordal graph $G$ satisfies $\chi_a(G) = \mathrm{tw}(G) + 1$.*

### Theorem

*If $\mathcal{C}$ is a subclass of the weakly chordal graphs for which* Treewidth *can be solved constructively (producing $H \supseteq G$) in $f_{\mathcal{C}}(n, m)$ time for every $G \in \mathcal{C}$, then an optimal acyclic coloring can be constructed in $O(f_{\mathcal{C}}(n, m) + |E(H)|)$ time for every $G \in \mathcal{C}$.*