SMOOTHING OF MIXED COMPLEMENTARITY PROBLEMS

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Mixed Complementarity Problems

Given a continuously differentiable mapping $f : \mathbf{R}^n \mapsto \mathbf{R}^n$, we want to find an $x \in \mathbf{R}^n$ such that

$$\begin{array}{lll} f_i(x) = 0 & \text{if} & x_i \in (l_i, u_i), \\ f_i(x) \geq 0 & \text{if} & x_i = l_i, \\ f_i(x) \leq 0 & \text{if} & x_i = u_i. \end{array}$$

General approach

Define a residual $r : \mathbf{R}^n \mapsto \mathbf{R}^n$ such that

$$r(x) = 0 \qquad \Longleftrightarrow \qquad x \text{ solves the MCP}$$

Variational Inequalities

If $\pi_C : \mathbf{R}^n \mapsto \mathbf{R}^n$ is the projection into the closed convex set C, then

$$(y - x)^T f(x) \geq 0, \qquad y \in C,$$

if and only if

$$x = \pi_C(x - f(x))$$

Boxes

The projection $\pi_B : \mathbf{R}^n \mapsto \mathbf{R}^n$ into the box

$$B = \{x \in \mathbf{R}^n : l \leq x \leq u\},$$

is given by

$$\pi_B(x) = \text{mid}(l, u, x),$$

where $\text{mid}(\cdot, \cdot, \cdot)$ is the median operator.

The Theoretical Holy Grail

Global convergence

If x^* is regular and a limit point of $\{x_k\}$, then x^* solves the MCP.

Local convergence

Assume that

- x^* is a limit point of $\{x_k\}$
- x^* solves the MCP
- x^* is regular

Under these assumptions $\{x_k\}$ converges to x^* either Q-linearly (with a guaranteed rate) or Q-superlinearly.

The Computational Holy Grail

- Algorithm applicable to general f.
- Only one linear system solve per Jacobian evaluation
- Truncated Newton strategies
$$\|r'(x_k)s_k + r(x_k)\| \leq \eta \|r(x_k)\|$$
allowed
- Structure of the Jacobian of the residual should preserve the sparsity (if any) of the Jacobian of f
- Invariance with respect to changes of scale such as $f \mapsto \alpha f$

Unconstrained Non-Smooth Equations

Xiao and Harker [1994]

$$r(x) = f(x_+) + x - x_+$$

Generalization to Mixed Complementarity

PATH: Ralph [1994], Dirkse and Ferris [1995]

$$r(x) = f[\pi_B(x)] + x - \pi_B(x)$$

Constrained Non-Smooth Equations

NE/SQP: Pang and Gabriel [1993]

$$r(x) = \min\{x, f(x)\}$$

$$\min\left\{\tfrac{1}{2}\|r(x)\|^2 : x \geq 0\right\}$$

Generalization to Mixed Complementarity

QPCOMP: Billups and Ferris [1995]

$$r(x) = \min\{x - l, \max\{x - u, f(x)\}\}$$
$$r(x) = \mathrm{mid}(x - l, x - u, f(x))$$

$$\min\left\{\tfrac{1}{2}\|r(x)\|^2 : l \leq x \leq u\right\}$$

Constrained Smooth Equations

Moré [1994]

$$r(x, w) = \begin{pmatrix} f(x) - w \\ \mathrm{diag}(x)w \end{pmatrix}$$

$$\min\left\{\tfrac{1}{2}\|r(x, w)\|^2 : x \geq 0,\ w \geq 0\right\}$$

Generalization to Mixed Complementarity

$$r(x, w, v) = \begin{pmatrix} f(x) - w + v \\ \mathrm{diag}(x - l)w \\ \mathrm{diag}(x - u)v \end{pmatrix}$$

$$\min\left\{\tfrac{1}{2}\|r(x, w, v)\|^2 : l \leq x \leq u,\ (w, v) \geq 0\right\}$$

Unconstrained Semi-Smooth Equations

Luca, Facchnei, and Kanzow [1995]

Facchnei and Kanzow [1995]

$$r(x) = \phi(x, f(x))$$

where

$$\phi(\alpha, \beta) = \sqrt{(\alpha^2 + \beta^2)} - (\alpha + \beta)$$

Generalization to Mixed Complementarity

Billups [1995]

$$r(x) = \phi(x - l, \phi(u - x, f(x))$$

Smooth System of Equations

Chen and Mangasarian [1995]

$$r(x) = x - (x - f(x))_+$$

Let $q_\lambda : \mathbf{R} \mapsto \mathbf{R}$ be a smooth approximation to the function $\alpha \mapsto \alpha_+$.

$$r_\lambda(x) = x - q_\lambda(x - f(x))$$

Generalization to Mixed Complementarity

Chen and Mangasarian [1995]

$$r_\lambda(x) = \begin{pmatrix} f(x) - w + v \\ x - l - q_\lambda(x - l - w) \\ u - x - q_\lambda(u - x - v) \end{pmatrix}$$

Smooth System of Equation – II

Gabriel and Moré [1995]

$$r(x) = x - \text{mid}(l, u, x - f(x))$$

$$r(x) = \text{mid}(x - l, x - u, f(x))$$

Let $p_\lambda : \mathbf{R} \mapsto \mathbf{R}$ be a smooth approximation to the function $\alpha \mapsto \text{mid}(l, u, \alpha)$.

$$r_\lambda(x) = x - p_\lambda(l, u, x - f(x))$$

$$r_\lambda(x) = p_\lambda(x - l, x - u, f(x))$$

Smoothing History

Smoothing in complementarity

Chen and Mangasarian [1995]

Smoothing in stochastic optimization

Katkovnik and Kulchitskii [1972]
Rubinstein [1981,1983,1986]
Kreimer and Rubinstein [1992]
Ermoliev, Norkin, and Wets [1994]

Smoothing in global optimization

Piela, Kostrowicki, and Scheraga [1989]
Wu [1994]
Moré and Wu [1995]

Smoothing Functions

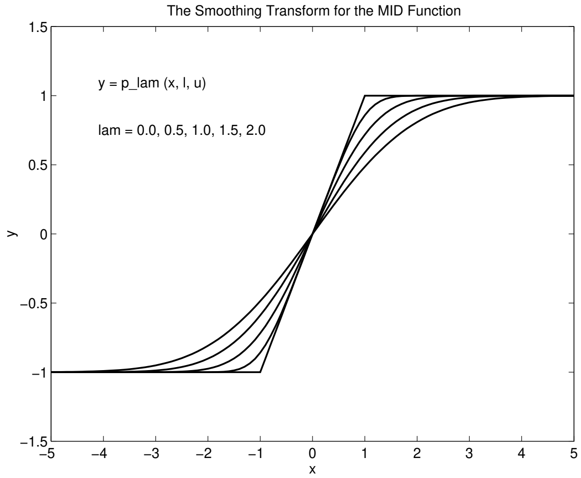
Let $\rho : \mathbf{R} \mapsto \mathbf{R}_+$ be a density function with bounded absolute mean, that is,

$$\kappa = \int_{-\infty}^{+\infty} |s|\rho(s)\, ds < +\infty.$$

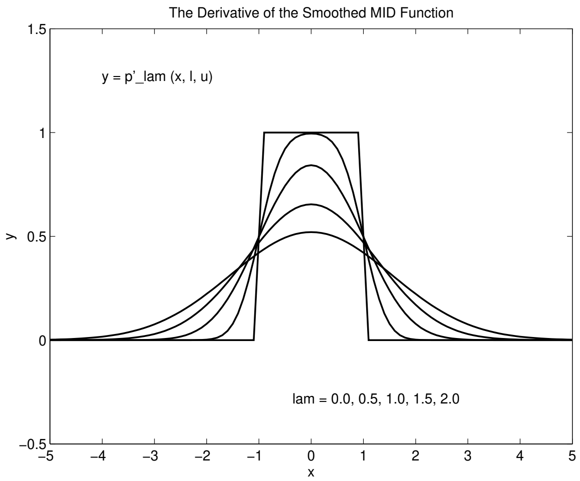
For any $\lambda > 0$ and $\alpha_l < \alpha_u$, define the smooth approximation to the mid function by

$$p_\lambda(\alpha) = \int_{-\infty}^{+\infty} \mathrm{mid}(\alpha_l, \alpha_u, \alpha - \lambda s)\rho(s)\, ds.$$

Smoothing - p_λ as a function of λ



Smoothing - p'_λ as a function of λ



Smoothing Properties

$$\alpha_l \le p_\lambda(\alpha) \le \alpha_u$$

$$\lim_{\alpha \to +\infty} p_\lambda(\alpha) = \alpha_u$$

$$\lim_{\alpha \to -\infty} p_\lambda(\alpha) = \alpha_l$$

$$p'_\lambda(\alpha) \in [0, 1]$$

$$\lim_{\lambda \to 0} p'_\lambda(\alpha) = 1, \quad \alpha \in (\alpha_l, \alpha_u)$$

$$\lim_{\lambda \to 0} p'_\lambda(\alpha) = 0, \quad \alpha \notin [\alpha_l, \alpha_u]$$

$$|p_\lambda(\alpha) - \mathrm{mid}(\alpha_l, \alpha_u, \alpha)| \le \kappa\lambda$$

Why Smoothing?

- Reliable and efficient (Billups, Dirkse, and Ferris [1995])
- Re-use of standard software

Basic Algorithm

Use any reliable nonlinear equation solver on

$$r_\lambda(x) = 0$$

for a small $\lambda > 0$, where

$$r_\lambda(x) = \text{mid}(x - l, x - u, f(x))$$

Sparsity is preserved since

$$r'_\lambda(x) = (I - D(x)) + D(x)f'(x)$$

where

$$D(x) = \text{diag}(p'_\lambda(x - f(x))$$

Continuation

Generate a path $x(\lambda)$ for $\lambda > 0$ where $x(\lambda)$ is an approximate solution to the least squares problem

$$\min\left\{\tfrac{1}{2}\|r_\lambda(x)\|^2 : x \in \mathbf{R}^n\right\}.$$

Assume, for example, that

$$\|r'_{k+1}(x_{k+1})^T r_{k+1}(x_{k+1})\| \le \eta\|r'_k(x_k)^T r_k(x_k)\|.$$

In general, we assume that

$$\lim_{k\to+\infty} r'_k(x_k)^T r_k(x_k) = 0.$$

Regularity Conditions

For any $x \in \mathbf{R}^n$ define the index sets

$$\mathcal{I} = \{i : l_i < x_i - f_i(x) < u_i\},$$
$$\mathcal{B} = \{i : x_i - f_i(x) = l_i\} \cup \{i : x_i - f_i(x) = u_i\},$$
$$\mathcal{E} = \{i : x_i - f_i(x) < l_i\} \cup \{i : x_i - f_i(x) > u_i\}.$$

Definition

A vector $x \in \mathbf{R}^n$ is *regular* if $[f'(x)]_\mathcal{K}$ is nonsingular for all $\mathcal{K}$ such that $\mathcal{I} \subset \mathcal{K} \subset \mathcal{I} \cup \mathcal{B}$, and

$$[f'(x)]_{\mathcal{I}\cup\mathcal{B}}/[f'(x)]_\mathcal{I}$$

is a P_0-matrix.

Global Convergence

Theorem

All limit point of $\{r'_k(x_k)\}$ are nonsingular if

- $\{\lambda_k\}$ converges to zero,
- x^* is regular,
- $\{x_k\}$ converges to x^*.

Theorem

Any limit point x^* of $\{x_k\}$ solves the MCP if

- $\{\lambda_k\}$ converges to zero,
- x^* is regular,
- $\{r'_k(x_k)^T r_k(x_k)\}$ converges to zero