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CIM-EARTH: Philosophy, Models, and Case Studies

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Abstract

General equilibrium models have been used for decades to obtain insights into the economic implications of policies and decisions. Such models offer a treatment of human behavior grounded in economic theory that also permits the integration of physical constraints on human activities. In this paper, we discuss our Community Integrated Model of Economic and Resource Trajectories for Humankind (CIM-EARTH), including a justification of our open-source philosophy and details of our computable general equilibrium and dynamic stochastic models. Case studies on the international consequences of unilateral carbon policy and solving stochastic optimal growth problems in parallel are used to illustrate the use of these models.

1 Introduction

Computable general equilibrium (CGE) models (Johansen, 1960, Robinson, 1991, Sue Wing, 2004) and their stochastic counterparts, dynamic stochastic general equilibrium (DSGE)

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models (Del Negro and Schorfheide, 2003), form the backbone of policy analysis programs around the world and have been used for decades to obtain insights into the economic implications of policies (Bhattacharyya, 1996, Shoven and Whalley, 1984, de Melo, 1988). Indeed, hundreds of such models have been built since their introduction (Devarajan and Robinson, 2002, Conrad, 2001). By computing market prices and the levels of supply and demand subject to productions constraints, taxes, and transportation costs, these models can be used to explore such policy-relevant questions as the impact of new tax policies or increased fossil energy costs on consumers. Moreover, these models form a core component when studying the interaction between economic activity and the Earth system with an integrated assessment model (IAM) (Dowlatabadi and Morgan, 1993, Weyant, 2009).

Despite successes, however, these economic models have limitations (Scrieciu, 2007). Models may not incorporate the industrial or process detail required to answer questions of interest; costs estimates from different models often differ considerably (Vuuren et al., 2009, Weyant, 1999, 2006, Friedlingstein et al., 2006, Lee, 2006); and little or no quantification of the uncertainty inherent in their estimates is performed. To understand the distributional impacts of a carbon emission policy, for example, one needs to represent the industries, regions, and income groups that may be affected and the complex interactions between different policies in different regions.

Many limitations of current economic models are due to computational and methodological constraints that can be overcome by leveraging recent advances in computer architecture, numerical methods, and economics research. For example, contemporary models use mathematical formulations, numerical methods, and computer systems that restrict the size of the models that can be solved in a reasonable time to a few tens of thousands of equations. Thus, it becomes impractical to add important detail such as increased industrial, geographic, or temporal resolution; capital vintages; overlapping generations; or stochastic dynamics. Yet more modern formulations and solvers, and more powerful computer systems, offer the potential to solve systems of equations that are several orders of magnitudes larger. Thus, we can in principle create models that encompass more details of importance to decision makers and characterize the model uncertainty. These results can then be used to identify policies that are robust to model uncertainty.

Motivated by these considerations, we are developing a new modeling framework: the Community Integrated Model of Economic And Resource Trajectories for Humankind (CIM-EARTH). Our goal is to facilitate and encourage the creation, execution, and testing of new economic models with significantly greater fidelity and sophistication than is the norm today. We envision the framework as combining (a) a high-level programming notation that permits the convenient formulation of a wide range of models with different purposes and characteristics; (b) a flexible implementation that permits the efficient solution of these models using the most advanced numerical methods and, where appropriate, high-performance computer systems; and (c) a suite of associated tools for parameter estimation, uncertainty quantification, and model validation.

We seek not only to provide access to better economic formulations and numerical methods but to also encourage the development and use of open models, that is, models that are both made freely available to all under terms that permit modification and redistribution and that are designed to facilitate study, modification, application, contribution, and redistribution by others. Open models play an important role in encouraging the application of the scientific method of reproducible research within economics and policy studies; increasing transparency of policy studies; and increasing participation in economic modeling.

In this paper, we describe our open-source philosophy and architecture, our general equilibrium models, and our initial implementation and its application to some case studies. Section 2 discusses our philosophy and architecture, the foundation upon which our models are built. Section 3 describes a basic CGE model and provides a small example with two industries, one consumer, and four markets to illustrate how these models are specified, the tax instruments available in our framework, and the myopic dynamics. Section 4 describes dynamic stochastic models through the use of optimal growth problems and our parallel dynamic programming methodology. Section 5 details the full CIM-EARTH v0.1 instance, including our dynamic trajectories for capital, labor productivity, and resource usage; the results obtained from a study of carbon leakage and the impact of parametric uncertainty; and preliminary results for our parallel dynamic programming methods on stochastic optimal growth problems. We conclude in Section 6 with future directions.

2 Philosophy

Open-source software is computer code made available under a license that permits others to read the software, modify it, and redistribute the modifications. Open-source concepts grew out of a research ethos that believed in the free exchange of ideas and viewed software as just another embodiment of ideas. As the range of uses for software has grown, however, so too have the motivations for open source. A second, increasingly common motivation is transparency, as when critics argue that the software for electronic voting machines should be open source, so that anyone can look for faulty assumptions and coding errors. A third common motivation is competitiveness, as when corporations invest in open-source Linux to combat Microsoft's dominance of the operating system market. A fourth motivation is often cost: many argue that open-source software reduces costs to both producers and consumers by encouraging contributions from a distributed community. Beyond the world of software, Chesbrough (2003) argues for the benefits of open innovation and Felten (accessed January 2010) for the "freedom to tinker" that results from open designs and technologies.

All of these arguments are highly relevant to economic and policy studies, given the complexity of the systems being studied and the magnitude of the decisions that model projections may influence. Unfortunately, while it is common in economic *research* for model data, equations, and implementations to be made available to other researchers, this convention is far from common in policy studies. For example, a recent U.S. Environmental Protection Agency analysis of the Waxman-Markey Discussion Draft of the American Clean Energy and Security Act uses two economy-wide models, ADAGE (Ross, 2008) and IGEM (Wilcoxen, 1988), an integrated assessment model, MiniCAM (Kim et al., 2006), an agricultural model, FASOMGHG (Adams et al., 2005), and an electricity industry model, IPM (U.S. Environmental Protection Agency Clean Air Markets Division). These models, however, are not open. No outside party can study, validate, run, or modify the models or meaningfully compare and contrast the results with other studies. For instance, the EPA study assumes a 5% discount rate, no international carbon leakage, and monotonic increases in energy efficiency. What happens if we change these assumptions? Broad study and de-

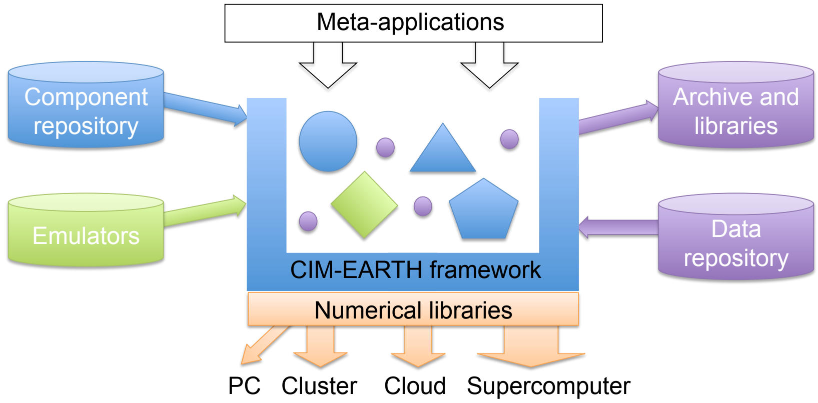

Figure 1: The CIM-EARTH framework allows for the rapid creation of applications via the composition of existing and new components. Other tools facilitate integration of data from multiple sources. "Meta-applications" can be invoked to quantify model uncertainty, for example.

bate of these questions are, in our view, essential for scientific progress and transparent and effective policymaking.

Transparent policy and the scientific method demand, first of all, that software and data are *accessible* and *understandable*. If, in addition, we design software to be *modifiable* and *extensible*, then we also facilitate the reuse of methodologies and tools: a model produced by one researcher can be tested by others with different data and compared with other models and extended in new directions. In this way, the barriers to entry for newcomers to a research field can be reduced, and thus the diversity and quality of the ideas explored can increase. In a world of closed models, a researcher with a new idea, such as a new approach to modeling international carbon leakage as a result of carbon taxes, cannot easily evaluate its effectiveness without developing an entire model from scratch. In a world of open models, such experimentation becomes significantly easier.

The most successful open models can form the basis for communities of contributors and users who develop different components and apply the resulting model to different problems. For example, in climate modeling, the Community Climate System Model (CCSM), developed by a team led by the National Center for Atmospheric Research (NCAR), is a highly successful open model. CCSM is downloadable by anyone and is used by many investigators worldwide. It is the foundation for a vibrant community of both users and developers, who experiment with alternative formulations of different components and test the model in different scenarios. Clearly the existence of this community is not due solely to openness—it is a product also of substantial funding and of hard work by NCAR staff—but it is hard to imagine such a community forming and prospering if CCSM were closed.

Software engineering plays an important role in developing an accessible, understandable, modifiable, and extensible system. Our overall architecture shown in Figure 1 uses a modular design, proven numerical libraries from sources such as TAO (Benson et al., accessed January 2010) and PETSc (Balay et al., 1997), and high-level specification languages. A parallel scripting language such as Swift (Zhao et al., 2007) can be used to define "meta-applications" such as sampling studies involving large ensembles or the coupling of multiple component models. An application is specified via a high-level language that defines the type of model (deterministic or stochastic, myopic or forward looking), the size of the model (regions, industries, consumers, time periods), the details for the industries and consumers (production and utility functions) and their parametrization (elasticities of substitution), and the coupling with other system components. In defining this language, we build on experience with general mathematical programming systems such as AMPL (Fourer et al., 2003) and GAMS (Brooke et al., 1988) and systems designed more specifically to support CGE modeling such as GEMPACK (Harrison and Pearson, 1994) and MPSGE (Rutherford, 1999). Each system is widely used, but has limitations. MPSGE models, for example, cannot include stochastic dynamics. Our tools currently compile an application specification to an AMPL model that can be read, modified, and solved.

Establishing a fully successful culture of open models in economics research and policymaking is a complex issue, and we recognize that real success in open modeling requires more than good software engineering. Above all, it requires a transformation of disciplinary culture, so that researchers become comfortable producing research using models that have been constructed by many contributors, funding agencies become comfortable paying for the development and sustenance of such models, and appointment and promotion committees become comfortable with interdisciplinary papers having many authors. These changes have occurred in other disciplines, such as climate science, and we hope to set an example in economic modeling that others can follow.

3 Computable General Equilibrium Models

Computable general equilibrium models determine prices and quantities over time for commodities such that supply equals demand for each good (Ballard et al., 1985, Ginsburgh and Keyzer, 1997, Scarf and Shoven, 1984). Such models have the following features:

- Many *industries* that hire labor, rent capital, and buy inputs to produce outputs. Each industry chooses a feasible production schedule to maximize its profit, the revenue received by selling its outputs minus the costs of producing them.

- Many *consumers* that choose what to buy and how much to work subject to the constraint that purchases cannot exceed income. Each consumer chooses a feasible consumption schedule to maximize his happiness as measured by a utility function.

- Many *markets* where producers and consumers trade that set wage rates and commodity prices to "clear" the markets. In particular, if the price of a commodity is positive, then supply must equal demand.

The primary modeling challenge is estimating the production and utility functions that characterize the physical and economic processes constraining the supply and demand decisions of industries and consumers. For our CGE models, we use constant elasticity of substitution (CES) production and utility functions. We detail the calibrated share model using a simple example to fix notation and discuss the inclusion of taxes and subsidies. We then describe the myopic dynamic model and our computational framework and numerical methods.

3.1 Calibrated Share Model

Calibrated constant elasticity of substitution functions (Boehringer et al., 2003) have the form

$$\frac{y}{\bar{y}} = \left(\sum_i \theta_i \left(\gamma_i \frac{x_i}{\bar{x}_i} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} ,$$

where $\frac{y}{\bar{y}}$ is the ratio between the output of the industry to a base year value, $\frac{x_i}{\bar{x}_i}$ are the ratios of the input commodities to their base year values, γ_i are the efficiency units that determine how effectively these factors can be used, θ_i are the share parameters with $\theta_i > 0$ and $\sum_i \theta_i = 1$, and σ controls the degree to which the inputs can be substituted for one another. When $\sigma = 0$, we obtain the Leontief production function

$$\frac{y}{\bar{y}} = \min_i \left\{ \gamma_i \frac{x_i}{\bar{x}_i} \right\} ,$$

in which the inputs are perfectly complementary; an increase in output requires an increase in all inputs. When $\sigma = 1$, we obtain the Cobb-Douglas production function

$$\frac{y}{\bar{y}} = \prod_i \left(\gamma_i \frac{x_i}{\bar{x}_i} \right)^{\theta_i} .$$

These functions are typically combined in a nested fashion where each nest describes the substitutability among commodity bundles.

The optimization problems solved by the industries and consumers and the market clearing conditions are then expressed in terms of the dimensionless variables

$$\mathbf{p} = \frac{p}{\bar{p}} , \quad \mathbf{x} = \frac{x}{\bar{x}} , \quad \mathbf{y} = \frac{y}{\bar{y}} .$$

These dimensionless variables represent the change in prices and quantities from their base-year values. The share parameters are then calibrated so that in the base year $\mathbf{p} = 1$, $\mathbf{x} = 1$, and $\mathbf{y} = 1$. That is, we choose shares that replicate the base-year revenue and expenditure data.

We now develop a simple instance with two industries, one consumer, and four markets to illustrate how these models are specified. The industries produce *materials* and *energy*, respectively. The consumer supplies *capital* and *labor* and demands materials and energy. The four markets are materials, energy, capital, and labor. We consistently use y for *supply* variables and x for *demand* variables, use subscripts to label the commodity or factor being supplied or demanded, and use superscripts to label either the industry (materials or energy) or the consumer supplying/demanding the commodity/factor. The variables in the model are described in Table 1 and the parameters in Table 2. A derivation of the calibrated model using expenditure and revenue data from the standard model using price and quantity data can be found in Elliott et al. (2009a).

3.1.1 Industries

Industries maximize profit, revenue minus expenditures, subject to production constraints. The materials industry in the simple instance demands materials, energy, capital, and labor,

Table 1: Variables in the simple calibrated CGE example.

$\mathbf{p}_m$	change in materials price
$\mathbf{p}_e$	change in energy price
$\mathbf{p}_K$	change in energy price
$\mathbf{p}_L$	change in energy price
$\mathbf{y}_m$	change in quantity of materials supplied by materials industry
$\mathbf{y}_e$	change in quantity of energy supplied by energy industry
$\mathbf{y}_K$	change in quantity of capital supplied by consumer
$\mathbf{y}_L$	change in quantity of labor supplied by consumer
$\mathbf{x}_m^m$	change in quantity of materials demanded by materials industry
$\mathbf{x}_e^m$	change in quantity of energy demanded by materials industry
$\mathbf{x}_K^m$	change in quantity of capital demanded by materials industry
$\mathbf{x}_L^m$	change in quantity of labor demanded by materials industry
$\mathbf{x}_K^e$	change in quantity of capital demanded by energy industry
$\mathbf{x}_L^e$	change in quantity of labor demanded by energy industry
$\mathbf{x}_m^c$	change in quantity of materials demanded by consumer
$\mathbf{x}_e^c$	change in quantity of energy demanded by consumer

Table 2: Parameters in the simple calibrated CGE example.

σ_{me}^m	elasticity of substitution among materials and energy for materials industry
σ_{KL}^m	elasticity of substitution among capital and labor for materials industry
σ^m	elasticity of substitution among (materials, energy) and (capital, labor) bundles
σ_{KL}^e	elasticity of substitution among capital and labor for energy industry
σ_{me}^c	elasticity of substitution among materials and energy for consumer
σ^c	elasticity of substitution among (materials, energy) bundle and savings for consumer
$\bar{e}_m^m$	base-year expenditure on materials demanded by materials industry
$\bar{e}_e^m$	base-year expenditure on energy demanded by materials industry
$\bar{e}_K^m$	base-year expenditure on capital demanded by materials industry
$\bar{e}_L^m$	base-year expenditure on labor demanded by materials industry
$\bar{e}_K^e$	base-year expenditure on capital demanded by energy industry
$\bar{e}_L^e$	base-year expenditure on labor demanded by energy industry
$\bar{e}_m^c$	base-year expenditure on materials demanded by consumer
$\bar{e}_e^c$	base-year expenditure on energy demanded by consumer
$\bar{e}_s^c$	base-year expenditure on share parameter for savings demanded by consumer
$\bar{r}_m$	base-year revenue from sales of materials
$\bar{r}_e$	base-year revenue from sales of energy
$\bar{r}_K$	base-year revenue from sales of capital
$\bar{r}_L$	base-year revenue from sales of labor

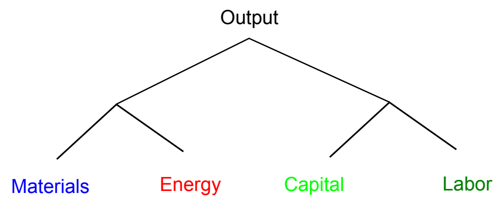

Figure 2: Basic nest for production function.

while the energy industry demands only capital and labor. In particular, the materials industry solves the optimization problem

$$
\begin{array}{lrcl}
\max\limits_{\mathbf{y}_m \geq 0, \mathbf{x}_i^m \geq 0} & \multicolumn{3}{l}{\bar{r}_m \mathbf{p}_m \mathbf{y}_m - \bar{e}_m^m \mathbf{p}_m \mathbf{x}_m^m - \bar{e}_e^m \mathbf{p}_e \mathbf{x}_e^m - \bar{e}_K^m \mathbf{p}_K \mathbf{x}_K^m - \bar{e}_L^m \mathbf{p}_L \mathbf{x}_L^m} \\
\text{s.t.} & \mathbf{y}_m & \leq & \left(\theta_{KL}^m (\mathbf{x}_{KL}^m)^{\rho^m} + \theta_{me}^m (\mathbf{x}_{me}^m)^{\rho^m}\right)^{\frac{1}{\rho^m}} \\
& \mathbf{x}_{KL}^m & \leq & \left(\theta_K^m (\mathbf{x}_K^m)^{\rho_{KL}^m} + \theta_L^m (\mathbf{x}_L^m)^{\rho_{KL}^m}\right)^{\frac{1}{\rho_{KL}^m}} \\
& \mathbf{x}_{me}^m & \leq & \left(\theta_m^m (\mathbf{x}_m^m)^{\rho_{me}^m} + \theta_e^m (\mathbf{x}_e^m)^{\rho_{me}^m}\right)^{\frac{1}{\rho_{me}^m}} ,
\end{array}
\tag{1}
$$

where $\rho^m = \frac{\sigma^m - 1}{\sigma^m}$, $\rho_{KL}^m = \frac{\sigma_{KL}^m - 1}{\sigma_{KL}^m}$, and $\rho_{me}^m = \frac{\sigma_{me}^m - 1}{\sigma_{me}^m}$. The production function constraints in (1) are depicted graphically by the tree structure shown in Figure 2, with each node representing a production function with its own elasticity of substitution that aggregates the inputs from below into a commodity bundle. The root node then aggregates the two intermediate commodity bundles into the total materials output.

The energy industry solves a similar, but simpler optimization problem since it demands only capital and labor:

$$
\begin{array}{ll}
\max\limits_{\mathbf{y}_e \geq 0, \mathbf{x}_i^e \geq 0} & \bar{r}_e \mathbf{p}_e \mathbf{y}_e - \bar{e}_K^e \mathbf{p}_K \mathbf{x}_K^e - \bar{e}_L^e \mathbf{p}_L \mathbf{x}_L^e \\
\text{s.t.} & \mathbf{y}_e \leq \left(\theta_K^e (\mathbf{x}_K^e)^{\rho_{KL}^e} + \theta_L^e (\mathbf{x}_L^m)^{\rho_{KL}^m}\right)^{\frac{1}{\rho_{KL}^m}} .
\end{array}
$$

3.1.2 Consumers

Consumers maximize their individual utility subject to a budget constraint; expenditures cannot exceed income. The consumer in the simple instance demands materials and energy, while supplying capital and labor. The supply of capital and labor is an endowed commodity; the consumer begins the period with a certain labor endowment and capital accumulated

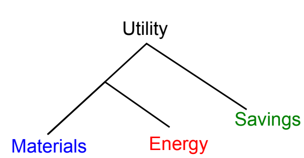

Figure 3: Basic nest for utility function.

from past savings. In particular, the consumer solves the optimization problem

$$
\begin{array}{lll}
\max\limits_{0\le \mathbf{y}_j^c\le 1, \mathbf{x}_i^c\ge 0} & \mathbf{x}^c & \\
\text{s.t.} & \mathbf{x}^c & \le \left(\theta_{me}^c(\mathbf{x}_{me}^c)^{\rho^c} + \theta_S^c(\mathbf{x}_S^c)^{\rho^c}\right)^{\frac{1}{\rho^c}} \\
 & \mathbf{x}_{me}^c & \le \left(\theta_m^c(\mathbf{x}_m^c)^{\rho_{me}^c} + \theta_e^c(\mathbf{x}_e^c)^{\rho_{me}^c}\right)^{\frac{1}{\rho_{me}^c}} \\
 & \multicolumn{2}{l}{\bar{e}_S^c\mathbf{x}_S^c + \bar{e}_m^c\mathbf{p}_m\mathbf{x}_m^c + \bar{e}_e^c\mathbf{p}_e\mathbf{x}_e^c \le \bar{r}_K^c\mathbf{p}_K\mathbf{y}_K^c + \bar{r}_L^c\mathbf{p}_L\mathbf{y}_L^c + \Pi_m + \Pi_e\,,}
\end{array}
\tag{2}
$$

where $\rho^c = \frac{\sigma^c-1}{\sigma^c}$, $\rho_{me}^c = \frac{\sigma_{me}^c-1}{\sigma_{me}^c}$, and Π_m and Π_e are the materials and energy industry profits returned to the consumer as a dividend, respectively. The savings demanded by the consumer, x_S^c, is necessary for myopic dynamic models to approximate the future utility of consumption. These savings are enter the economy as capital in the next time step. In practice, we choose σ^c to be one so that the CES function aggregating savings and the (materials, energy) bundle reduces to the Cobb-Douglas function, which implies that a fixed share of consumer income goes to savings each year. The utility function constraints in (2) are depicted graphically by the tree structure shown in Figure 3.

3.1.3 Markets

The market clearing conditions are as follows:

$$
\begin{array}{lclcl}
0 \le \mathbf{p}_m & \perp & \bar{r}_m\mathbf{y}_m & \ge & \bar{e}_m^m\mathbf{x}_m^m + \bar{e}_m^c\mathbf{x}_m^c \\
0 \le \mathbf{p}_e & \perp & \bar{r}_e\mathbf{y}_e & \ge & \bar{e}_e^m\mathbf{x}_e^m + \bar{e}_e^c\mathbf{x}_e^c \\
0 \le \mathbf{p}_L & \perp & \bar{r}_L\mathbf{y}_L & \ge & \bar{e}_L^m\mathbf{x}_L^m + \bar{e}_L^e\mathbf{x}_L^e \\
0 \le \mathbf{p}_K & \perp & \bar{r}_K\mathbf{y}_K & \ge & \bar{e}_K^m\mathbf{x}_K^m + \bar{e}_K^e\mathbf{x}_K^e\,.
\end{array}
$$

The complementarity condition signified by $\perp$ implies that one of the two inequalities in each expression must be saturated. That is, either supply equals demand and the price is nonnegative, or supply exceeds demand and the price is zero. In particular, a zero price means that the market for the good or factor collapses.

3.1.4 Calibration

Assuming revenues equal expenditures in all industry objective functions, consumer budget constraints, and market clearing conditions, we can choose values for the share parameters so that $\mathbf{p} = 1$, $\mathbf{y} = 1$, and $\mathbf{x} = 1$ solves the problem. That is, the prices and quantities do not deviate from their base-year levels. This process of choosing the share parameters based on base-year data is referred to as calibration to a base year. In particular, by choosing

$$
\begin{array}{lclclcl}
\theta_m^m & = & \frac{\bar{e}_m^m}{\bar{e}_m^m+\bar{e}_e^m} & \qquad & \theta_K^e & = & \frac{\bar{e}_K^e}{\bar{e}_K^e+\bar{e}_L^e} \\
\theta_e^m & = & \frac{\bar{e}_e^m}{\bar{e}_m^m+\bar{e}_e^m} & & \theta_L^e & = & \frac{\bar{e}_L^e}{\bar{e}_K^e+\bar{e}_L^e} \\
\theta_K^m & = & \frac{\bar{e}_K^m}{\bar{e}_K^m+\bar{e}_L^m} & & \theta_K^c & = & \frac{\bar{e}_K^c}{\bar{e}_K^c+\bar{e}_L^c} \\
\theta_L^m & = & \frac{\bar{e}_L^m}{\bar{e}_K^m+\bar{e}_L^m} & & \theta_L^c & = & \frac{\bar{e}_L^c}{\bar{e}_K^c+\bar{e}_L^c} \\
\theta_{KL}^m & = & \frac{\bar{e}_{KL}^m}{\bar{e}_{KL}^m+\bar{e}_{me}^m} & & \theta_{me}^c & = & \frac{\bar{e}_{me}^c}{\bar{e}_{me}^c+\bar{e}_S^c} \\
\theta_{me}^m & = & \frac{\bar{e}_{me}^m}{\bar{e}_{me}^m+\bar{e}_{me}^m} & & \theta_S^c & = & \frac{\bar{e}_S^c}{\bar{e}_{KL}^c+\bar{e}_S^c}\,,
\end{array}
$$

where $\bar{e}_{KL}^m = \bar{e}_K^m + \bar{e}_L^m$, $\bar{e}_{me}^m = \bar{e}_m^m + \bar{e}_e^m$, and $\bar{e}_{KL}^c = \bar{e}_K^c + \bar{e}_L^c$, one can show that $\mathbf{p} = 1$, $\mathbf{y} = 1$, and $\mathbf{x} = 1$ is a solution.

3.2 Taxes and Subsidies

Taxes are an important part of an economy and any CGE model. Import and export taxes play an important role in determining the sizes of bilateral trade flows; domestic taxes can reallocate economic activity into more socially advantageous efforts; and environmental taxes can be levied to encourage carbon-neutral behaviors and slow the emission of CO_2 and other harmful pollutants into the atmosphere. Here we detail how taxes are included in the production, consumption, and investment equations. Subsidies are simply a negative tax rate. Tax revenues are aggregated into region-specific tax accounts.

3.2.1 Ad Valorem and Excise Taxes

Each industry in the model pays a tax on the value of the goods and factors demanded. Some taxes, such as the federal gasoline tax, are applied to volumes rather than value. These taxes modify the expenditure terms in the optimization problems solved by the industries. Using the materials industry from the simple example, we calculate the expenditure on energy inputs as

$$((\bar{e}_e^m + \bar{s}_e^m)\,\mathbf{p}_e + \bar{t}_e^m)\,\mathbf{x}_e^m\,,$$

where $\bar{s}_e^m$ is the ad valorem tax expenditure and $\bar{t}_e^m$ is the excise tax expenditure. The distinction between ad valorem and excise taxes matters only as the change in commodity prices stray from unity and the difference is strongly dependent on the excise tax expenditure. For example, if the price of a commodity taxed in the base year at 10% doubles, then the tax revenues will be off by approximately 5% if the incorrect tax representation is used.

3.2.2 Production Taxes

Production taxes are paid by industries on the goods they produce. Using the materials industry from the simple example, the revenue from materials production becomes

$$((\bar{r}_m^m - \bar{s}_m^m)\,\mathbf{p}_m - \bar{t}_m^m)\,\mathbf{y}_m\,,$$

where $\bar{s}_m^m$ and $\bar{t}_m^m$ are the ad valorem and excise tax expenditures, respectively. Excise production taxes may be needed to study the effects of a producer-level carbon tax. For example, if one charges an emissions tax based on the amount of coal mined rather than the amount of coal burned to generate energy, then we would need an excise production tax on mined coal. While carbon may not be priced in this way, the analysis extends to this case.

3.2.3 Income Taxes

Income taxes are subtracted from the consumer incomes at the point of payment. These taxes have the same form as production taxes but are levied on the consumer revenue terms. Using consumer capital from the simple example, we calculate the modified revenue term as

$$((\bar{r}_K^c - \bar{s}_K^c)\,\mathbf{p}_K - \bar{t}_K^c)\,\mathbf{y}_K\,,$$

where $\bar{s}_K^c$ and $\bar{t}_K^c$ are the ad valorem and excise tax expenditures, respectively. We note that while excise taxes on labor and capital are not likely to be imposed, the modeling framework does not prevent their inclusion.

3.2.4 Import and Export Duties

For international trade, we treat domestic and imported goods as distinct products. Each region contains an importer for each commodity that buys goods internationally and sells them domestically. Because the importer inputs commodities from many regions, we need to distinguish between import and export duties, since the duties are paid at the destination or origination points, respectively. That is, we must distribute the revenue to the correct region. Using the materials commodity from the simple example, we calculate the expenditures for the materials importer in region r' as

$$\left(\left(\bar{e}^{i,r'}_{m,r} + \bar{s}^{i,r'}_{m,r} + \bar{s}^{e,r'}_{m,r}\right)\mathbf{p}_{m,r}\right) + \bar{t}^{i,r'}_{m,r} + \bar{t}^{e,r'}_{m,r}\mathbf{x}^{i,r'}_{m,r} \, ,$$

where $\bar{s}^{i,r'}_{m,r}$ and $\bar{t}^{i,r'}_{m,r}$ are the ad valorem and excise import duty amounts for materials imported from region r, respectively, and $\bar{s}^{e,r'}_{m,r}$ and $\bar{t}^{e,r'}_{m,r}$ are the ad valorem and excise export duty amounts for materials exported by region r.

3.2.5 Carbon Taxes

Carbon taxes are excise taxes placed on the inputs and outputs of producers and consumers. Since carbon emissions are free in most of the world, no data is typically available for industry expenditures on carbon emissions in the base year and we need to compute the taxable carbon emissions. Using the materials industry from the simple example, we calculate the expenditure on energy with a carbon tax as

$$(\bar{e}^m_e \mathbf{p}_e + t^m_e \bar{f}^m_e)\mathbf{x}^m_e \, ,$$

where t^m_e is the tax rate per emissions unit and $\bar{f}^m_e$ is the taxable base-year emissions units generated by the materials industry from the use of energy. The emissions factors for simple carbon taxes are usually based on available energy use volume data.

Carbon taxes on imports and exports are also used for border-tax adjustments on emissions. Using the materials commodity from the simple example, we calculate the border-tax adjustment for the materials importer in region r' as

$$(\bar{e}^{i,r'}_{m,r}\mathbf{p}_{m,r} + t^{i,r'}_{m,r}\bar{f}^{i,r'}_{m,r} + t^{e,r'}_{m,r}\bar{f}^{e,r'}_{m,r})\mathbf{x}^{i,r'}_{m,r} \, ,$$

where $t^{i,r'}_{m,r}$ and $t^{e,r'}_{m,r}$ are the import and export duties per emissions unit, respectively, and $\bar{f}^{i,r'}_{m,r}$ and $\bar{f}^{e,r'}_{m,r}$ are the taxable base-year emissions units for the materials importer. Export duties are negative when the exporting country refunds the carbon taxes on their exports.

Measuring the emissions units in this case is hard given the difficult carbon accounting introduced by the fact that the commodity in question may not be produced in the taxing region. We calculate the carbon content by assuming conservation of carbon. Therefore, the carbon content of the output is the sum of the carbon content of the inputs used in the production processes

$$C_{j_r}\bar{y}_{j_r}\mathbf{y}_{j_r} = \sum_{i_r} C_{i_r}\bar{x}^{j_r}_{i_r}\mathbf{x}^{j_r}_{i_r} \, ,$$

where C_* is the carbon content per unit of the commodity, i_r is the set of commodities used in the production of good j_r, $\bar{x}^{j_r}_{i_r}$ is the base-year volume of commodity i used in the production of commodity j in region r, and $\bar{y}_{j_r}$ is the base-year volume of commodity j produced by the industry in region r, and $\mathbf{x}_{i_r}^{j_r}(t)$.

This expression is written in terms of quantities, whereas most available data is in terms of expenditures. Therefore, rather than compute the carbon content per commodity unit, we compute the total carbon budget for the industry measured in terms of the base-year quantities. In particular, we make the substitution

$$\bar{f}_{j_r} = C_{j_r}\bar{y}_{j_r}$$

to obtain the equivalent system

$$\bar{f}_{j_r}\mathbf{y}_{j_r} = \sum_{i_r} \bar{f}_{i_r} \frac{\bar{x}_{i_r}^{j_r}}{\bar{y}_{i_r}} \mathbf{x}_{i_r}^{j_r} .$$

We typically do not know the base-year volumes $\bar{x}_{i_r}^{j_r}$ and $\bar{y}_{i_r}$. In those cases where we do know the volume data for the base year, however, we directly compute the ratio. In all other cases, we compute the ratio from available expenditure data,

$$\frac{\bar{x}_{i_r}^{j_r}}{\bar{y}_{i_r}} = \frac{\bar{p}_{i_r}\bar{x}_{i_r}^{j_r}}{\bar{p}_{i_r}\bar{y}_{i_r}} = \frac{\bar{e}_{i_r}^{j_r}}{\bar{R}_{i_r}} \equiv \Phi_{i_r}^{j_r} ,$$

where the expenditure and revenue data for each industry, $\bar{e}_{i_r}^{j_r}$ and $\bar{R}_{i_r}$, respectively, are known. Note that if the volume and expenditure data are consistent, the ratios computed from either will be identical. Therefore, we have

$$\bar{f}_{j_r}\mathbf{y}_{j_r} = \sum_{i_r} \bar{f}_{i_r} \Phi_{i_r}^{j_r} \mathbf{x}_{i_r}^{j_r} . \tag{3}$$

We estimate the total carbon budget $\bar{f}_{j_r}$ for each industry j in region r by solving the system of equations (3) for given Φ, $\mathbf{x}$, and $\mathbf{y}$. These amounts are then used in the carbon tax computations. However, this system has more variables than equations due to the land, labor, and capital factors. In our model, we ignore the contribution of these factors to the carbon content by fixing their amounts to zero. We are then left with a square system of equations having zero as a solution. Therefore, we fix the carbon amounts for the energy industries using energy volume data and standard conversion factors and solve the reduced system for the remaining emission factors.

3.2.6 Endogenous Tax Rates

Endogenous taxes rates are required to implement cap-and-trade policies. In this case, the tax rate is determined within the model so that the cap is not violated. We discuss an endogenous carbon emissions tax, but other endogenous taxes can be added to the model. The mechanism for setting the rate is to create a market for emissions having a fixed supply, with the price of emissions determined so that the demand does not exceed the supply.

In the simple example, an endogenous tax on the emissions from energy consumption would introduce the constraints

$$0 \le t_e \quad \perp \quad F_e \ge \bar{f}_e^m \mathbf{x}_e^m + \bar{f}_e^c \mathbf{x}_e^c ,$$

where t_e is the endogenous tax rate on energy; F_e is the cap on emissions from energy; and $\bar{f}_e^m$ and $\bar{f}_e^c$ are the taxable base year emissions units generated by the materials industry and consumer from the use of energy, respectively. In a calibrated model having no endogenous carbon tax in the base year, we set $F_e = \bar{f}_e^m + \bar{f}_e^c$. Analysis of the entire CGE problem shows that the tax rate in the base year is then zero. That is, under a business-as-usual scenario, there is no tax. By using a fraction of the base-year emissions, a positive tax rate is obtained.

3.3 Myopic Dynamic Models

The simplest dynamic CGE models are *myopic*, in which the industries and consumers look only at their current state and do not consider the future. In this case, we solve a sequence of static CGE models with dynamic trajectories for the factor endowments, efficiency units, and emission factors. The primary drivers of economic development are capital accumulation, labor productivity, and resource usage. We use exogenous time-series forecasts of important economic drivers constructed by extrapolation from historical data, with forecasts constrained by physical restrictions such as expected fossil reserve availability. The construction of these trajectories is documented in Section 5.1.

3.4 Computational Framework

Because the optimization problems solved by the industries and consumers are convex in their own variables and satisfy a constraint qualification, we can replace each with an equivalent complementarity problem obtained from the first-order optimality conditions by adding Lagrange multipliers on the constraints. If we assume all functions are general constant elasticity of substitution functions, the first-order conditions for the material industry are

$$\begin{array}{lll}
\Pi_m & \perp & \Pi_m + \bar{e}_m^m \mathbf{p}_m \mathbf{x}_m^m + \bar{e}_e^m \mathbf{p}_e \mathbf{x}_e^m + \bar{e}_K^m \mathbf{p}_K \mathbf{x}_K^m + \bar{e}_L^m \mathbf{p}_L \mathbf{x}_L^m - \bar{r}_m \mathbf{p}_m \mathbf{y}_m = 0 \\
0 \le \mathbf{y}_m & \perp & \lambda_m - \bar{r}_m \mathbf{p}_m \ge 0 \\
0 \le x_{KL}^m & \perp & \lambda_{KL}^m - \theta_{KL}^m \left(\theta_{KL}^m (\mathbf{x}_{KL}^m)^{\rho^m} + \theta_{me}^m (\mathbf{x}_{me}^m)^{\rho^m}\right)^{\frac{1}{\rho^m}-1} (\mathbf{x}_{KL}^m)^{\rho^m - 1} \lambda_m \ge 0 \\
0 \le x_{me}^m & \perp & \lambda_{me}^m - \theta_{me}^m \left(\theta_{KL}^m (\mathbf{x}_{KL}^m)^{\rho^m} + \theta_{me}^m (\mathbf{x}_{me}^m)^{\rho^m}\right)^{\frac{1}{\rho^m}-1} (\mathbf{x}_{me}^m)^{\rho^m - 1} \lambda_m \ge 0 \\
0 \le x_K^m & \perp & \bar{e}_K^m \mathbf{p}_K - \theta_K^m \left(\theta_K^m (\mathbf{x}_K^m)^{\rho_{KL}^m} + \theta_L^m (\mathbf{x}_L^m)^{\rho_{KL}^m}\right)^{\frac{1}{\rho_{KL}^m}-1} (\mathbf{x}_K^m)^{\rho_{KL}^m - 1} \lambda_{KL}^m \ge 0 \\
0 \le x_L^m & \perp & \bar{e}_L^m \mathbf{p}_L - \theta_L^m \left(\theta_K^m (\mathbf{x}_K^m)^{\rho_{KL}^m} + \theta_L^m (\mathbf{x}_L^m)^{\rho_{KL}^m}\right)^{\frac{1}{\rho_{KL}^m}-1} (\mathbf{x}_L^m)^{\rho_{KL}^m - 1} \lambda_{KL}^m \ge 0 \\
0 \le x_m^m & \perp & \bar{e}_m^m \mathbf{p}_m - \theta_m^m \left(\theta_m^m (\mathbf{x}_m^m)^{\rho_{me}^m} + \theta_e^m (\mathbf{x}_e^m)^{\rho_{me}^m}\right)^{\frac{1}{\rho_{me}^m}-1} (\mathbf{x}_m^m)^{\rho_{me}^m - 1} \lambda_{me}^m \ge 0 \\
0 \le x_e^m & \perp & \bar{e}_e^m \mathbf{p}_e - \theta_e^m \left(\theta_m^m (\mathbf{x}_m^m)^{\rho_{me}^m} + \theta_e^m (\mathbf{x}_e^m)^{\rho_{me}^m}\right)^{\frac{1}{\rho_{me}^m}-1} (\mathbf{x}_e^m)^{\rho_{me}^m - 1} \lambda_{me}^m \ge 0 \\
0 \le \lambda_m & \perp & \left(\theta_{KL}^m (\mathbf{x}_{KL}^m)^{\rho^m} + \theta_{me}^m (\mathbf{x}_{me}^m)^{\rho^m}\right)^{\frac{1}{\rho^m}} - \mathbf{y}_m \ge 0 \\
0 \le \lambda_{KL}^m & \perp & \left(\theta_K^m (\mathbf{x}_K^m)^{\rho_{KL}^m} + \theta_L^m (\mathbf{x}_L^m)^{\rho_{KL}^m}\right)^{\frac{1}{\rho_{KL}^m}} - \mathbf{x}_{KL}^m \ge 0 \\
0 \le \lambda_{me}^m & \perp & \left(\theta_m^m (\mathbf{x}_m^m)^{\rho_{me}^m} + \theta_e^m (\mathbf{x}_e^m)^{\rho_{me}^m}\right)^{\frac{1}{\rho_{me}^m}} - \mathbf{x}_{me}^m \ge 0\,,
\end{array}$$

for the energy industry are

$$
\begin{array}{lll}
\Pi_e & \perp & \Pi_e + \bar{e}^e_K \mathbf{p}_K \mathbf{x}^e_K + \bar{e}^e_L \mathbf{p}_L \mathbf{x}^e_L - \bar{r}_e \mathbf{p}_e \mathbf{y}_e = 0 \\
0 \leq \mathbf{y}_e & \perp & \lambda_e - \bar{r}_e \mathbf{p}_e \\
0 \leq \mathbf{x}^e_K & \perp & \bar{e}^e_K \mathbf{p}_K - \theta^e_K \left(\theta^e_K (\mathbf{x}^e_K)^{\rho^e_{KL}} + \theta^e_L (\mathbf{x}^m_L)^{\rho^m_{KL}} \right)^{\frac{1}{\rho^m_{KL}} - 1} (\mathbf{x}^e_K)^{\rho^e_{KL} - 1} \lambda_e \geq 0 \\
0 \leq \mathbf{x}^e_L & \perp & \bar{e}^e_L \mathbf{p}_L - \theta^e_L \left(\theta^e_K (\mathbf{x}^e_K)^{\rho^e_{KL}} + \theta^e_L (\mathbf{x}^m_L)^{\rho^m_{KL}} \right)^{\frac{1}{\rho^m_{KL}} - 1} (\mathbf{x}^e_L)^{\rho^e_{KL} - 1} \lambda_e \geq 0 \\
0 \leq \lambda_e & \perp & \left(\theta^e_K (\mathbf{x}^e_K)^{\rho^e_{KL}} + \theta^e_L (\mathbf{x}^m_L)^{\rho^m_{KL}} \right)^{\frac{1}{\rho^m_{KL}}} - \mathbf{y}_e \geq 0\,,
\end{array}
$$

and for the consumer are

$$
\begin{array}{lll}
0 \leq x_c & \perp & \lambda_c - 1 \geq 0 \\
0 \leq x^c_{me} & \perp & \lambda^c_{me} - \theta^c_{me} \left(\theta^c_{me} (\mathbf{x}^c_{me})^{\rho^c} + \theta^c_S (\mathbf{x}^c_S)^{\rho^c} \right)^{\frac{1}{\rho^c} - 1} (\mathbf{x}^c_{me})^{\rho^c - 1} \lambda_c \geq 0 \\
0 \leq x^c_S & \perp & \bar{e}^c_S \mu_c - \theta^c_S \left(\theta^c_{me} (\mathbf{x}^c_{me})^{\rho^c} + \theta^c_S (\mathbf{x}^c_S)^{\rho^c} \right)^{\frac{1}{\rho^c} - 1} (\mathbf{x}^c_S)^{\rho^c - 1} \lambda_c \geq 0 \\
0 \leq x^c_m & \perp & \bar{e}^c_m \mathbf{p}_m \mu_c - \theta^c_m \left(\theta^c_m (\mathbf{x}^c_m)^{\rho^c_{me}} + \theta^c_e (\mathbf{x}^c_e)^{\rho^c_{me}} \right)^{\frac{1}{\rho^c_{me}} - 1} (\mathbf{x}^c_m)^{\rho^c_{me} - 1} \lambda^c_{me} \geq 0 \\
0 \leq x^c_e & \perp & \bar{e}^c_e \mathbf{p}_e \mu_c - \theta^c_e \left(\theta^c_m (\mathbf{x}^c_m)^{\rho^c_{me}} + \theta^c_e (\mathbf{x}^c_e)^{\rho^c_{me}} \right)^{\frac{1}{\rho^c_{me}} - 1} (\mathbf{x}^c_e)^{\rho^c_{me} - 1} \lambda^c_{me} \geq 0 \\
0 \leq y^c_K \leq 1 & \perp & -\bar{r}^c_K \mathbf{p}_K \mu_c \\
0 \leq y^c_L \leq 1 & \perp & -\bar{r}^c_L \mathbf{p}_L \mu_c \\
0 \leq \lambda_c & \perp & \left(\theta^c_{me} (\mathbf{x}^c_{me})^{\rho^c} + \theta^c_S (\mathbf{x}^c_S)^{\rho^c} \right)^{\frac{1}{\rho^c}} - \mathbf{x}^c \geq 0 \\
0 \leq \lambda^c_{me} & \perp & \left(\theta^c_m (\mathbf{x}^c_m)^{\rho^c_{me}} + \theta^c_e (\mathbf{x}^c_e)^{\rho^c_{me}} \right)^{\frac{1}{\rho^c_{me}}} - \mathbf{x}^c_{me} \geq 0 \\
0 \leq \mu_c & \perp & \bar{r}^c_K \mathbf{p}_K \mathbf{y}^c_K + \bar{r}^c_L \mathbf{p}_L \mathbf{y}^c_L - \bar{e}^c_S \mathbf{x}^c_S - \bar{e}^c_m \mathbf{p}_m \mathbf{x}^c_m - \bar{e}^c_e \mathbf{p}_e \mathbf{x}^c_e + \Pi_m + \Pi_e \geq 0\,.
\end{array}
$$

The correct first-order optimality conditions are used when Leontief and Cobb-Douglas functions are present. These optimality conditions in combination with the market clearing conditions

$$
\begin{array}{lll}
0 \leq \mathbf{p}_m & \perp & \bar{r}_m \mathbf{y}_m \geq \bar{e}^m_m \mathbf{x}^m_m + \bar{e}^c_m \mathbf{x}^c_m \\
0 \leq \mathbf{p}_e & \perp & \bar{r}_e \mathbf{y}_e \geq \bar{e}^m_e \mathbf{x}^m_e + \bar{e}^c_e \mathbf{x}^c_e \\
0 \leq \mathbf{p}_L & \perp & \bar{r}_L \mathbf{y}_L \geq \bar{e}^m_L \mathbf{x}^m_L + \bar{e}^e_L \mathbf{x}^e_L \\
0 \leq \mathbf{p}_K & \perp & \bar{r}_K \mathbf{y}_K \geq \bar{e}^m_K \mathbf{x}^m_K + \bar{e}^e_K \mathbf{x}^e_K
\end{array}
$$

form a square complementarity problem that can be solved by applying a generalized Newton method, such as PATH (Dirkse and Ferris, 1995, Ferris and Munson, 1999, 2000).

PATH is a sophisticated implementation of a Josephy-Newton method that solves a linear complementarity problem at each iteration using a variant of Lemke's method to obtain a direction and then searches along this direction to obtain sufficient decrease in a merit function. Many enhancements have been made to the code, including the addition of preprocessing techniques to automatically improve the model formulation and crashing methods to rapidly approximate the active set.

The initial version of our framework has been implemented in the AMPL modeling language (Fourer et al., 2003). This language is convenient for expressing large optimization problems and includes convenient notation for sets and algebraic constraints. It also computes all the derivative information needed by the PATH solve when calculating a solution. We have coded the calibrated share form with all the tax and subsidy instruments and implemented the calibration procedure given expenditure data. The models are processed to check consistency and eliminate small industries. Once the processing is complete, either a scalar AMPL model can be emitted and checked by the user, or the model can be solved by applying the PATH algorithm. The construction of the first-order optimality conditions for the industry and consumer problems is completely automated. Summary reports are written to user-defined files.

4 Dynamic Stochastic Models

In this section, we describe optimal growth problems that illustrate the economic elements of dynamic stochastic general equilibrium models. We then outline the computational strategies we use to solve them.

4.1 Optimal Growth Problems

The simplest optimal growth model is the deterministic, discrete-time infinite-horizon optimal growth model, which solves the problem:

$$V(k_0) = \begin{cases} \max\limits_{c,l,k} & \sum\limits_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ \text{s.t.} & k_{t+1} = F(k_t, l_t) - c_t \quad \forall t \geq 0\,, \end{cases}$$

where k_t is the capital stock at time t with k_0 given; l_t is the labor supply; c_t is the consumption; $F(k, l) = k + f(k, l)$, where $f(k, l)$ is the aggregate net production function; $u(c_t, l_t)$ is the utility function; and β is the discount factor.

In the comparable stochastic optimal growth model, we let θ denote the current productivity level and $f(k, l, \theta)$ denote net output. Define $F(k, l, \theta) = k + f(k, l, \theta)$, and assume θ follows the stochastic law $\theta_{t+1} = g(\theta_t, \varepsilon_t)$, where ε_t are i.i.d. disturbances. Then the infinite-horizon discrete-time optimization problem becomes

$$V(k_0, \theta_0) = \begin{cases} \max\limits_{c,l,k} & E\left\{\sum\limits_{t=0}^{\infty} \beta^t u(c_t, l_t)\right\} \\ \text{s.t.} & k_{t+1} = F(k_t, l_t, \theta_t) - c_t + \epsilon_t \quad \forall t \geq 0 \\ & \theta_{t+1} = g(\theta_t, \varepsilon_t) \qquad\qquad\quad \forall t \geq 0\,, \end{cases}$$

where k_0 and θ_0 are given and $E\{\dots\}$ is the expectation conditional on current information, θ represents the productivity level, and ϵ_t and ε_t are independent i.i.d. disturbances. Both models model can be extended to include heterogeneous types of capital and labor.

A generic, infinite-horizon, optimal decision-making problem has the following general form:

$$V(x_0) = \max_{a_t \in \mathcal{D}(x_t)} E\left\{\sum_{t=0}^{\infty} \beta^t u(x_t, a_t)\right\}\,,$$

where x_t is the state process with initial state x_0, $\mathcal{D}(x_t)$ is the set of possible actions, a_t is the action taken, and β is the discount factor.

4.2 Dynamic Programming

Optimal growth models are examples of dynamic programming problems that can be formulated as solutions to the Bellman equation. The dynamic programming (DP) model for the generic infinite-horizon problem is

$$V(x) = \max_{a \in \mathcal{D}(x)} u(x, a) + \beta E\{V(x^+) \mid x, a\},$$

where x is the state variable, a is the action variable, x^+ is the next-stage state conditional on the current-stage state x and the action a, and $V(x)$ is the value function. We solve these dynamic program problems with value function iteration methods that incorporate efficient methods for optimization, quadrature, and function approximation.

The value function is typically a continuous function if the state and control variables are continuous. Since computers cannot model arbitrary continuous functions, we use a finitely parameterizable collection of functions to approximate the value function, $V(x) \approx \hat{V}(x; \mathbf{b})$, where $\mathbf{b}$ is a vector of parameters. The functional form $\hat{V}$ may be a linear combination of polynomials, may represent a rational function or neural network, or may be some other parameterization specially designed for the problem. After the functional form is fixed, we focus on finding the vector of parameters, $\mathbf{b}$, such that $\hat{V}(x; \mathbf{b})$ approximately satisfies the Bellman equation (Judd, 1998).

The following is the value function iteration algorithm for infinite-horizon problems.

Algorithm 4.1 *Value Function Iteration for Infinite-Horizon Problems*

Initialization. Choose the approximation grid, $X = \{x_i : 1 \leq i \leq m\}$, and choose a functional form for $\hat{V}(x; \mathbf{b})$. Make an initial guess $\hat{V}(x; \mathbf{b}^0)$, and choose a stopping tolerance $\tau > 0$. Let $t = 0$.

Step 1. Maximization step. Compute

$$v_i = \max_{a_i \in \mathcal{D}(x_i)} u(x_i, a_i) + \beta E\{\hat{V}(x_i^+; \mathbf{b}^t) \mid x_i, a_i\}$$

for each $x_i \in X$, $1 \leq i \leq m$.

Step 2. Fitting step. Using the appropriate approximation method, compute the $\mathbf{b}^{t+1}$ such that $\hat{V}(x; \mathbf{b}^{t+1})$ approximates the (x_i, v_i) data.

Step 3. Termination step. If $\|\hat{V}(x; \mathbf{b}^{t+1}) - \hat{V}(x; \mathbf{b}^t)\| < \tau$, then stop. Otherwise, increment t, and go to Step 1.

An approximation scheme consists of two parts: basis functions and approximation nodes. Approximation nodes can be chosen as uniformly spaced nodes, Chebyshev nodes, or some other specified nodes. From the viewpoint of basis functions, approximation methods can be classified as either spectral or finite-element methods. A spectral method uses globally nonzero basis functions $\phi_j(x)$ such that $\hat{V}(x) = \sum_{j=0}^{n} c_j \phi_j(x)$ is the degree-n approximation. We present Chebyshev polynomial approximation as an example of spectral methods. In contrast, a finite-element method uses locally basis functions $\phi_j(x)$ that are nonzero over subdomains of the approximation domain. For detailed discussions of approximation methods, see Cai (2009) and Judd (1998).

Chebyshev Polynomial Approximation Chebyshev polynomials on $[-1,1]$ are defined as $T_j(x) = \cos(j\cos^{-1}(x))$, while general Chebyshev polynomials on $[a,b]$ are defined as $T_j\left(\frac{2x-a-b}{b-a}\right)$ for $j = 0,1,2,\ldots$. These polynomials are orthogonal under the weighted inner product:

$$\langle f,g\rangle = \int_a^b f(x)g(x)w(x)dx$$

with the weighting function

$$w(x) = \left(1 - \left(\frac{2x-a-b}{b-a}\right)^2\right)^{-1/2}.$$

We approximate V with the least-squares polynomial approximation with respect to the weighting function, that is, a degree-n polynomial $\hat{V}_n(x)$, such that

$$\hat{V}_n(x) \in \arg\min_{\deg(\hat{V})\le n} \int_a^b \left(V(x) - \hat{V}_n(x)\right)^2 w(x)dx\,.$$

The least-squares degree-n polynomial approximation $\hat{V}_n(x)$ on $[-1,1]$ has the form

$$\hat{V}_n(x) = \frac{1}{2}c_0 + \sum_{j=1}^{n} c_j T_j(x),$$

where

$$c_j = \frac{2}{\pi}\int_{-1}^{1} \frac{V(x)T_j(x)}{\sqrt{1-x^2}}dx \quad \forall\, j = 0,1,\ldots,n$$

are the Chebyshev least-squares coefficients.

Multidimensional Tensor Chebyshev Approximation In a d-dimensional approximation problem, let $a = (a_1,\ldots,a_d)$ and $b = (b_1,\ldots,b_d)$ with $b_i > a_i$ for $i = 1,\ldots,d$. Let $x = (x_1,\ldots,x_d)$ with $x_i \in [a_i,b_i]$ for $i = 1,\ldots,d$. For simplicity, we denote this set as $x \in [a,b]$. Let $\alpha = (\alpha_1,\ldots,\alpha_d)$ be a vector of nonnegative integers. Let $T_\alpha(z)$ denote the tensor product $T_{\alpha_1}(z_1)\cdots T_{\alpha_d}(z_d)$ for $z = (z_1,\ldots,z_d) \in [-1,1]^d$. Let $(2x-a-b)./(b-a)$ denote the vector $\left(\frac{2x_1-a_1-b_1}{b_1-a_1},\ldots,\frac{2x_d-a_d-b_d}{b_d-a_d}\right)$. Then the degree-$n$ tensor Chebyshev approximation for $V(x)$ is

$$\hat{V}_n(x) = \sum_{0\le\alpha_i\le n,1\le i\le d} c_\alpha T_\alpha\left((2x-a-b)./(b-a)\right).$$

Multidimensional Complete Chebyshev Approximation Tensor product approximations are expensive to use. Instead we use the degree-n complete Chebyshev approximation for $V(x)$, which is

$$\hat{V}_n(x) = \sum_{0\le|\alpha|\le n} c_\alpha T_\alpha\left((2x-a-b)./(b-a)\right),$$

where $|\alpha|$ denotes $\sum_{i=1}^{d} \alpha_i$ for the nonnegative integer vector $\alpha = (\alpha_1, \ldots, \alpha_d)$. We know the number of terms with $0 \leq |\alpha| = \sum_{i=1}^{d} \alpha_i \leq n$ is $\binom{n+d}{d}$ for the degree-n complete Chebyshev approximation in $\mathbb{R}^d$, while the number of terms for the tensor Chebyshev approximation is $(n+1)^d$. The complexity of computation of a degree-n complete Chebyshev polynomial is much less than the complexity of computing a tensor Chebyshev polynomial in $\mathbb{R}^d$.

4.3 Parallelization

Numerical dynamic programming problems can require weeks or months of computation to solve high-dimensional problems because of the "curse of dimensionality" arising from the number of optimization problems in each maximization step of Algorithm 4.1. That is, we must compute

$$v_i = \max_{a_i \in \mathcal{D}(x_i)} u(x_i, a_i) + \beta E\{\hat{V}(x_i^+; \mathbf{b}^t) \mid x_i, a_i\}$$

for each continuous state point x_i in the finite set $X_t \subset \mathbb{R}^d$. However, this maximization step is naturally parallelizable. Using modern parallel architectures, we can therefore reduce the computation time to solve these problems.

We use the Master-Worker paradigm for our parallel numerical DP algorithms. This paradigm consists of two entities: a master and many workers. The master manages decomposing the problem into small tasks, queueing and distributing the tasks among the workers, and collecting the results. The workers each receive a task from the master, perform the task, and then send the result back to the master. A file-based, remote I/O scheme can be used as the message-passing mechanism between the master and the workers.

5 Case Studies

We illustrate the capabilities of a first version of this framework by presenting the results of a study of the impact of carbon leakage and border-tax adjustments and the numerical performance of parallel dynamic programming methods on stochastic optimal growth models.

5.1 CIM-EARTH CGE Model

We begin with a detailed description of the CIM-EARTH v0.1 model we have built for testing and development. The model is written in the AMPL modeling language (Fourer et al., 2003); scalar versions of some instances used in the case studies are available at `www.cim-earth.org`.

5.1.1 Regions and Industries

The regional and industrial resolution of the CIM-EARTH v0.1 model is shown in Table 3. This particular aggregation was chosen to study carbon leakage, the impact of a unilateral carbon emissions policy on the global movement of industrial production capacity away from that region. Therefore, the model contains more detailed resolution in the energy-intensive industries and in the industries that provide transport services to importers to move goods around the world.

Table 3: Aggregate regions and industries for the 16×16 model used here.

Regions	**Industries** (per region)
Oceania	Agriculture and forestry
Southeast Asia	Coal
Japan	Oil
Rest of East Asia	Natural gas
India	Iron and steel
Rest of South Asia	Chemicals
Russia, Georgia & Asiastan	Nonferrous metals
Middle East & N. Africa	Cement/mineral products
Sub-Saharan Africa	Other manufacturing
Western Europe	Refined petroleum
Rest of Europe	Electricity
Brazil	Land transport
Mexico	Air transport
Rest of Latin America	Sea transport
USA	Government services
Canada	Other services

This model does not contain a government consumer; it contains only a producer of government goods and services, which include defense, social security, health care, and education. Industries and consumers demand these government goods and services. The government producer is treated like any other producer and is subject to ad valorem and excise taxes. All taxes collected by a region are returned to consumers in that region.

Trade among regions is handled through importers of each commodity in each region. The importers buy commodities both domestically and internationally, transport these commodities, and sell the imports to producers and consumers. Domestic production and imports are treated as separate commodities traded in different markets. We use three types of transportation: land transportation, including freight by trucks and pipelines; air transportation; and sea transportation. Each of these transport services is an industry/commodity in the model. However, since the importers do not care about the origination of the transport services, we model international transportation as a homogeneous commodity having one global price; each region supplies some amount of transportation services.

5.1.2 Production Functions

The production functions in each region have the nested structure shown in Figure 4. As before, each node represents a CES function aggregating the production factor branches coming into it from below. We use elasticities of substitution taken from the CGE literature for the producers and consumers. In particular, we use the same elasticities of substitution as the EPPA (MIT Joint Program on the Science and Policy of Global Change, accessed January 2010) and GTAP (Ianchovichina and McDougall, 2000) models.

The importers are modeled like other producers using the nested CES production function shown at the bottom right of Figure 4. We use a Leontief production function to

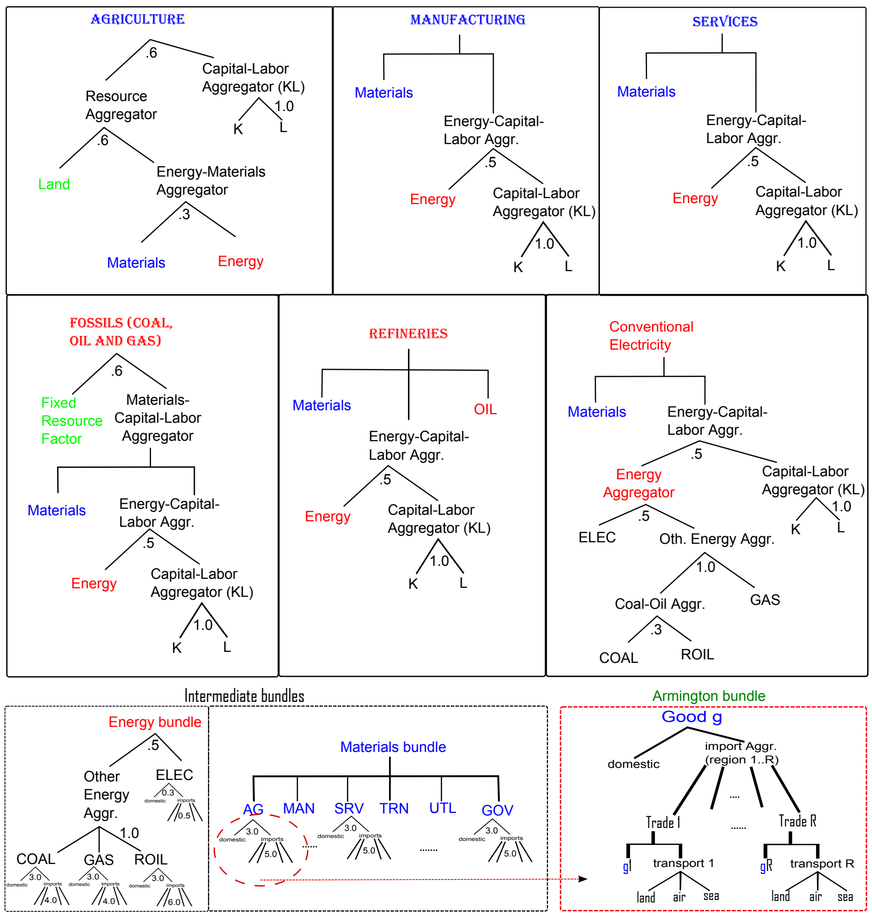

Figure 4: Full model production functions. Each node of the tree represents a production function. Nodes with vertical line factor inputs use Leontief functions. The other nodes are labeled with their elasticities of substitution.

aggregate between the imported good and the relevant total transport margin so that the amount of transport demanded scales with the amount of the good imported. We use a subnest to represent the importer use of air, land, and sea transport with a small elasticity of substitution, $\sigma = 0.2$.

We use the GTAP database for the base-year revenues and expenditures. In particular, our share parameters are calibrated with the GTAP v7 database of global expenditure values for 2004 (Gopalakrishnan and Walmsley, 2008). Emission amounts are obtained from the energy volume information in GTAP-E (Burniaux and Truong, 2002).

5.1.3 Dynamic Trajectories

We solve a myopic general equilibrium model with dynamic trajectories for the factor endowments and efficiency units. Thus far we have prototyped exogenous statistical trend forecasts for labor endowment, labor productivity, energy efficiency, agricultural land endowment, land productivity (yield), and fossil resource availability and extraction technology. These simple dynamics provide a stable basis for model testing. We now describe several examples of these dynamic trajectories in more detail.

Capital Accumulation We use a perfectly fluid model of capital with a 4% yearly depreciation rate. Investment contributes to consumer utility, with investment amounts calibrated to historical data. Investment enters the consumer utility function in a Cobb-Douglas nest with the government services and consumption bundles, implying that a fixed share of consumer income in each year goes to government services, investment, and consumption. In particular, the consumer buys the output from an industry that produces capital goods. This industry behaves as any other, demanding material goods and services in order to produce the capital good. This industry, however, does *not* demand capital, labor, or energy. By far the largest expenditure of the capital goods industry is on construction services, reflecting the fact that most capital is buildings, with sizable demands from other industries, such as machinery, transport equipment, and computing equipment.

The capital endowment in the next period is obtained from the dynamic equation

$$\bar{\mathbf{y}}^c_{K,t+1} = (1-\delta)\bar{\mathbf{y}}^c_{K,t} + \frac{\bar{x}^c_{I,0}}{\bar{y}^c_{K,0}}\mathbf{x}_{I,t}$$

where the capital depreciation rate, δ, is exogenously specified and the ratio in the equation is available from data, with the boundary condition $\bar{\mathbf{y}}^c_{K,0} = 1$.

Labor Productivity Another primary economic driver is population growth and increased labor productivity. We use population data from 1950 to 2008, with forecasts to 2050, from the 2008 United Nations population database (U.N. Population Division). Historical economic activity rates, the fraction of people that participate in the economy either with a job or looking for a job, from 1980 to 2006 are taken from the International Labor Organization (International Labor Organization Department of Statistics), along with projections to 2020. We combine these projections to estimate the labor endowments in each region.

Increasing productivity is modeled by inclusion of a productivity factor γ_L multiplying the labor endowment in the consumer problem, where $\gamma_L(t)$ is the labor productivity in

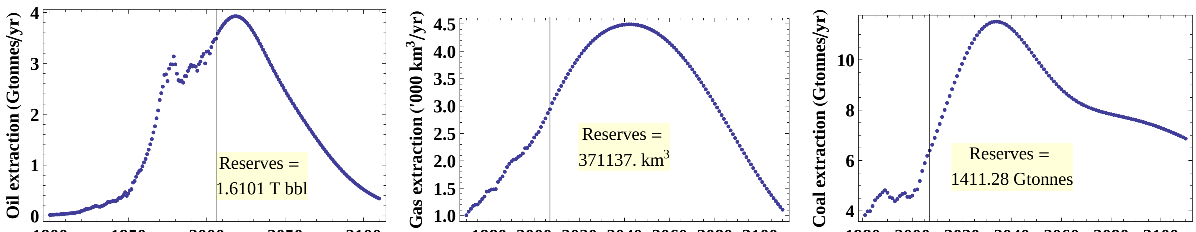

Figure 5: World conventional oil (left), gas (middle), and coal (right) depletion profiles.

year t and is tuned exogenously to match forecasts extrapolated from historical trends. In particular, we extrapolate the year-on-year growth rates in labor productivity to have constant mean and variance consistent with historical data from the U.S. Bureau of Labor Statistics International Labor Comparisons Division and from the OECD Statistics Division. Since this data covers only a subset of countries and economic regions, we group regions into three categories based on their recent growth rates and assume that regions in each category with unknown data share the same mean productivity rates as the remaining regions with known data.

Resource Usage Crude fossil extraction, reserves, depletion, and backstops are vital in understanding how energy demand is met. Based on a simple fossil resource depletion model, we forecast Gaussian extraction curves fit to historical data for model regions independently, constrained to give future extraction equal to existing fossil reserves. This model combines forecasts of new reserve discoveries with advancing extraction technologies to predict the extraction curves. This representation is straightforward for conventional fossil fuel industries because the technologies are well developed and detailed global historical data exists. Figure 5 shows the sum of all regional extrapolations for oil, gas, and coal resources.

The remaining global conventional crude oil in our trajectory is about 1.6 trillion barrels (Tbbl), which is near the median of expert estimates in the standard literature. The 2007 WEC Survey of Energy Resources (Caill, 2007) estimates global remaining resources of conventional crude plus proved reserves at about 1.8 Tbbl, though questions remain as to how much will be ultimately extractable. We have used simple, symmetric curves for these fits, implying a smooth fall-off of extraction rates as reserves deplete. The remaining global conventional gas in our trajectory is about 371 trillion m^3, which is near the 2007 WEC estimate of 386 trillion m^3.

Forecasts for coal depletion are more ambiguous, with high estimated resources to proved reserves ratio and serious questions about what percentage will be technologically recoverable and at what rates. The sum of coal reserves we use in our trajectory is 1.4 trillion tonnes of ultimately extractable coal resource remaining in the ground. The estimate amounts to an assumption of only about 25% of the existing coal resources being ultimately recoverable, which is at the low end of estimated recoverable resources. This outcome is considered relatively unlikely, though not impossible.

Table 4: 2020 fossil fuel CO_2 difference accounting for the AB scenario with a uniform 105\$/tC (USD per tonne carbon) tax starting in 2012 versus the BAU scenario. Changes in measured carbon flows are in millions of tonnes. Percent changes are shown in (); reductions in carbon flows relative to baseline are shown in green, increases in red.

AB-105 vs. BAU	Annex B USA	EU	RUS	JAZ	CAN	Non Annex B CHK	LAM	ROW	Prod.
USA	-1936.98 (-29.03)	-65.95 (-22.87)	-2.69 (-25.51)	-29.30 (-30.31)	-44.76 (-25.10)	-53.06 (-31.05)	-85.71 (-34.06)	-45.45 (-34.14)	-2263.89 (-29.02)
EU	-82.63 (-25.18)	-1232.11 (-22.22)	-16.69 (-21.64)	-13.14 (-21.22)	-5.66 (-19.03)	-33.85 (-27.12)	-20.26 (-27.33)	-126.70 (-33.61)	-1531.05 (-23.14)
RUS	-46.03 (-38.15)	-178.93 (-34.72)	-930.46 (-29.44)	-10.40 (-35.69)	-1.76 (-35.15)	-96.66 (-47.31)	-22.29 (-50.73)	-80.12 (-42.02)	-1366.67 (-32.01)
JAZ	-11.07 (-17.67)	-10.70 (-17.25)	-0.70 (-21.06)	-534.67 (-29.37)	-1.53 (-24.11)	-63.32 (-30.32)	-3.09 (-26.20)	-32.44 (-31.74)	-657.53 (-28.87)
CAN	-61.35 (-23.87)	-7.30 (-20.46)	-0.25 (-18.62)	-1.75 (-20.71)	-118.22 (-23.23)	-4.65 (-23.28)	-2.48 (-23.57)	-3.12 (-24.07)	-199.12 (-23.29)
CHK	42.43 (5.64)	50.14 (6.39)	5.54 (8.49)	36.98 (7.61)	5.42 (7.26)	29.81 (0.24)	12.47 (7.53)	27.82 (3.65)	210.60 (1.34)
LAM	131.44 (43.03)	27.27 (19.91)	3.54 (36.87)	1.41 (6.93)	6.51 (35.15)	2.96 (4.63)	101.93 (5.26)	4.50 (7.72)	279.55 (10.97)
ROW	61.73 (19.36)	161.39 (23.17)	8.61 (16.72)	71.94 (24.62)	3.80 (14.96)	41.92 (6.29)	11.45 (15.47)	291.76 (3.80)	652.60 (6.66)
Cons.	-1902.47 (-21.58)	-1256.19 (-15.58)	-933.10 (-27.61)	-478.93 (-17.01)	-156.20 (-18.44)	-176.86 (-1.26)	-7.99 (-0.31)	36.24 (0.39)	-4875.51 (-9.78)

5.1.4 Carbon Policy Study

We report in Elliott et al. (2010) on a study of carbon leakage under various global scenarios of climate change mitigation policy. We consider four classes of scenarios in this study: a business-as-usual scenario with no climate policy (BAU); a fully global scenario where every country prices carbon (UN); a scenario in which only Annex B countries price carbon (AB); and a scenario with carbon pricing in Annex B regions and full border-tax adjustments applied to imports and exports (AB-BTA). In each of the scenarios with carbon pricing, we use uniform carbon prices with values ranging from 10\$/tC to 175\$/tC (2004 USD per tonne carbon).

To compute the carbon content of the imports for the AB-BTA scenario, we approximately solve (3) by constructing a hierarchy of carbon contributions: primary direct carbon is emitted from consumption of crude energy commodities (coal, oil, and gas); secondary direct carbon from the consumption of processed crude petroleum products such as gasoline and petroleum coke; primary indirect carbon from consumption of electricity; and secondary indirect carbon from consumption of energy intensive materials such as steel, chemicals, cement, and nonferrous metals. We also include other secondary indirect carbon contributions from less energy-intensive industries. Full details on this approximation can be found in Elliott et al. (2010).

We then define a carbon flow matrix to collect the results of the simulations. Table 4 shows the carbon difference matrix the AB scenario with a carbon price of 105\$/tC (AB-105) relative to the BAU scenario. The upper-left block of the matrix shows decreased trade among the taxing regions, while the lower-right block shows increased trade among the nontaxing regions. Increases in imports of carbon from the nontaxing regions due to carbon leakage are shown in the lower-left block. In particular, carbon consumption for all

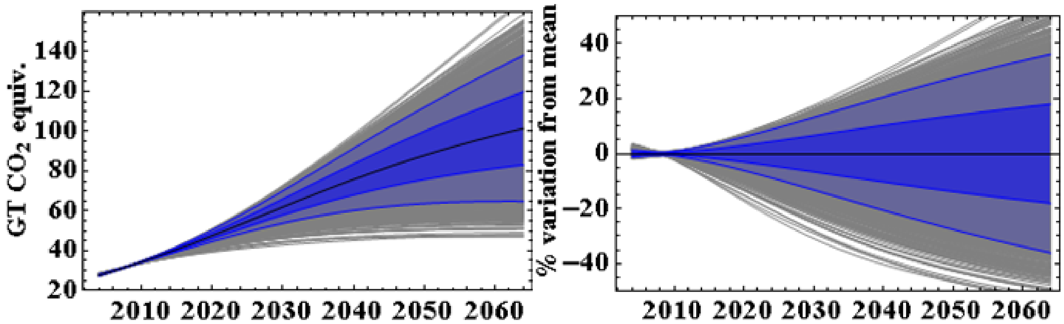

Figure 6: (Left) Global emissions from fossil fuel consumption for 5,000 model runs with perturbed substitution elasticities and (right) the relative sensitivity of global CO_2 emissions. Each gray line is a single simulated trajectory, and the black line is the mean. The dark and light blue shaded areas encompass one and two standard deviations from the mean, respectively.

taxing regions (direct and virtual) falls much more slowly than carbon production due to carbon leakage. Depending on how the goal for an emissions target is defined, the fact can change the necessary carbon price target by as much as 15-20%. The addition of border-tax adjustments has a small, but not insubstantial, effect.

The results of this study are uncertain due to the propagation of errors from the input data set, the GTAP I/O matrices, and key model parameters, the elasticities of substitution. Using an instance of the model having slightly different dynamic trajectories, we report in Elliott et al. (2009b) on a study of these errors. When studying the errors due to misspecification of the elasticity of substitution parameters, we synthesized estimates for capital-labor substitution from several sources and simulated large portions of the relevant parameter space, focusing on parameter distributions centered on Cobb-Douglas for ease of comparison with other recent studies (Sokolov et al., 2009). In addition to the 16 capital-labor substitutions for each industry, we also looked at the 16 Armington trade elasticities, the 16 substitution elasticities for the import/domestic consumption decision, and 23 other substitutions parameters from the nested functions depicted in Figure 4, such as energy-materials substitution and the substitutability between fossil energy inputs such as coal and natural gas. We considered symmetric Gaussian distributions with standard deviations set universally at 20% of mean values taken from a variety of sources (Balistreri et al., 2003, Paltsev et al., 2005, Liu et al., 2004). We performed an ensemble simulation using 5,000 uncorrelated draws from the full multivariate distribution and another ensemble of 1,000 draws to study a key subset, the Armington elasticities. In all the studies we explored model output variables at a wide range of regional and industrial scales in order to get a full view of the impacts of data and parameter error on forecast results. Figure 6 shows an example of such a sensitivity measurement for global CO_2 emissions.

5.2 Parallel Stochastic Optimal Growth Problems

DPSOL is parallel dynamic programming solver being developed by Cai and Judd (Cai, 2009, Cai and Judd, 2010). This solver uses the Condor system, an open-source software framework for high-throughput, distributed parallelization of computationally intensive tasks on a farm of computers developed by the Condor team at the University of Wisconsin-Madison. Condor acts as a resource manager for allocating and managing the computers available in the pool of machines.

This algorithm is implemented by using the Condor Master-Worker (MW) implementation. The Condor MW implementation circumvents the typical parallel programming hurdles, such as load balancing, termination detection, and distributing algorithm control information to the compute nodes. Moreover, the computation in the MW system is fault-tolerant: if a worker fails in executing a portion of the computation, the master simply distributes that portion of the computation to another worker, which can be an additional worker available in the pool of computers. Furthermore, the user can request any number of workers, independent of the number of tasks. If the user requests m workers and there are $n \times m$ tasks, then the fast computers will process more tasks than the slow machines, eliminating the load-balancing problem when n is large.

We use the multidimensional stochastic optimal growth problem to illustrate the efficiency of the DPSOL algorithm. In particular, we solve the problem

$$\begin{aligned} V(k,\theta) \quad = \quad & \max_{c\geq 0, l\geq 0, k^+, \theta^+} && u(c,l,k^+) + \beta E\{V(k^+,\theta^+) \mid k,\theta,c,l\} \\ & \text{s.t.} && k^+ = F(k,l,\theta) - c \\ & && k^+ \in [\underline{k}, \bar{k}]\,, \end{aligned}$$

where $k, k^+ \in \mathbb{R}^d$ are the continuous state vectors, $\theta, \theta^+ \in \Theta = \{\theta_j = (\theta_{j,1}, \ldots, \theta_{j,d}) : 1 \leq j \leq N\}$ are discrete state vectors, c and l are the actions, $\beta \in (0,1)$ is the discount factor, u is the utility function, and $E\{\cdot\}$ is the expectation operator. Here k^+ and θ^+ are the next-stage states dependent on the current-stage states and actions, and they are random.

We let k, θ, k^+, θ^+, c, and l are d-dimensional vectors with $d = 10$, making a DP problem with ten continuous states and ten states taking on discrete values. We choose $\beta = 0.8$, $[\underline{k}, \bar{k}] = [0.5, 3.0]^d$, $F(k,l,\theta) = k + \theta A k^{\alpha} l^{1-\alpha}$ with $\alpha = 0.25$, and $A = \frac{1-\beta}{\alpha\beta} = 1$, and

$$u(c,l,k^+) = \sum_{i=1}^{d} \left[\frac{c_i^{1-\gamma}}{1-\gamma} - B\frac{l_i^{1+\eta}}{1+\eta} \right] + \sum_{i\neq j} \mu_{ij}(k_i^+ - k_j^+)^2,$$

with $\gamma = 2$, $\eta = 1$, $\mu_{ij} = 0$, and $B = (1-\alpha)A^{1-\gamma} = 0.75$. Moreover, we assume $\theta_1^+, \ldots, \theta_d^+$ are independent and the possible values of θ_i and θ_i^+ are $a_1 = 0.9$ and $a_2 = 1.1$, and the probability transition matrix from θ_i to θ_i^+ is

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix},$$

for each $i = 1, \ldots, d$. That is,

$$\Pr[\theta^+ = (a_{j_1}, \ldots, a_{j_d}) \mid \theta = (a_{i_1}, \ldots, a_{i_d})] = P_{i_1,j_1} P_{i_2,j_2} \cdots P_{i_d,j_d}\,,$$

where P_{i_α,j_α} is the (i_α, j_α) element of P, for any $i_\alpha, j_\alpha = 1, \ldots, 2$ with $\alpha = 1, \ldots, d$. Therefore,

$$\begin{aligned} & E\{V(k^+,\theta^+) \mid k, \theta = (a_{i_1}, \ldots, a_{i_d}), c, l\} \\ = \quad & \textstyle\sum_{j_1,j_2,\ldots,j_d=1}^{2} P_{i_1,j_1} P_{i_2,j_2} \cdots P_{i_d,j_d} V(k_1^+, \ldots, k_d^+, a_{j_1}, \ldots, a_{j_d})\,. \end{aligned}$$

Table 5: Results for parallel stochastic optimal growth problem with $d = 10$.

Wall clock time for all 4 VFIs	20,019 seconds
Total CPU time used by all workers	1,166,767 seconds
Mean CPU time for the workers	18,230 seconds
Standard deviation CPU time for the workers	469 seconds
Number of (different) workers	64
Overall parallel performance	92.64%

Our value function iteration method approximates the continuous dimensions with a degree three complete Chebyshev polynomial approximation with $1 + 2d^2 + 4d(d-1)(d-2)/3 = 1161$ nodes. Use use NPSOL to solve the individual optimization problems. Since the number of possible values of θ_i is 2 for $i = 1, \ldots, d$, the total number of tasks for one value function iteration is $2^d = 1024$. Under Condor, we assign 64 workers to do this parallel work. After running 4 value function iterations (VFIs), we obtain the results in Table 5, where the last line shows that the parallel efficiency of our parallel numerical DP method is very high.

6 Future Extensions

Our modeling framework is meant to be extensible by both developers and users to suit their modeling needs. Many extensions are planned, and others are under development. In particular, we are planning to introduce fully dynamic computable general equilibrium models and to augment the set of building blocks available to assemble a model for particular studies. These building blocks include capital and product vintages (Benhabib and Rustichini, 1991, Cadiou et al., 2003, Salo and Tahvonen, 2003) and overlapping consumer generations (Auerbach and Kotlikoff, 1987), which are necessary when studying distributional impacts (Fullerton, 2009, Fullerton and Rogers, 1993). We plan to add features such as private learning, research and development, and technology adoption (Boucekkine and Pommeret, 2004, Futagami and Iwaisako, 2007, Hritonenko, 2008, Zou, 2006), which add differential equations to the optimal control problem solved by each industry. Consumer modeling will be extended to include household production functions, nonseparable utility functions, and heterogeneous beliefs. We will also augment our policies to include floors and ceilings on cap-and-trade programs, permit banking and expiration, dynamic tax rates, and revenue recycling policies. Future enhancements to the approximation, optimization, and quadrature methods in DPSOL will allow us to solve significantly larger problems dynamic stochastic problems.

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