Stability and Performance Analysis of NLP Sensitivity-Based NMPC Controllers

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Carnegie Mellon

CAPD CENTER

Industrial-Size NMPC -ExxonMobil, BASF, Honeywell, ABB, Cybernetica-

- **Rigorous Dynamic Models ~ 4,000 DAEs**
- **On-Line Solution Times ~ 2-3 min**

- Ideal NMPC: Instantaneous Optimal Feedback, Not Achievable in Practice

- Real-Time NMPC: *Good* Practical Performance, *Conservative* Stability Analysis

Results

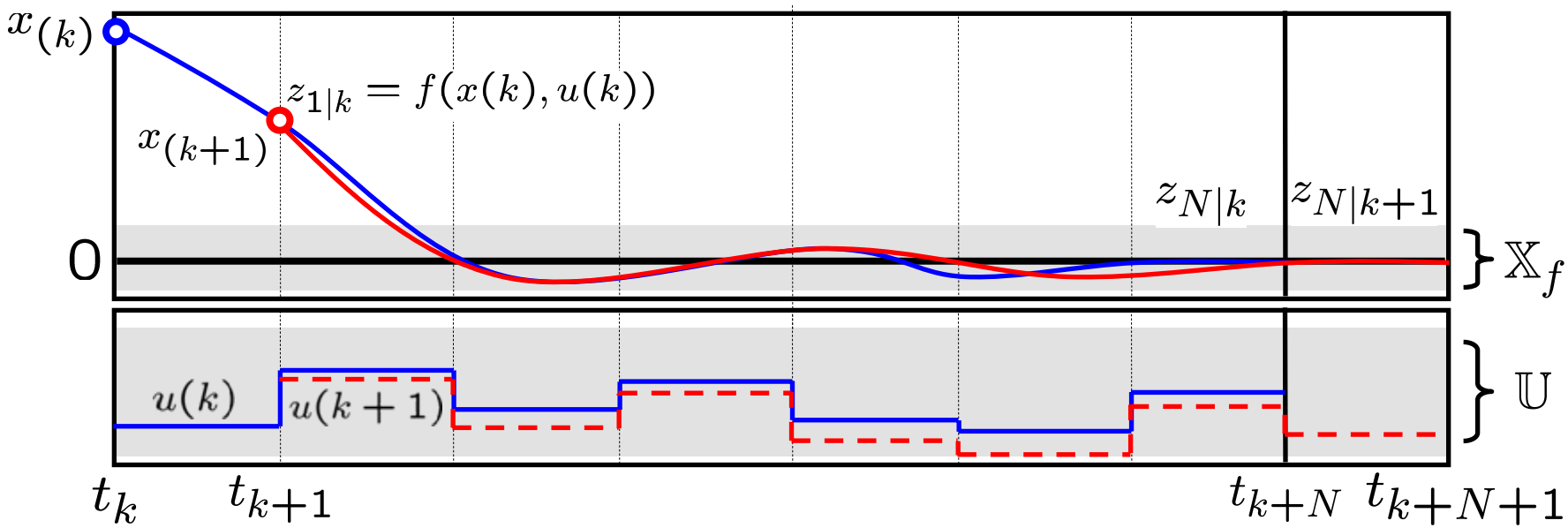
- Advanced Step NMPC Controller - **Instantaneous Feedback for Industrial-Size NMPC**

- **Concise Stability Analysis**
- **Approximates Robustness Margins of Ideal NMPC**

Ideal NMPC

Ideal NMPC – Nominal Case

Perfect Model



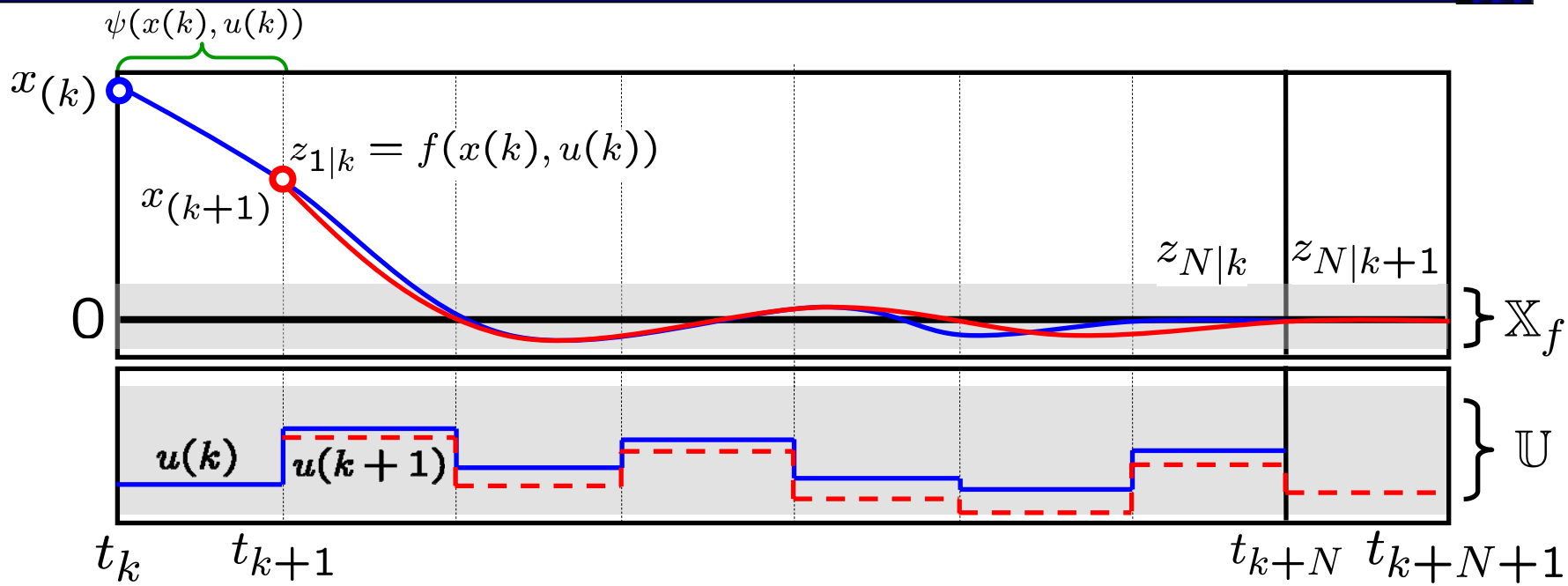
$$\mathbf{P}(x_{(k)}) \quad \begin{aligned} \min_{v_{l|k}} \quad & J(x(k)) \\ \text{s. t. } z_{l+1|k} &= f(z_{l|k}, v_{l|k}) \\ \boxed{z_{0|k} = x(k),} \\ z_{N|k} &\in \mathbb{X}_f, \; v_{l|k} \in \mathbb{U}. \end{aligned}$$

$$\longrightarrow \quad J(x_{(k)}) \qquad u_{(k)} = h^{id}(x_{(k)})$$

$$\mathbf{P}(x_{(k+1)}) \quad \begin{aligned} \min_{v_{l|k+1}} \quad & J(x(k+1)) \\ \text{s. t. } z_{l+1|k+1} &= f(z_{l|k+1}, v_{l|k+1}) \\ \boxed{z_{0|k+1} = x(k+1)} \\ z_{N|k+1} &\in \mathbb{X}_f, \; v_{l|k+1} \in \mathbb{U}. \end{aligned}$$

$$\longrightarrow \quad J(x_{(k+1)}) \qquad u_{(k+1)} = h^{id}(x_{(k+1)})$$

Ideal NMPC – Nominal Case



- **Recomputation Enforces Terminal Constraint**

$$J(x(k+1)) - J(x(k)) \leq -\psi(x(k), u(k))$$

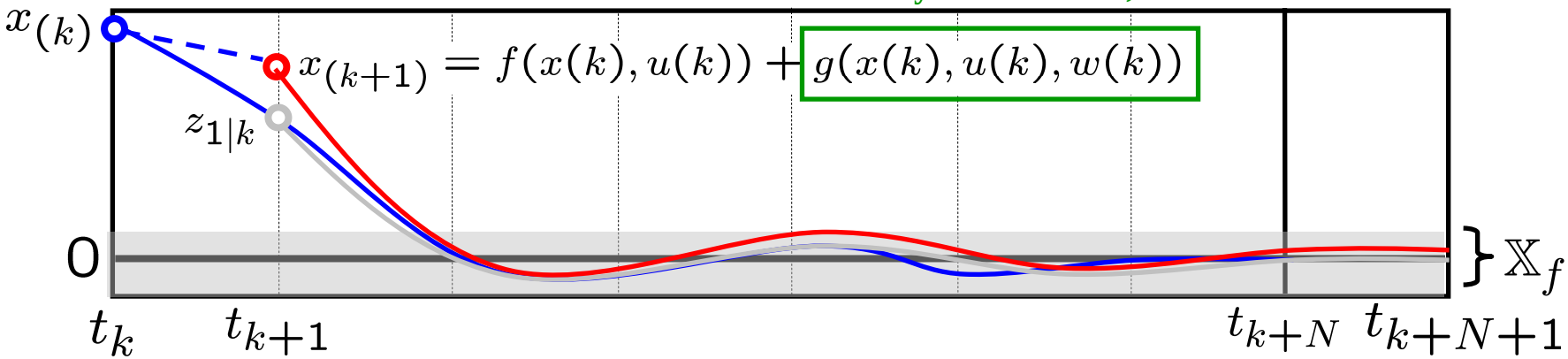
- **Cost Function is a Lyapunov Function, Nominal Stability**
- **Assumption: Solution of** $\mathrm{P}(x)$ **Instantaneous** ← **Not In Practice**
- **Model is Perfect:** $x(k+1)$ **Always Known IN ADVANCE at Time** t_k
- **Solve** $\mathrm{P}(x(k+1))$ **Between Sampling Times** t_k **and** t_{k+1} *Findeisen, et.al. 2004*

Do We Need Nominal Stability Proof for Real-Time NMPC?

Ideal NMPC – Robust Case

Plant-Model Mismatch

Uncertainty – Parameters, Noise -



What Is

$$\mathbf{P}(x(k)) \longrightarrow J(x_{(k)})$$

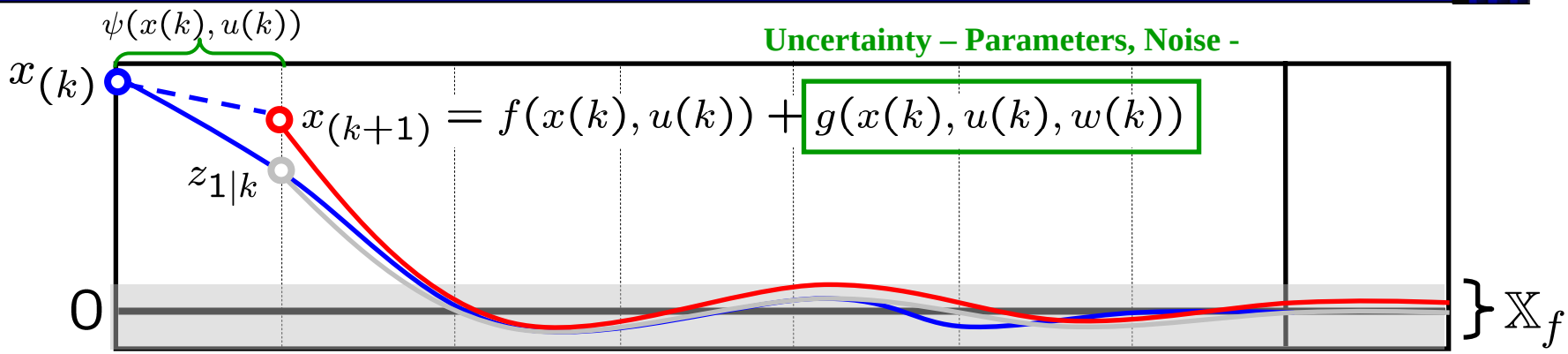
What Would Be

$$\mathbf{P}(z_{1|k}) \longrightarrow J(z_{1|k})$$

What Will Be

$$\mathbf{P}(x(k+1)) \longrightarrow J(x_{(k+1)})$$

Ideal NMPC – Robust Case



Plant-Model Mismatch $x_{(k+1)} - z_{1|k} = g(x(k), u(k), w(k))$

Cost Mismatch $\epsilon(x_{(k+1)}) = J(x_{(k+1)}) - J(z_{1|k})$

- **Recomputation Enforces Terminal Constraint**

$$J(x(k+1)) - J(x(k)) \;\leq\; -\psi(x(k), u(k))$$

- **If** $|\epsilon(x_{(k+1)})| \leq L_J |g(x(k), u(k), w(k)|$ **then** $J(\cdot)$ **is an ISS Lyapunov Function**

Magni & Scattolini, 2007

- **Ideal NMPC provides Inherent Robustness**
- $x(k+1)$ **Cannot be Known in Advance at Time** t_k
- **Cannot Solve** $\mathbf{P}(x_{(k+1)})$ **IN ADVANCE but Can Solve** $\mathbf{P}(z_{1|k})$ **IN ADVANCE**

How To React On-Line?

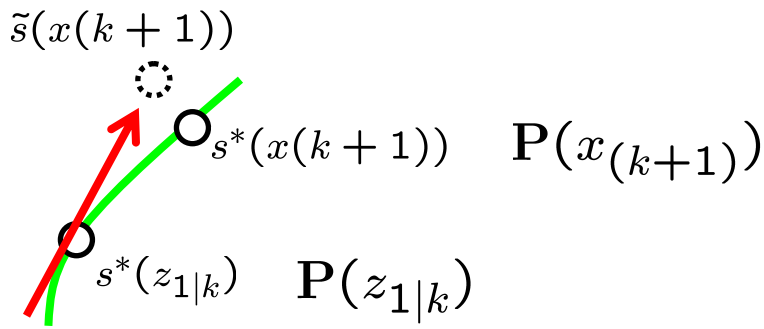
Nonlinear Programming Sensitivity

Optimal Control Problem is Parametric on Initial State $\mathbf{P}(x)$

NLP Sensitivity

- **NLP Sensitivity** - **Existence and Differentiability of Solution Path** - *Fiacco, 1983*

Fast Perturbed Newton Step



KKT System at Nominal Solution

Perturb $x_{(k+1)}$

KKT Matrix →

$$\mathbf{K}(z_{1|k})\Delta s(x_{(k+1)}) = -\varphi(s^*(z_{1|k}),\ z_{1|k}\ ).$$

$$\tilde{s}(x_{(k+1)}) - s^*(z_{1|k})$$

KKT Conditions

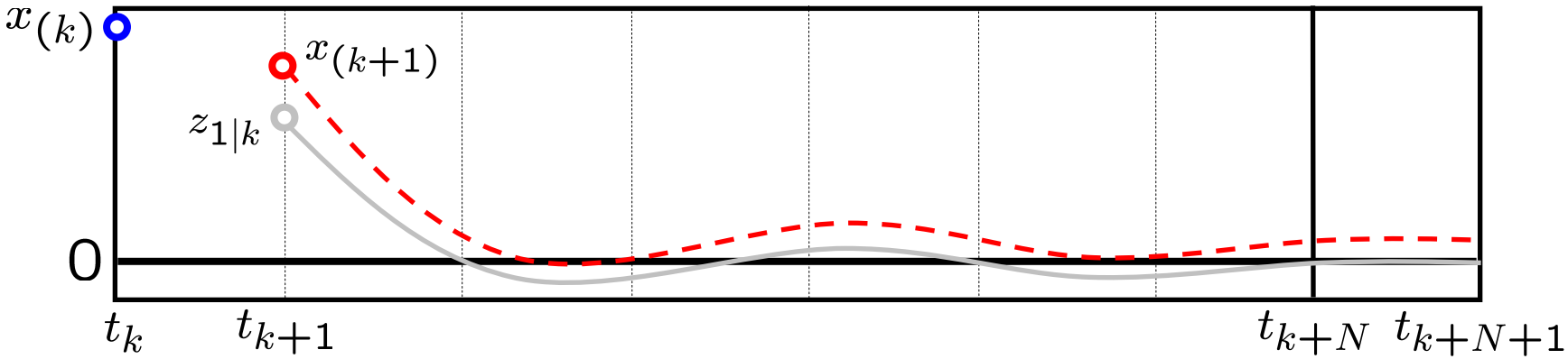
$$|\tilde{s}(x_{(k+1)}) - s^*(x_{(k+1)})| = O|x_{(k+1)} - z_{1|k}|^2$$

- **Assumption** - **Perturbation** $|x_{(k+1)} - z_{1|k}|$ (**Small**) **Does not Induce Active-Set Changes**

Advanced-Step NMPC

Predict and Correct

AS-NMPC - Algorithm



t_k - **Predict** $z_{1|k} = f(x_{(k)}, u_{(k)})$

$t < t_{k+1}$ - **At Solution of** $\mathrm{P}(z_{1|k})$ **Hold KKT System**

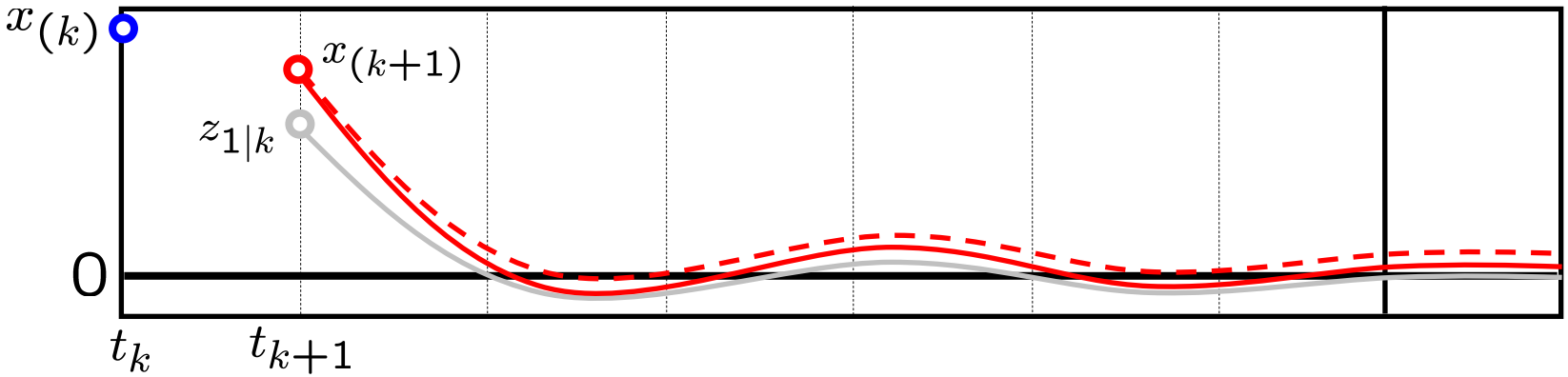
t_{k+1} - **Estimate** $x_{(k+1)}$

- **Perturb KKT System** $\tilde{s}(x(k+1))$ - **Extract** $u_{(k+1)}$

- **Predict and Start Background Problem**

AS-NMPC Control Law $u = h^{as}(x)$

AS-NMPC - Properties



- **Properties of AS-NMPC**

AS-NMPC **Ideal NMPC**

$$|\tilde{s}(x_{(k+1)}) - s^*(x_{(k+1)})| \leq L|x_{(k+1)} - z_{1|k}|^2$$

$$|h^{as}(x(k)) - h^{id}(x(k))| \leq L_u|g(x_{(k)}, u_{(k)}, w_{(k)})|^2$$

- **Perfect Model** $g(x_{(k)}, u_{(k)}, w_{(k)}) = 0 \qquad h^{as}(x) = h^{id}(x)$

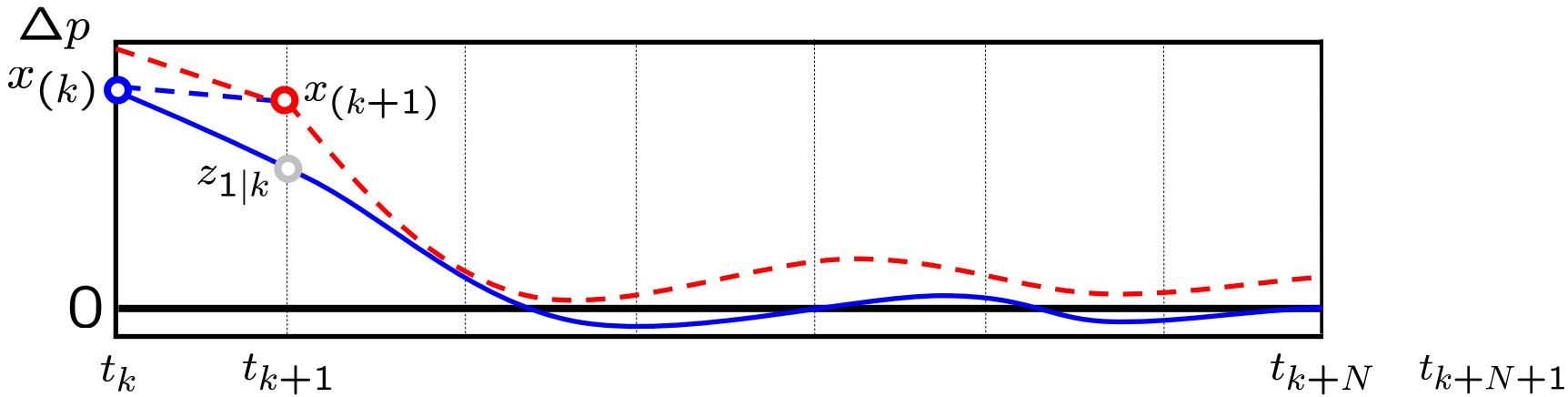
- $\mathbf{P}(\cdot)$ **Is a Quadratic Program -MPC-** $\tilde{s}(x) = s^*(x) \qquad h^{as}(x) = h^{id}(x)$

- **Uncertainty Attenuated through State Estimation**

$$g(x_{(k)}, u_{(k)}, w_{(k)}) \to 0 \qquad h^{as}(x) \to h^{id}(x)$$

AS-NMPC - Enhanced

- **Avoid Simulation Step** $z_{1|k} = f(x_{(k)}, u_{(k)})$
- **Include In Formulation of** $\mathrm{P}(\cdot)$



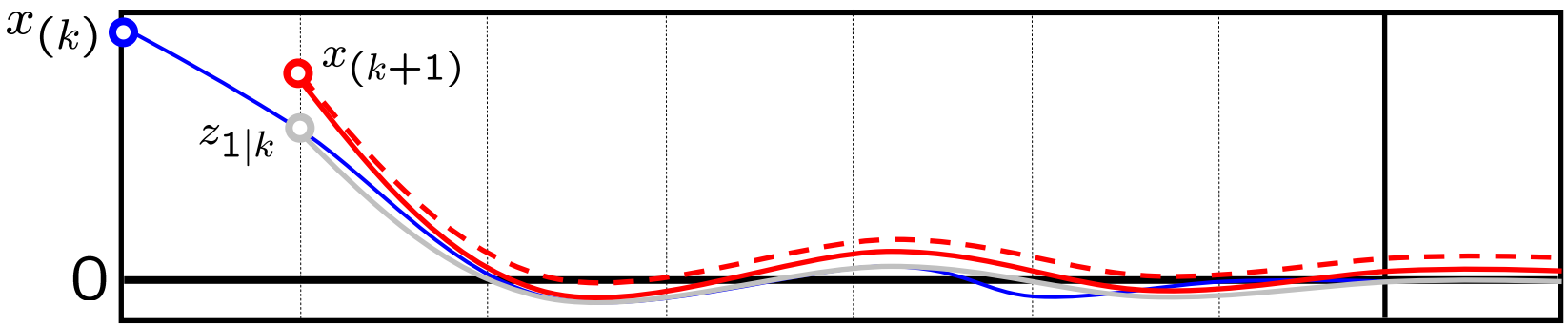
- **AS-NMPC Problem** $\mathrm{P}_{N+1}(x(k), u(k))$

$$
\begin{aligned}
\min_{v_{l|k}} \quad & \bar{J}(x(k), u(k)) \\
\text{s.t.} \; z_{l+1|k} &= f(z_{l|k}, v_{l|k}) \\
x(k+1) \longrightarrow \quad \boxed{z_{1|k}} &= f(z_{0|k}, u(k)) \\
z_{0|k} &= \boxed{x(k)} \longleftarrow p \\
z_{N|k} &\in \mathbb{X}_f, \; v_{l|k} \in \mathbb{U}
\end{aligned}
$$

Current Control

- **Shifting Strategy: Perturb Initial Conditions and Force** $z_{1|k} = x(k+1)$

AS-NMPC –Robust Stability



What Is

$$\mathbf{P}(x(k), h^{as}(x(k))) \longrightarrow \bar{J}(x_{(k)}, h^{as}(x_{(k)}))$$

What Would Be

$$\mathbf{P}(z_{1|k}, h^{id}(z_{1|k})) \longrightarrow \bar{J}(z_{1|k}, h^{id}(z_{1|k}))$$

$$\updownarrow \quad \epsilon_s(x(k+1))$$

Uncertainty

What Will Be –Ideally-

$$\mathbf{P}(x(k+1), h^{id}(x(k+1))) \longrightarrow \bar{J}(x_{(k+1)}, h^{id}(x_{(k+1)}))$$

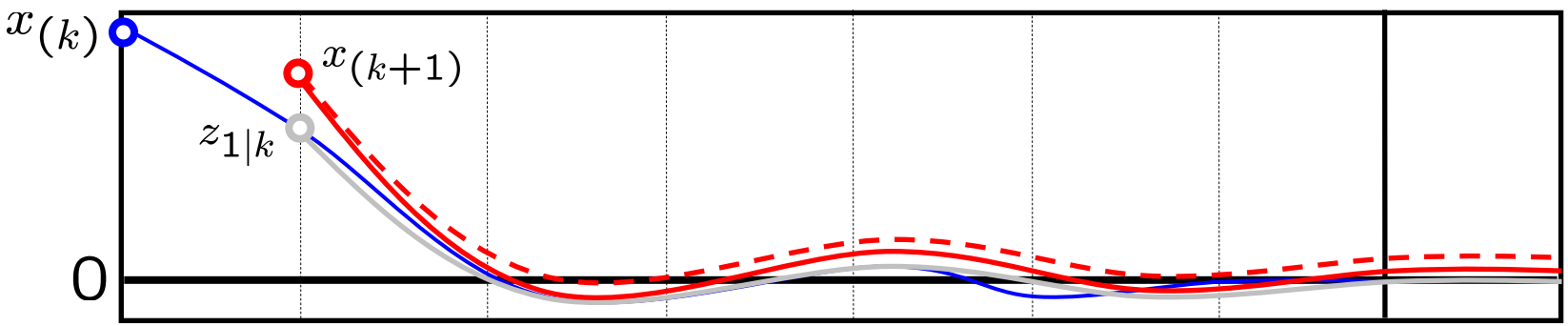
$$\updownarrow \quad \epsilon_{as}(x(k+1))$$

NLP Sensitivity Errors

What Will Be

$$\mathbf{P}(x(k+1), h^{as}(x(k+1))) \dashrightarrow J(x_{(k+1)}, h^{as}(x_{(k+1)}))$$

AS-NMPC –Robust Stability



- **Recomputation Enforces Terminal Constraint**

$$\bar{J}(x(k+1), h^{as}(x(k+1))) - \bar{J}(x(k), h^{as}(x(k))) \leq -\psi(x(k), u(k))$$

$$|\epsilon_s(x_{(k+1)})| \leq L_J |g(x(k), u(k), w(k)|$$

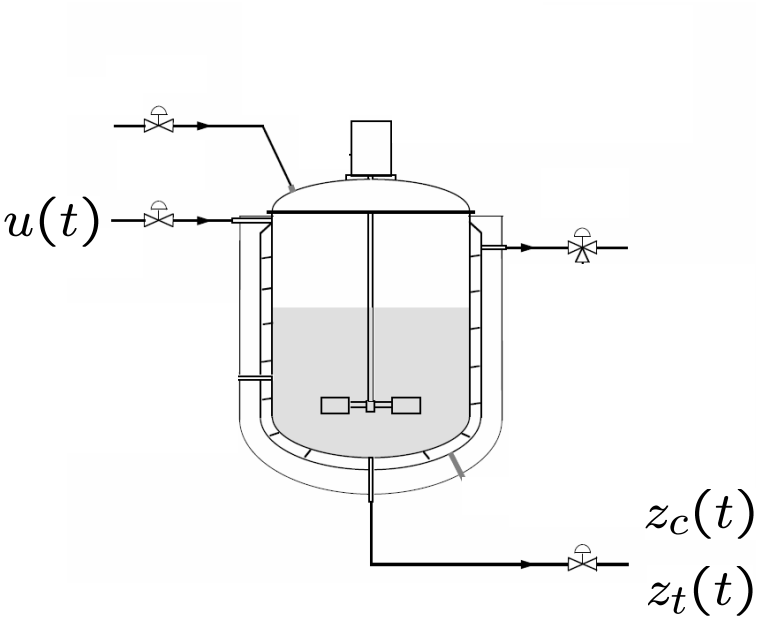
$$|\epsilon_{as}(x_{(k+1)})| \leq L_{Jas} |g(x(k), u(k), w(k)|^2$$

AS-NMPC Control Law is ISS

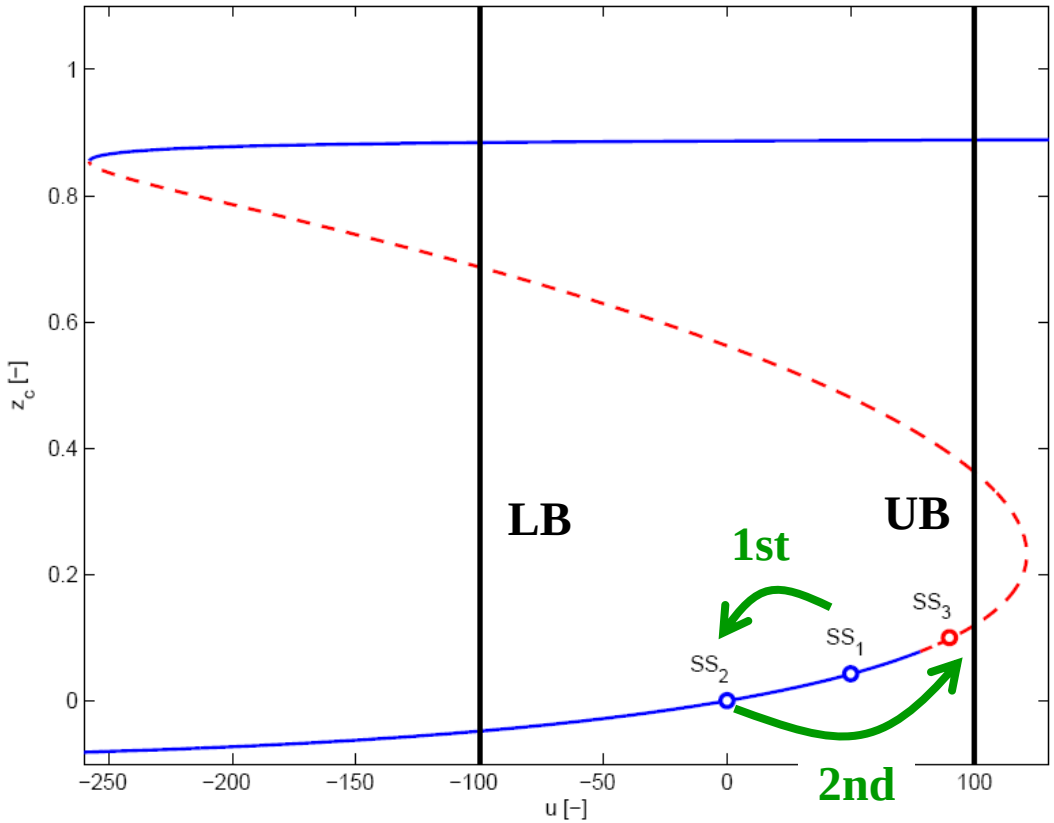
Robustness of AS-NMPC Degrades *Gracefully* due to NLP Sensitivity Errors

Case Study

- **Impact of Computational Delay**
- **Contrast Robustness Margins AS-NMPC vs. Ideal NMPC**
- **Model Mismatch on Residence Time and Noise**

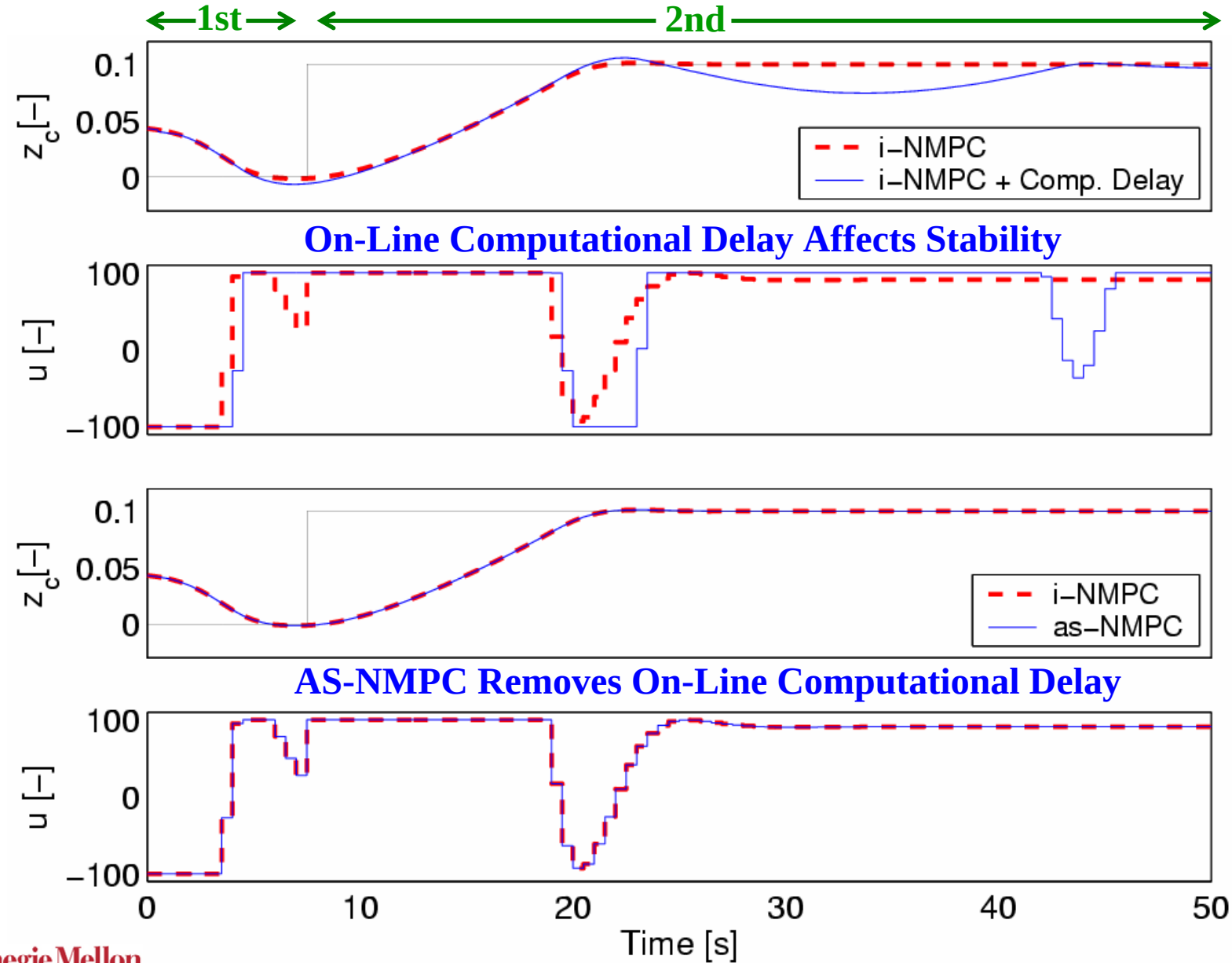


Take Reactor to Extreme Conditions



Case Study – Nominal Case

Effect of Computational Delay

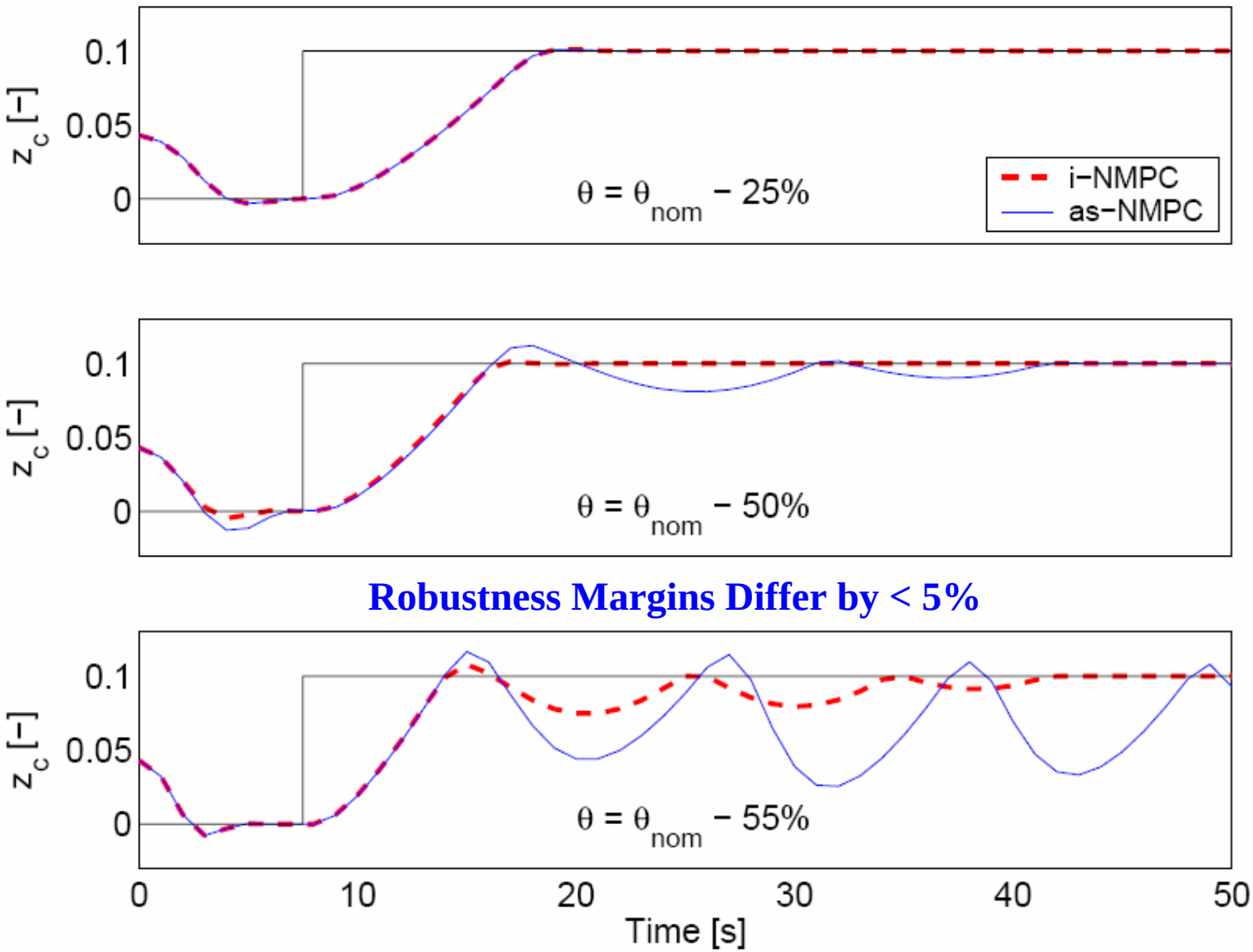


On-Line Computational Delay Affects Stability

AS-NMPC Removes On-Line Computational Delay

Case Study – Robust Case

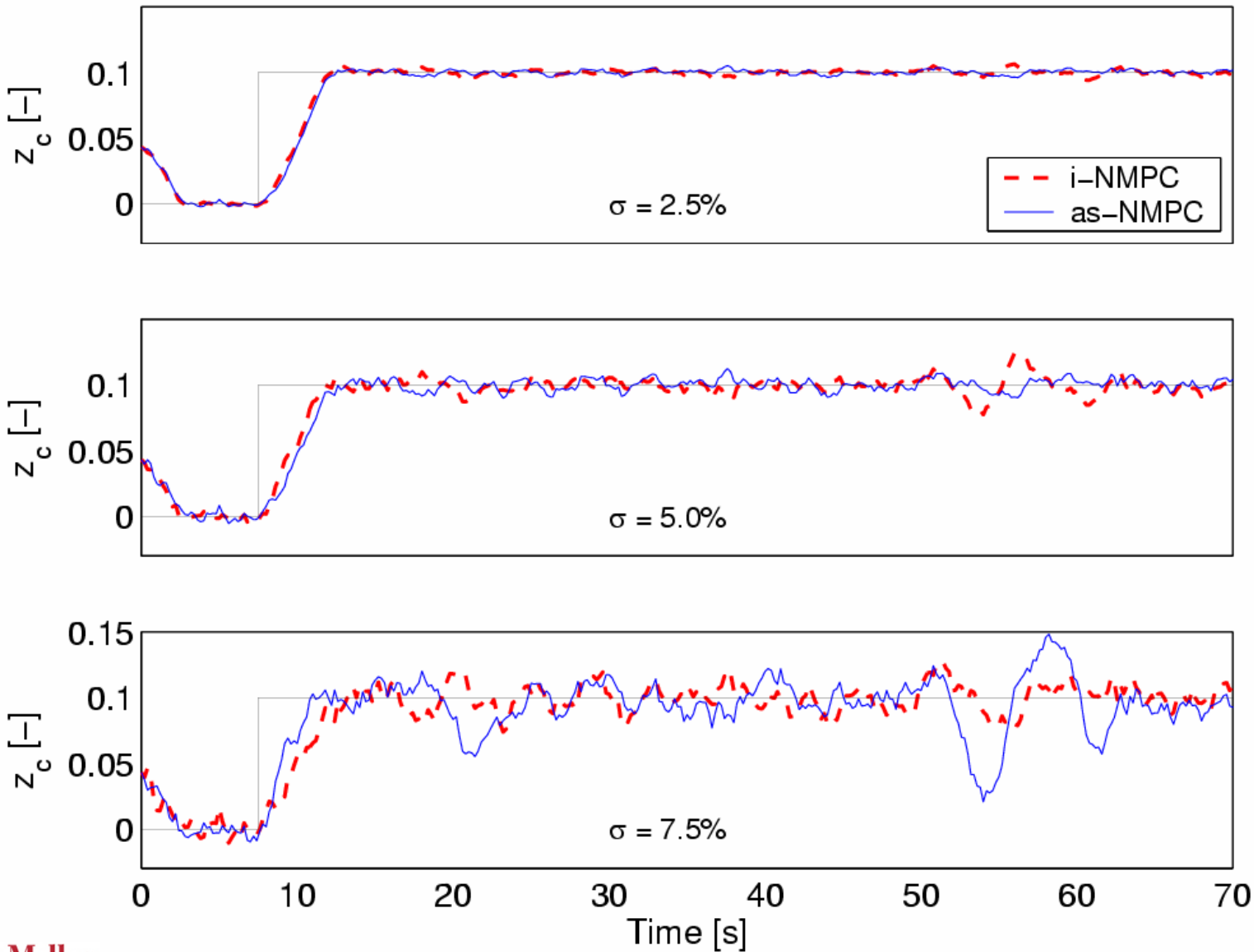
Robust Case – Residence Time: Plant Much Faster than What Controller Predicts



Robustness Margins Differ by < 5%

Case Study

Robust Case – Model Mismatch + Estimation Noise



Conclusions and Future Work

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Conclusions

AS-NMPC Controller

- Instantaneous Feedback
- Approximates Robustness Margins of Ideal NMPC

Attenuate Uncertainty Crucial

Fast NMPC = Fast Solver + Real-Time Strategy + State Estimation

Future Work

- Improve Robustness of AS-NMPC
  - Couple to Fast MHE Strategy *Z. Laird & Biegler, 2007*

Very Large Models – Background Problem Solves in 2-3 Sampling Times

- Multi-Step AS-NMPC – Performance and Stability

- **Special Message:**
  - **Perturbed Newton Step Trivially Implementable**
  - **Independent of Solution Method and NLP Solver**
  - **Less Accurate –Bounded Error- with Quasi-Newton Solver** ***Diehl et.al., 2001***

AS-NMPC Implementable NOW with Standard Optimization Tools

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