The PATH Not Taken: Lemke's Method for Strictly Positive Matrices[1]

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The PATH Not Taken

Two problem formulations diverged

The grassy fork wanting wear

Preventing a cycle back

Scheme less traveled

"The Road Not Taken", Robert Frost, 1920

Two roads diverged in a yellow wood,
And sorry I could not travel both
And be one traveler, long I stood
And looked down one as far as I could
To where it bent in the undergrowth;

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$$0 \leq x \quad \perp \quad 1 - Mx \geq 0$$

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$$\begin{array}{lclcl} 0 \le w & \perp & -Mw + v & \ge 0 \\ 0 \le v & \perp & e^T w & \ge 1 \end{array}$$

  - $x^* = \frac{w^*}{v^*}$
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  - PATH with regular starts solves only small problems

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    - Better question: why does PATH even work on some problems?

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- A dual regularized variational formulation

$$\begin{array}{llll} 0 \le w & \perp & -Mw + v & \ge 0 \\ v \text{ free} & \perp & -e^T w + \frac{1}{\alpha} v & = -1 \end{array}$$

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- Find a nontrivial solution to

$$0 \leq x \quad \perp \quad 1 - Mx \geq 0$$

  - For any solution $e^T x^* \geq \frac{1}{\max_{i,j} M_{i,j}}$
- A strictly positive formulation (inspired by CPS 1992)

$$0 \leq w \quad \perp \quad (\alpha E - M)w - 1 \geq 0$$

  - $x^* = \frac{w^*}{\alpha e^T w^* - 1}$ (if M strictly semimonotone)
  - $w^* = \frac{x^*}{\alpha e^T x^* - 1}$ (if $\alpha > \max_{i,j} M_{i,j} > 0$)

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    - Better question: why does PATH fail on some problems?

Then took the other, as just as fair,
And having perhaps the better claim,
Because it was grassy and wanted wear;
Though as for that the passing there
Had worn them really about the same,

The grassy fork wanting wear

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- Implement method in arbitrary precision arithmetic (Bailey et.al.)
  - Dense matrices and linear algebra
  - Rank-revealing QR factorization using Householder transformations
  - Refactor and compute iterate after each pivot
  - Minimum ratio test needs to break ties
    - Use maximum direction value (devex)
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- Conclude that instances are degenerate in infinite precision
  - Zero step length fixable by devex or Bland's rule
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  - Cycles of nonzero length in piecewise linear path are not
  - Linear path furcates and forms a complicated figure-eight cycle
  - PATH can detect cycles, but we must prevent them

And both that morning equally lay
In leaves no step had trodden black.
Oh, I kept the first for another day!
Yet knowing how way leads on to way,
I doubted if I should ever come back.

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- Conjecture: with high probability, randomized problem is nondegenerate
- Numerical results for the sparse instances
  - Lemke and PATH solve all problems with either randomization
  - Methods use a small number of pivots to find a solution

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  - In eight digit arithmetic
    - Degeneracy observed with both randomizations
    - Even using the first randomization tried
    - Lemke and PATH still solve them though

I shall be telling this with a sigh

Somewhere ages and ages hence:

Two roads diverged in a wood, and I –

I took the one less traveled by,

And that has made all the difference.

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  - Handle Lemke and regular starts in a rigorous manner
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- Concluding message
  - Bad formulations can lead to insights
  - Good formulations can lead to better insights
  - PATH requires rigorous degeneracy resolution

ICCOPT 2013

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Actively seeking session organizers for
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Contact Francisco Facchinei or Todd Munson

`http://eventos.fct.unl.pt/iccopt2013`