Derivative-based solution of the optimization problem(s) in DeMarco's model

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Cascading Network Failure

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Cascading Network Failure

- To predict cascading failure in large-scale networks, solid understanding of propagations of failures in small-scale networks is vital.
- This allows optimal redistribution of loads and network design.

Network Model(s): I

- Eight-node, Eleven-branch circuit is used as a toy model.
- Model state

$$\mathbf{x} = \begin{bmatrix} \phi \\ \mathbf{q} \\ \gamma \end{bmatrix} \in \mathbb{R}^{26} \tag{1}$$

where

1) ϕ is a vector of nodal flux differences ($\phi \in \mathbb{R}^{7}$),
2) $\mathbf{q}$ is the vector of nodal charges on capacitors ($\mathbf{q} \in \mathbb{R}^{8}$),
3) γ is the failure state of branches ($\gamma \in \mathbb{R}^{11}$),

Network Model(s): II

- DeMarco's original model equations

$$
\begin{aligned}
d\phi &= E_r^T C^{-1} \mathbf{q} \, dt &(2)\\
d\mathbf{q} &= \left(-E_r H_r L^{-1} H_r^T \phi - G C^{-1} \mathbf{q} + i_{\mathrm{in}} \right) \, dt, &(3)
\end{aligned}
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- Stochastic Version of the model

$$\begin{bmatrix} d\phi \\ dq \end{bmatrix} = \mathbf{M} + \mathbf{P} + \mathbf{U} \tag{4}$$
$$= \begin{bmatrix} E_r^T C^{-1}\mathbf{q}(t)dt \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -E_r H_r L^{-1} H_r^T \phi(t)dt \end{bmatrix} + \begin{bmatrix} 0 \\ -GC^{-1}\mathbf{q}(t)dt + \sqrt{2G\tau}\, dW_t \end{bmatrix}.$$

  - i_{in} : the current input, τ : system's temperature,
  - C, E_r, H_r, L, G : constant matrices,
  - M, P, U : system's mass, potential, Ornstein Uhlenbeck process.

Time stepping: A Splitting Solver

The integrator is a composition of maps

$$\begin{bmatrix}\phi(t)\\ \mathbf{q}(t)\end{bmatrix} = \mathbf{P}_{\frac{t}{2}} \circ \mathbf{M}_{\frac{t}{2}} \circ \mathbf{U}_t \circ \mathbf{M}_{\frac{t}{2}} \circ \mathbf{P}_{\frac{t}{2}} \left(\begin{bmatrix}\phi(0)\\ \mathbf{q}(0)\end{bmatrix} \right); \quad (5)$$

$$\mathbf{M}_t \left(\begin{bmatrix}\phi(0)\\ \mathbf{q}(0)\end{bmatrix} \right) = \begin{bmatrix}\phi(0) + E_r^T C^{-1} \mathbf{q}(0) t\\ \mathbf{q}(0)\end{bmatrix}, \quad (6)$$

$$\mathbf{P}_t \left(\begin{bmatrix}\phi(0)\\ \mathbf{q}(0)\end{bmatrix} \right) = \begin{bmatrix}\phi(0)\\ \mathbf{q}(0) - E_r H_r L^{-1} H_r^T \phi(0) t\end{bmatrix}, \quad (7)$$

$$\mathbf{U}_t \left(\begin{bmatrix}\phi(0)\\ \mathbf{q}(0)\end{bmatrix} \right) = \begin{bmatrix}\phi(0)\\ e^{-GC^{-1}t} \mathbf{q}(0) + \sqrt{\tau C \left(I - e^{-2GC^{-1}t} \right)}\; \mathbf{d}\end{bmatrix}, \quad (8)$$

Where t is the step size, and $\mathbf{d}$ is the stochastic force.

Optimization problems

Since we are interested in failure, we ask how the white noise might steer the system towards increasing energy.

- The exact problem:

$$\min_{\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_N} \mathcal{J}(\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_N) = \sum_{i=1}^{N} \mathbf{d}_i^T \mathbf{d}_i \qquad (9)$$

subject to

$$\max\left(E_N\right) > \epsilon. \qquad (10)$$

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- A proxy problem: replace constraint with:

$$\mathbb{I}_N^T E_N > \epsilon, \quad (\text{Or} \quad E_N^T E_N > \epsilon) \qquad (11)$$

  where E_N is the energy function, and $\mathbb{I}_N$ is a vector of all ones.

Energy Function

$$E_N(\phi_N) = E_N\left(\begin{bmatrix}(\phi_N)_1\\(\phi_N)_2\\(\phi_N)_3\\(\phi_N)_4\\(\phi_N)_5\\(\phi_N)_6\\(\phi_N)_7\end{bmatrix}\right) = \begin{bmatrix}\frac{1}{2}L_1^{-1}\gamma_1\left(-(\phi_N)_1\right)^2\\ \frac{1}{2}L_2^{-1}\gamma_2\left(-(\phi_N)_2\right)^2\\ \frac{1}{2}L_3^{-1}\gamma_3\left((\phi_N)_2-(\phi_N)_3\right)^2\\ \frac{1}{2}L_4^{-1}\gamma_4\left((\phi_N)_1-(\phi_N)_3\right)^2\\ \frac{1}{2}L_5^{-1}\gamma_5\left((\phi_N)_3-(\phi_N)_4\right)^2\\ \frac{1}{2}L_6^{-1}\gamma_6\left((\phi_N)_3-(\phi_N)_5\right)^2\\ \frac{1}{2}L_7^{-1}\gamma_7\left((\phi_N)_1-(\phi_N)_5\right)^2\\ \frac{1}{2}L_8^{-1}\gamma_8\left((\phi_N)_4-(\phi_N)_5\right)^2\\ \frac{1}{2}L_9^{-1}\gamma_9\left((\phi_N)_4-(\phi_N)_7\right)^2\\ \frac{1}{2}L_{10}^{-1}\gamma_{10}\left((\phi_N)_5-(\phi_N)_6\right)^2\\ \frac{1}{2}L_{11}^{-1}\gamma_{11}\left((\phi_N)_6-(\phi_N)_7\right)^2\end{bmatrix} \tag{12}$$

Probability of Failure(s)

We can compute the probability of one failure (at time t_N) from optimal noise vectors $\mathbf{d}_1, \mathbf{d}_1, \ldots \mathbf{d}_N$ via

$$P(\text{Failure}|\mathbf{d}_1, \mathbf{d}_1, \ldots \mathbf{d}_N) = e^{-\frac{\sum_{i=1}^{N} \mathbf{d}_i^T \mathbf{d}_i}{\tau}} \tag{13}$$

Multiple failure is an enumeration problem solved by exhaustive search.

Derivative Information: I

We have reformulated the splitting solver as linear discrete map:

$$\mathbf{x}_i = \mathbf{A}\mathbf{x}_{i-1} + \mathbf{B}\mathbf{d}_i, \tag{14}$$

The blocks of $\mathbf{A}$ are:

$$\mathbf{A}_{1,1} = I + (E_r^T C^{-1})(I + e^{-GC^{-1}h})(-E_r H_r L^{-1} H_r^T)(\frac{h^2}{4}) \tag{15}$$

$$\mathbf{A}_{1,2} = (E_r^T C^{-1})(I + e^{-GC^{-1}h})(\frac{h}{2}) \tag{16}$$

$$\mathbf{A}_{2,1} = \left((-E_r H_r L^{-1} H_r^T)(E_r^T C^{-1})(\frac{h^2}{4}) + I\right)(I + e^{-GC^{-1}h})(-E_r H_r L^{-1} H_r^T)(\frac{h}{2}) \tag{17}$$

$$\mathbf{A}_{2,2} = (-E_r H_r L^{-1} H_r^T)(E_r^T C^{-1})(I + e^{-GC^{-1}h})(\frac{h^2}{4}) + (e^{-GC^{-1}h}) \tag{18}$$

$$\mathbf{A}_{1,3} = \mathbf{A}_{2,3} = \mathbf{A}_{3,1} = \mathbf{A}_{3,2} = \mathbf{0};\ \mathbf{A}_{3,3} = I \tag{19}$$

Derivative Information: II

B reads

$$\mathbf{B} = \begin{bmatrix} (E_r^T C^{-1})\sqrt{\tau C(I - e^{-2GC^{-1}h})(\frac{h}{2})} \\ \left((-E_r H_r L^{-1} H_r^T)(E_r^T C^{-1})(\frac{h^2}{4}) + I\right) \sqrt{\tau C(I - e^{-2GC^{-1}h})} \\ \mathbf{0} \end{bmatrix} \tag{20}$$

the derivatives read

$$\begin{aligned} \nabla_{\mathbf{x}_{i-k}} \mathbf{x}_i &= \mathbf{A}^k \quad \forall k = 1, 2, \ldots, i-1 && (21) \\ \nabla_{\mathbf{d}_{i-k}} \mathbf{x}_i &= \mathbf{A}^k B \quad \forall k = 0, 1, \ldots, i-1 && (22) \\ \nabla_{\mathbf{d}_i} E_N &= (\nabla_{\phi_N} E_N)(\nabla_{\mathbf{d}_i} \phi_N) \\ &= (E_{N\phi})(\nabla_{\mathbf{d}_i} \phi_N) \quad \forall i = 1, 2, \ldots, N && (23) \end{aligned}$$

$E_{N\phi} = \frac{dE_N}{d\phi_N} \in \mathbb{R}^{11\times 7}$ is the Jacobian of the energy functional w.r.t flux differences.

Derivative Information: III

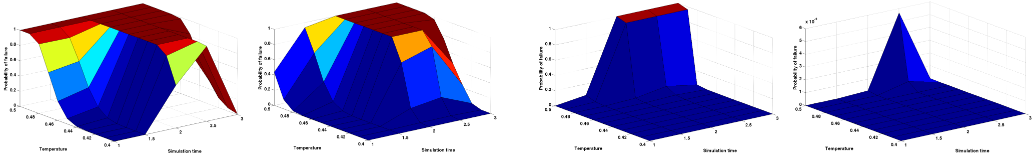
- Gradient of the cost function:

$$\nabla_{[\mathbf{d}_1^T, \mathbf{d}_2^T, \ldots, \mathbf{d}_N^T]^T} \mathcal{J} = 2 \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \\ \vdots \\ \mathbf{d}_N \end{bmatrix} . \quad (24)$$

- Gradient of the constraint(s):

$$\nabla_{\phi_N} \left(\mathbb{I}_N^T E_N \right) = E_{N\phi}^T \mathbb{I}_N ; \quad \nabla_{\phi_N} \left(E_N^T E_N \right) = 2 \begin{bmatrix} \left(\nabla_{\mathbf{d}_1} E_N \right)^T E_N \\ \left(\nabla_{\mathbf{d}_2} E_N \right)^T E_N \\ \vdots \\ \left(\nabla_{\mathbf{d}_N} E_N \right)^T E_N \end{bmatrix} \quad (25)$$

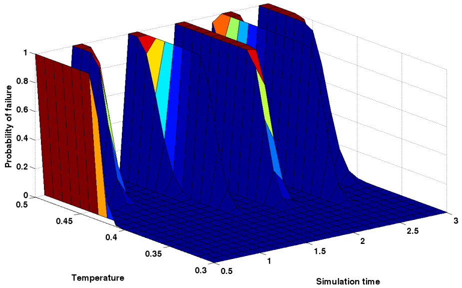
Results I: Probability of failure(s)



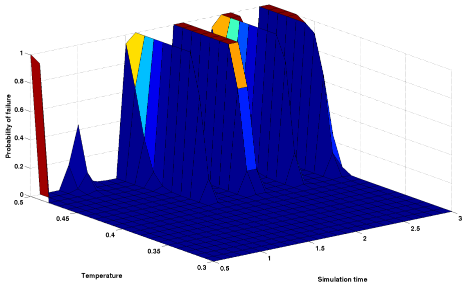
(a) One failure (b) Two failures (c) Three failures (d) Four failures

Figure: Probability of line failure(s). One, two, three, and four failures are plotted.

Results III: Probability of failure(s)



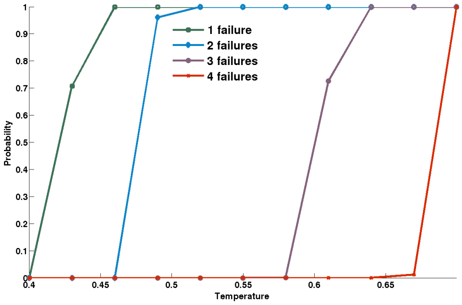
(a) One failure



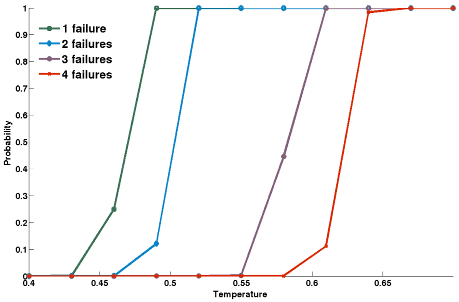
(b) Two failures

Figure: Probability of line failure(s) on higher resolution grid

Results IV: Probability of failure(s)



(a) Simulation time = 0.5



(b) Simulation time = 1

Figure: Relation between probability of branch failures and system's temperature.

Results V: Computational Time

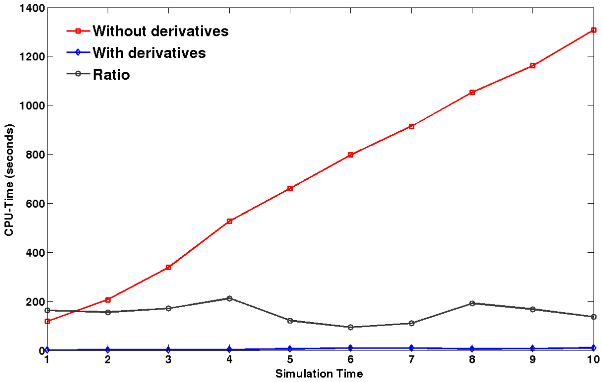


Figure: CPU-time of the optimization step for one failure case with and without derivative information.

Thanks

Questions?