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Stability of Multiobjective Predictive Control: An Utopia-Tracking Approach ⋆

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Abstract

We propose an utopia-tracking strategy to handle multiple conflicting objectives in model predictive control. The controller minimizes the distance of its vector of objectives to that of the compromise solution: the point along the steady-state Pareto front closest to the utopia point, where all the objectives are independently minimized. We establish conditions for asymptotic stability and propose numerical implementation variants. One of the key advantages of the approach is that it avoids the computation of Pareto fronts in real time environments. In addition, the approach can handle general objectives of different nature such as economic and regularization objectives.

Key words: stability, predictive control, multiobjective, Lyapunov, utopia, Pareto, economic.

1 Introduction

Conflicting objectives arise naturally in model predictive control (MPC). Trade-offs include tracking performance and robustness or economic performance and sustainability. Specific domains where reconciling objectives is critical include chemical and energy systems [13,16,17]. A key technical challenge in dealing with multiple objectives is that the Pareto front is computationally expensive to build, particularly in multiple dimensions. In addition, even when such a front is built, expert knowledge is still needed to obtain a preferred solution. Traditional approaches such as weighting and expert systems are limited because system conditions and priorities change under different operating modes. It is thus desired to allow the MPC controller to handle trade-offs automatically as conditions change.

Stability is another technical issue arising in multiobjective MPC. In [1], the MPC control action is chosen among the set of Pareto optimal solutions based on a time-varying, state-dependent decision criterion. In [14], the control action minimizes the maximum of a finite number of objectives. In [11], the MPC controller switches objectives depending on the value of the state vector under stabilizing constraints. This type of expert knowledge is also used in [9], where a lexicographic formulation and logic are used to prioritize the objectives. In these works, the multiple objectives are assumed to be Lyapunov functions, as in traditional MPC formulations.

In this work, we propose a new strategy to handle multiple objectives. We call this *utopia-tracking* MPC. We establish conditions for nominal asymptotic stability and propose numerical implementation schemes. The key idea is to minimize the distance of the cost function to that of the steady-state utopia point (unreachable point given by the intersection of the minima of the independent objectives). A key property of the controller is that it can exploit the system dynamics to leave the steady-state Pareto front and get closer to the utopia point compared with any solution along the steady-state Pareto front. Stability is ensured by using a terminal constraint to a reachable point along the Pareto front. Our proposed approach is novel because it can handle general cost functions (e.g., economic, regularization, tracking) that are required to satisfy only a Lipschitz continuity property. In addition, the strategy does not require the construction of the Pareto front, nor does it require the selection of weighting factors.

The paper is structured as follows. We start with basic definitions in Section 2. Definitions of steady-state multiobjective optimization are presented in Section 3. In Section 4 we analyze the stability of the utopia-tracking controller. In Section 5 we discuss computational issues. We present a numerical study in Section 6 and close in Section 7 with conclusions and directions for future work.

2 Preliminaries

We consider a discrete-time dynamic system of the form $x_{k+1} = f(x_k, u_k)$, where $x_k \in \Re^{n_x}$ are the states and

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$u_k \in \Re^{n_u}$ are the controls. The mapping $f : \Re^{n_x \times n_u} \rightarrow \Re^{n_x}$ is assumed to be Lipschitz in both arguments with constant $L_f \geq 0$ and is assumed to satisfy $f(x^s, u^s) = x^s$ at an equilibrium point (x^s, u^s). The state and controls are required to satisfy the constraints $x_k \in \mathcal{X},\ u_k \in \mathcal{U}$ $\forall\, k$. The sets $\mathcal{X} \subseteq \Re^{n_x}$ and $\mathcal{U} \subseteq \Re^{n_u}$ are assumed to be compact and to contain the equilibrium point. We define the vector $\mathbf{u}_N^T := [u_0^T, ..., u_{N-1}^T]^T \in \Re^{N \times n_u}$.

Definition 1 *(***Admissible Set***) Given $N+1$ time steps $k = 0, ..., N$, the admissible set is given by*

$$\mathcal{W}_N := \{(x, \mathbf{u}_N) \,|\, x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ x_N = x^s\}. \tag{1}$$

We note that the admissible set depends on the equilibrium point and the horizon length. The set of admissible states $\mathcal{Z}_N$ is given by

$$\mathcal{Z}_N =: \{x \,|\, \exists\, \mathbf{u}_N\ s.t.\ (x, \mathbf{u}_N) \in \mathcal{W}_N\}. \tag{2}$$

Definition 2 **($\mathcal{K}$-Function [8])** *A continuous function $\alpha : \Re \rightarrow \Re$ is called a $\mathcal{K}$ function if $\alpha(s) = 0$ for $s = 0$ and $\alpha(s) > 0$ for $s > 0$, and it is strictly increasing.*

The p-norm $\|\cdot\|_p$ with $p \geq 1$ has the form $\|s\|_p = (\sum_{i=1}^{n_s} |s_i|^p)^{\frac{1}{p}}$ for vector $s \in \Re^{n_s}$ with elements $s_i,\ i = 1, ..., n_s$. We have that $\|s\|_p = 0$ if $s = 0$ and $\|s\|_p > 0$ otherwise for all $p \geq 1$. In addition, the p-norm is Lipschitz continuous with constant equal to 1. Well-known norms are the $\mathcal{L}_1, \mathcal{L}_2$ and the $\mathcal{L}_\infty$ norms: $\|s\|_1 = \sum_{i=1}^{n_s} |s_i|$, $\|s\|_2 = \sqrt{\sum_{i=1}^{n_s} (s_i)^2}$, and $\|s\|_\infty = \max\{|s_1|, ..., |s_{n_s}|\}$.

Definition 3 **(Lyapunov Function [10])** *A continuous function $V(\cdot) : \Re^{n_x} \rightarrow \Re$ is called a Lyapunov function if there exist an invariant set $\mathcal{X}$ and $\mathcal{K}$ functions $\alpha_L(\cdot), \alpha_U(\cdot)$, and $\Delta\alpha(\cdot)$ such that, $\forall x \in \mathcal{X}$,*

$$\alpha_L(\|x\|_p) \leq V(x) \leq \alpha_U(\|x\|_p) \tag{3a}$$
$$\Delta V(x) \leq -\Delta\alpha(\|x\|_p). \tag{3b}$$

3 Steady-State Multiobjective Optimization

Consider the multiobjective steady-state problem

$$\min_{x,u}\ [\Phi_1(x, u), \Phi_2(x, u), ..., \Phi_M(x, u)] \tag{4a}$$
$$\text{s.t. } x = f(x, u),\ x \in \mathcal{X}, u \in \mathcal{U}, \tag{4b}$$

where the objective (cost) functions $\Phi_i : \Re^{n_x \times n_u} \rightarrow \Re,\ i \in \mathcal{M} := \{1, ..., M\}$ are assumed to be Lipschitz continuous in both arguments. We define the objective vector as

$$\Phi(\cdot, \cdot)^T := [\Phi_1(\cdot, \cdot), \Phi_2(\cdot, \cdot), ..., \Phi_M(\cdot, \cdot)]^T, \tag{5}$$

with Lipschitz constant L_Φ. No further assumptions are needed about the properties of these functions. This is an important advantage over existing multiobjective MPC implementations [14,1]. The cost functions can be conflicting, so one cannot be minimized without increasing the other. This situation gives rise to the concept of a Pareto solution.

Definition 4 *(***Steady-State Pareto Solution [2]***) A feasible point (x_p, u_p) for the multiobjective problem (4) is said to be Pareto optimal if and only if there exists no other feasible point (x, u) such that $\Phi_i(x, u) \leq \Phi_i(x_p, u_p),\ \forall i \in \mathcal{M}$, and $\Phi_i(x, u) < \Phi_i(x_p, u_p)$ for at least one index $i \in \mathcal{M}$.*

The family of Pareto solutions forms the so-called Pareto front, which represents a limiting curve of performance in the cost space. In this work, we will not follow the traditional approach of constructing the Pareto front and then choosing a suitable point along it [9]. The first reason is that this seems impractical in real-time environments. The second reason is that expert knowledge is needed to select the point and the selection criterion might need to be changed as the conditions of the system change (e.g., prices). We overcome some of these limitations by following an *utopia*-tracking approach.

Definition 5 *(***Steady-State Utopia Point** *[6]) The steady-state utopia point is a point given by the solution $(x_i^{L,s}, u_i^{L,s})$ with coordinates $\Phi_i(x_i^{L,s}, u_i^{L,s}),\ i \in \mathcal{M}$ in the cost space. The coordinates are given by the solution of problems $i \in \mathcal{M}$,*

$$\min_{x,u}\ \Phi_i(x, u)\ s.t.\ x = f(x, u),\ x \in \mathcal{X}, u \in \mathcal{U}. \tag{6}$$

The utopia cost vector will be denoted as $\Phi^{L,s}$. The utopia point is unattainable because the costs are conflicting; however, it can still be used as a reference point. For instance, one can compute the closest point along the Pareto front to the utopia point (also known as the *compromise solution.*)

Definition 6 *(***Steady-State Compromise Solution***.) The steady-state compromise solution is a point (x^s, u^s) with cost $\Phi(x^s, u^s)$ given by the solution of the minimum distance problem*

$$\min_{x,u}\ \|\Phi(x, u) - \Phi^{L,s}\|_p\ s.t.\ x = f(x, u),\ x \in \mathcal{X}, u \in \mathcal{U}. \tag{7}$$

The individual costs of the compromise solution are given by $\Phi_i(x^s, u^s),\ i \in \mathcal{M}$. We denote the above problem as the *steady-state utopia-tracking problem.* A schematic representation of the utopia-tracking approach is presented in Figure 1. Note that for the single-objective case, the compromise solution and the utopia point coincide so that $\Phi_1(x^s, u^s) = \Phi_1^{L,s}$. The choice of the compromise solution as equilibrium point is not strictly necessary. Other choices include the Kalai-Smorodinsky solution, the egalitarian solution, and the Nash solution [5]. The compromise solution is attractive, however, because it can be easily computed.

4 Multiobjective Predictive Control

We start by making an assumption about controllability [3,7].

Definition 7 *(***Weak Controllability***.) There exists a $\mathcal{K}$-function $\gamma(\cdot)$ such that for every $x \in \mathcal{X}$, there exists $(x, \mathbf{u}_N) \in \mathcal{W}_N$ such that*

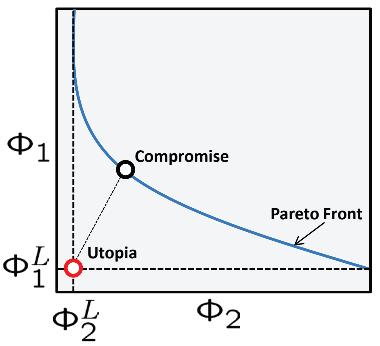

Fig. 1. Schematic representation of Pareto front, compromise solution, and utopia point.

$$\sum_{k=0}^{N-1} \|u_k - u^s\|_p \leq \gamma(\|x - x^s\|_p). \tag{8}$$

Lemma 1 *Consider the general MPC problem*

$$\min_{x_k,u_k} \sum_{k=0}^{N-1} \varphi(x_k, u_k, x^s, u^s) \tag{9a}$$
$$s.t.\ x_{k+1} = f(x_k, u_k),\ k \in \mathcal{N}^- \tag{9b}$$
$$x_0 = x_\ell \tag{9c}$$
$$x_T = x^s \tag{9d}$$
$$x_k \in \mathcal{X}, u_k \in \mathcal{U},\ k \in \mathcal{N}. \tag{9e}$$

Here, ℓ is the current time instant. The solution of this problem is given by the vector $\mathbf{u}_\ell^T := [u_0^T, ..., u_{N-1}^T]$ from where the control $u_\ell := u_0$ is injected into the system. We have $\mathcal{N} := \{0, ..., N\}$ and $\mathcal{N}^- := \mathcal{N} \setminus \{N\}$. The cost $\varphi : \Re^{n_x \times n_u} \to \Re$ is assumed to be non-negative and Lipschitz continuous such that there exists $L \geq 0$ satisfying

$$\varphi(x, u, x^s, u^s) \leq L(\|x - x^s\|_p + \|u - u^s\|_p). \tag{10}$$

If weak controllability and Lipschitz continuity of the system $f(\cdot, \cdot)$ hold, then there exists a $\mathcal{K}$-function $\alpha_U(\cdot)$ such that for all $(x, \mathbf{u}) \in \mathcal{W}_N$,

$$\sum_{k=0}^{N-1} \varphi(x_k, u_k, x^s, u^s) \leq \alpha_U(\|x - x^s\|_p). \tag{11}$$

Proof: The proof is an extension of the upper bounding strategy used in [3,7]. Applying the Lipschitz property assumed, the system is propagated forward in time and substituted in the objective. Under Lipschitz continuity of the system the result follows. □

We propose three strategies to deal with multiple objectives. In the first strategy (state-tracking MPC), the controller tracks directly the state of the compromise solution. In the second strategy (cost-tracking MPC), the controllers track the compromise solution in the cost space. The third strategy (utopia-tracking MPC) tracks the steady-state utopia point in the cost space using the compromise solution as terminal condition. We will see that tracking the costs is advantageous because it is possible to leave the Pareto front during the dynamic transition and get closer to the steady-state utopia point, thus maximizing economic performance.

4.1 State-Tracking MPC

We consider the state-tracking (ST) problem

$$\min_{x_k,u_k} \sum_{k=0}^{N-1} \|x_k - x^s\|_p + \|u_k - u^s\|_p \tag{12a}$$
$$\text{s.t. } (9b) - (9e). \tag{12b}$$

The control law resulting from the closed-loop solution of this problem is $u_\ell = h_{ST}(x_\ell)$, and the optimal cost is used as the value function $V_{ST}(x_\ell)$. Stability results for this controller are well known.

Theorem 2 *(***Stability of Tracking MPC***.) The minimum-distance steady-state point x^s under the control law $h_{ST}(x_\ell)$ given by the tracking MPC formulation* (12) *is an asymptotically stable equilibrium with region of attraction $\mathcal{Z}_N$.*

We note that the state-tracking MPC does not reach the steady-state point in an *economically* optimal manner. *We interpret economic performance as the distance to the utopia point since this is the limiting point.* The proposed multiobjective formulations of the following subsections can be used to avoid this limitation.

4.2 Cost-Tracking MPC

To address the limitations of tracking MPC in dealing with multiple objectives, we first propose the *cost-tracking* (CT) MPC controller:

$$\min_{x_k,u_k} \sum_{k=0}^{N-1} \|\Phi(x_k, u_k) - \Phi(x^s, u^s)\|_p \tag{13a}$$
$$\text{s.t. } (9b) - (9e). \tag{13b}$$

The closed-loop control law is given by $u_\ell = h_{CT}(x_\ell)$ with value function is $V_{CT}(x_\ell)$. The objective of the controller is to minimize the cost distance to the compromise steady-state solution. We will now prove that the value function can be used as a Lyapunov function to establish stability.

Assumption 1 *There exists a $\mathcal{K}$-function $\alpha_L(\cdot)$ such that $\|\Phi(x, u) - \Phi(x^s, u^s)\|_p \geq \alpha_L(\|x - x^s\|_p)$.*

Theorem 3 *Under weak controllability and Assumption 1, the steady-state x^s under the control law $h_{CT}(x_\ell)$ given by the multiobjective MPC formulation* (13) *is an asymptotically stable equilibrium point with region of attraction $\mathcal{Z}_N$.*

Proof: From Assumption 1, the balue function is bounded from below by a $\mathcal{K}$-function. Under weak controllability, Lemma 1 holds immediately with $L = L_\Phi$. Consequently, the value function is bounded from above by a $\mathcal{K}$-function. To show that the value function is nonincreasing, we establish

$$
\begin{aligned}
&V_{CT}(x_{\ell+1}) - V_{CT}(x_\ell) \\
&\quad = \sum_{k=\ell+1}^{\ell+N} \|\Phi(x_k,u_k) - \Phi(x^s,u^s))\|_p \\
&\qquad - \sum_{k=\ell}^{\ell+N-1} \|\Phi(x_k,u_k) - \Phi(x^s,u^s))\|_p \\
&\quad \le - \|\Phi(x_\ell,u_\ell) - \Phi(x^s,u^s)\|_p \\
&\quad \le -\alpha_L(\|x_\ell - x^s\|_p).
\end{aligned}
$$

The last inequality also follows from Assumption 1. The proof is complete. □

A key property of the cost-tracking approach is that the nature of the cost functions does not affect the upper bound property. Assumption 1 is the most restrictive assumption we have found that requires the stage cost to have a *unique* minimizer at (x_s, u_s). The lower bound condition can be guaranteed to hold locally under the satisfaction of the so-called strong second order condition. This condition requires that the optimal solution be well defined and locally unique. In other words, the cost is zero only at $x = x^s$ and strictly positive and nondecreasing for nonzero $\|x - x^s\|_p$. In [7], the authors propose to add a regularization term for the case in which the condition does not hold because of ill-conditioning of the cost function.

4.3 Utopia-Tracking MPC

We now propose the *utopia-tracking* (UT) formulation that minimizes directly the distance to the utopia point:

$$
\min_{x_k,u_k} \sum_{k=0}^{N-1} \left\|\Phi(x_k,u_k) - \Phi^{L,s}\right\|_p \tag{14a}
$$

$$
\text{s.t. } (9b) - (9e). \tag{14b}
$$

The closed-loop control law is given by $u_\ell = h_{UT}(x_\ell)$, and the value function is $V_{UT}(x_\ell)$. Since this controller minimizes the distance to the utopia directly, it can exploit the system dynamics to leave the steady-state Pareto front and get closer to the utopia point. The main technical difficulty in establishing stability of the UT controller is that the value function $V_{UT}(x)$ is nonzero at $x = x^s$ since the utopia point $\Phi^{L,s}$ is *unreachable*. Consequently, the value function does not qualify as a Lyapunov function. To establish stability for this formulation, we follow the approach proposed in [3]. We define the partial Lagrange function of the steady-state utopia-tracking problem (7):

$$
\mathcal{L}(x,u,\lambda) := \|\Phi(x,u) - \Phi^{L,s}\|_p + (x - f(x,u))^T\lambda, \tag{15}
$$

where $\lambda \in \Re^{n_x}$ is a Lagrange multiplier. At x^s, u^s, λ^s we have that the partial Lagrange function reaches a minimum given by $\mathcal{L}(x^s,u^s,\lambda^s) = \|\Phi(x^s,u^s) - \Phi^{L,s}\|$ since $0 = x^s - f(x^s,u^s)$. With this, an artificial origin is introduced if $(x,u) = (x^s,u^s)$. We need the following assumption.

Assumption 2 *(***Strong Duality***.) There exists a multiplier λ^s such that the pair u^s, x^s uniquely solves*

$$
\min_{x,u} \ \mathcal{L}(x,u,\lambda^s), \ \text{s.t. } (x,u) \in \mathcal{X} \times \mathcal{U}. \tag{16}
$$

From strong duality we have that $\mathcal{L}(x,u,\lambda^s) - \mathcal{L}(x^s,u^s,\lambda^s) \ge 0, \ \forall (x,u) \in \mathcal{X} \times \mathcal{U}$. We also have that there exists a $\mathcal{K}$-function $\alpha_L(\cdot)$ such that

$$
\mathcal{L}(x,u,\lambda^s) - \mathcal{L}(x^s,u^s,\lambda^s) \ge \alpha_L(\|x - x^s\|_p). \tag{17}
$$

We can now define the utopia-tracking MPC problem (14) in terms of the partial Lagrange function:

$$
\min_{u_k} \sum_{k=0}^{N-1} \left(\mathcal{L}(x_k,u_k,\lambda^s) - \mathcal{L}(x^s,u^s,\lambda^s)\right) \tag{18a}
$$

$$
\text{s.t. } (9b) - (9e). \tag{18b}
$$

As shown in [3] (see Lemma 2), formulations (18) and (14) are equivalent. Consequently, we can now use the optimal objective value of (18) as the value function $V_{UT}(x_\ell)$.

Theorem 4 *Under weak controllability and strong duality, the steady-state x^s under the control law $h_{UT}(x_\ell)$ given by utopia-tracking MPC formulation* (18) *is an asymptotically stable equilibrium point with region of attraction $\mathcal{Z}_N$.*

Proof: From strong duality, the value function is bounded from below by a $\mathcal{K}$-function. We now prove that it is bounded above. From strong duality we have that $\sum_{k=0}^{N-1} \mathcal{L}(x_k,u_k,\lambda^s) - \mathcal{L}(x^s,u^s,\lambda^s) = \sum_{k=0}^{N-1} |\mathcal{L}(x,u,\lambda^s) - \mathcal{L}(x^s,u^s,\lambda^s)|$. We also have

$$
\begin{aligned}
&|\mathcal{L}(x,u,\lambda^s) - \mathcal{L}(x^s,u^s,\lambda^s)| \\
&= \big|\|\Phi(x,u) - \Phi^{L,s}\|_p + (x - f(x,u))^T\lambda \\
&\quad - \left(\|\Phi(x^s,u^s) - \Phi^{L,s}\|_p + (x^s - f(x^s,u^s))^T\lambda^s\right)\big| \\
&\le \big|\|\Phi(x,u) - \Phi^{L,s}\|_p - \|\Phi(x^s,u^s) - \Phi^{L,s}\|_p\big| \\
&\quad + |(x - f(x,u))^T\lambda^s - (x^s - f(x^s,u^s))^T\lambda^s| \\
&\le (L_\Phi + (L_f + 1)\|\lambda^s\|_q)\left(\|x - x^s\|_p + \|u - u^s\|_p\right).
\end{aligned}
$$

Where the last inequality follows from Hölder's inequality, $\|\cdot\|_q$ is the q-norm, and $1/p + 1/q = 1$. Consequently, Lemma 1 holds with $L = L_\Phi + (L_f + 1)\|\lambda^s\|_q$. To show that value function is nonincreasing, we establish the following:

$$
\begin{aligned}
&V_{UT}(x_{\ell+1}) - V_{UT}(x_\ell) \\
&\quad = \sum_{k=\ell+1}^{\ell+N} \left(\mathcal{L}(x_k,u_k,\lambda^s) - \mathcal{L}(x^s,u^s,\lambda^s)\right) \\
&\qquad - \sum_{k=\ell}^{\ell+N-1} \left(\mathcal{L}(x_k,u_k,\lambda^s) - \mathcal{L}(x^s,u^s,\lambda^s)\right) \\
&\quad \le -\left(\mathcal{L}(x_\ell,u_\ell,\lambda^s) - \mathcal{L}(x^s,u^s,\lambda^s)\right) \\
&\quad \le -\alpha_L(\|x - x^s\|_p).
\end{aligned}
$$

The last inequality follows from strong duality. The proof is complete. □

The most restrictive assumption that we have found is strong duality, which is difficult to check in practice. This property guarantees that the Lagrange function has a unique minimizer at (x_s, u_s). In [7] the authors propose to add a regularization term to the cost function to promote local uniqueness.

5 Computational Considerations

We highlight that the coordinates of the utopia point can be computed off-line and in parallel. Consequently, this computation does not involve additional on-line costs for the controller. We also note that the UT controller does not need to compute the entire Pareto front, which is a significant computational advantage over existing multiobjective approaches.

The choice of the norm has implications on computational performance. For instance, the $\mathcal{L}_2$ norm is smooth, whereas $\mathcal{L}_1$ and $\mathcal{L}_\infty$ are not. Another issue is that the cost functions can have drastically different values. The solution of the individual problems (6) yield upper bounds $\Phi_i^{U,s},\ i \in \mathcal{M}$, given by the maximum of the costs not minimized. Consequently, we can use these to scale the controller cost without affecting its properties. The scaled $\mathcal{L}_2$ problem has the form

$$\min_{x_k,u_k} \sum_{k=0}^{N-1} \left\| \frac{\Phi(x_k,u_k)-\Phi^{L,s}}{\Phi^{U,s}-\Phi^{L,s}} \right\|_2 \tag{19a}$$
$$\text{s.t. (9b)} - \text{(9e)}. \tag{19b}$$

The square root in the objective function can introduce numerical ill-conditioning because the first derivative diverges as the argument approaches zero. To deal with this problem, we consider the following formulation:

$$\min_{x_k,u_k} \sum_{k=0}^{N-1} z_k \tag{20a}$$
$$\text{s.t. (9b)} - \text{(9e)} \tag{20b}$$
$$z_k^2 = \sum_{i\in\mathcal{M}} \left(\frac{\Phi_i(x_k,u_k)-\Phi_i^{L,s}}{\Phi_i^{U,s}-\Phi_i^{L,s}} \right)^2 \tag{20c}$$
$$z_k \geq 0,\ k \in \mathcal{N}^-, \tag{20d}$$

which is better-conditioned. An alternative is to minimize the squared norm. To reformulate the $\mathcal{L}_1$ variant, we introduce variables $y_{k,i}^+, y_{k,i}^- \geq 0,\ i \in \mathcal{M}$, and define the absolute value $y_{k,i}^+ - y_{k,i}^- = \frac{\Phi_i(x_k,u_k)-\Phi_i^{L,s}}{\Phi_i^{U,s}-\Phi_i^{L,s}}$. After scaling we have

$$\min_{x_k,u_k} \sum_{k=0}^{N-1} \sum_{i\in\mathcal{M}} (y_{k,i}^+ + y_{k,i}^-) \tag{21a}$$
$$\text{s.t. (9b)} - \text{(9e)} \tag{21b}$$
$$y_{k,i}^+ - y_{k,i}^- = \frac{\Phi_i(x_k,u_k)-\Phi_i^{L,s}}{\Phi_i^{U,s}-\Phi_i^{L,s}} \tag{21c}$$
$$y_{k,i}^+, y_{k,i}^- \geq 0,\ k \in \mathcal{N}^-,\ i \in \mathcal{M}, \tag{21d}$$

where $\sum_{k=0}^{N-1} \left\| \frac{\Phi(x_k,u_k)-\Phi^{L,s}}{\Phi^{U,s}-\Phi^{L,s}} \right\|_1 = \sum_{k=0}^{N-1} \sum_{i\in\mathcal{M}} (y_{k,i}^+ + y_{k,i}^-)$. We can reformulate the $\mathcal{L}_\infty$ variant as

$$\min_{x_k,u_k} \sum_{k=0}^{N-1} \eta_k + \rho \sum_{k=0}^{N-1} \sum_{i\in\mathcal{M}} (y_{k,i}^+ + y_{k,i}^-) \tag{22a}$$
$$\text{s.t. (9b)} - \text{(9e)} \tag{22b}$$
$$y_{k,i}^+ - y_{k,i}^- = \frac{\Phi_i(x_k,u_k)-\Phi_i^{L,s}}{\Phi_i^{U,s}-\Phi_i^{L,s}} \tag{22c}$$
$$y_{k,i}^+ + y_{k,i}^- \leq \eta_k,\ k \in \mathcal{N}^-,\ i \in \mathcal{M}, \tag{22d}$$

where $\sum_{k=0}^{N-1} \left\| \frac{\Phi(x_k,u_k)-\Phi^{L,s}}{\Phi^{U,s}-\Phi^{L,s}} \right\|_\infty = \sum_{k=0}^{N-1} \eta_k$ and $\rho > 0$ is a penalty parameter.

6 Numerical Case Study

We simulated the performance of the three proposed controllers using a free-radical polymerization reactor [12]. The dynamic model has the form

$$\dot{C}_m(t) = -(k_p + k_{fm}) C_m(t) P_0(t) + \frac{F}{V}(C_{m,in} - C_m(t)) \tag{23a}$$
$$\dot{C}_i(t) = -k_i C_i(t) + \frac{F_i(t)}{V} C_{i,in} - \frac{F}{V} C_i(t) \tag{23b}$$
$$\dot{D}_0(t) = (0.5 k_{tc} + k_{td}) P_0(t)^2 + k_{fm} C_m(t) P_0(t) - \frac{F}{V} D_0(t) \tag{23c}$$
$$\dot{D}_1(t) = M_m (k_p + k_{fm}) C_m(t) P_0(t) - \frac{F}{V} D_1(t). \tag{23d}$$

Here, $C_m(t)$ is the monomer concentration, $C_i(t)$ is the initiator concentration, $D_0(t)$ is the zeroth moment, and $D_1(t)$ is the first moment. These are the states. The control variable is the initiator flowrate $F_i(t)$, and $P_0(t) = \sqrt{2\eta_i k_i C_i(t)/(k_{td}+k_{tc})}$. The parameter values can be found in [12]. We assume that it is desired to maximize conversion $\Phi_1(t) = X(t) = (C_{m,in} - C_m(t))/C_{m,in}$ while simultaneously maximizing the profit $\Phi_2(t) = 2500 + 3500 X(t)^{0.6} + 9 \times 10^{-4} M_w(t)^{0.65} - 3000 F_i(t)^{0.5}$ where $M_w(t) = D_1(t)/D_0(t)$ is the polymer molecular weight. We converted the model into discrete time form using Euler discretization and solved the resulting problems using IPOPT [15]. All the controller implementations are available at `http://www.mcs.anl.gov/~vzavala`. Since the $\mathcal{L}_\infty$ formulation proved to be computationally more robust, it was used in all the experiments. We verified that the solutions were locally unique by monitoring the second-order conditions with IPOPT.

We tested the ST, CT, and UT controllers under two initial points at the extremes of the Pareto front. The two-dimensional cost trajectories for the CT and UT controllers are presented in Figure 2. In the transition from the lower end of the Pareto front, both controllers leave the Pareto front because they can exploit the system dynamics to get to the compromise solution. The UT controller is able to get much closer to the utopia point during the transition, and then converges to the compromise point. In other words, UT has much better performance than the CT counterpart. In the transition from

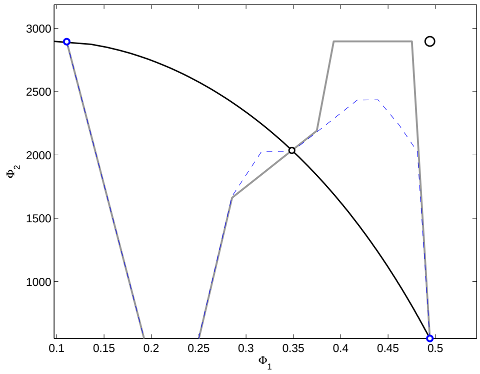

Fig. 2. Phase plot of utopia-tracking (gray line) and cost–tracking controllers (dotted line). The utopia point is the large dot.

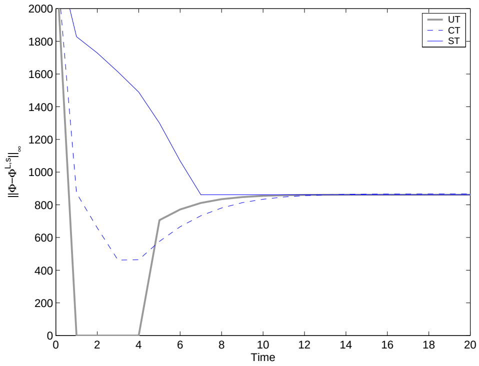

Fig. 3. Time evolution of the distance of the controllers trajectories to the utopia point.

the second initial point, the difference in performance is less pronounced. The reason is that the controllers are physically unable to visit the region surrounding the upper end of the Pareto front. This situation suggests that performance improvements depend on the initial state of the system and on the shape of the Pareto front. We also found that ST is stable but its performance is not competitive. In Figure 3 we present the time evolution of the distance for the three controllers to the utopia point ($\|\Phi - \Phi^{L,s}\|_\infty$). The performance of UT is superior, while the poorest performance is that of ST. The distance for UT to the utopia point is negligible during the dynamic transition and then it settles at it reaches the utopia point. The accumulated distances over time for UT, CT, and ST are 1.49×10^4, 1.69×10^4, and 2.25×10^4, respectively. The performance improvement of UT over ST is 33%.

7 Conclusions and Future Work

We proposed an utopia-tracking strategy to handle multiple conflicting objectives in model predictive control, established conditions for nominal asymptotic stability, and proposed numerical variants. The approach can handle general objectives that are required to satisfy only a Lipschitz continuity property. In addition, it does not require the construction of the Pareto front and avoids the need of adjusting weights. Directions of future work include stability under different terminal conditions and in the face of uncertainty. In addition, we plan to explore strategies to enlarge the region of attraction. Recent work presented in [4] can be extended to this case.

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