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MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS AND APPLICATIONS TO CONTROL

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Abstract: We discuss recent advances in mathematical programs with equilibrium constraints (MPECs). We describe the challenges posed by these problems and the current algorithmic solutions. We emphasize in particular the use of the elastic mode approach. We also present initial investigations in applications of MPECs to control problems.

Keywords: Complementarity Constraints, Equilibrium Constraints, Elastic Mode, Nonlinear Programming.

1. MPEC: APPLICATIONS AND CURRENT APPROACHES

The paper presents a summarizes of recent work on mathematical programs with equilibrium constraints (MPECs). When the equilibrium constraints occur from optimality conditions over polyhedral cones, the problem becomes a mathematical program with complementarity constraints (MPCC). In this work we will use the term MPEC to describe MPCC as well, since this term is far more popular in the current complementarity literature.

1.1 Formulation and difficulty

At the core of MPECs is the complementarity constraint. We say that variables a, b are complementary, or that they satisfy a complementarity constraint, if they satisfy the relationship $a \geq 0, b \geq 0, ab = 0$. We denote this relationship between the variables by $a \perp b$. Note that we can write the same relationship between vector components, provided that we use the scalar product to define the relevant product, that is, $ab = a^T b = 0$.

An MPEC is a nonlinear program that contains complementarity constraints between suitable variables, in addition to other type of constraints:

$$\begin{array}{rl} \min\limits_{x} & f(x) \\ \text{(MPEC) subject to} & g(x) \geq 0,\ h(x) = 0, \\ & 0 \leq G^T x \perp H^T x \geq 0. \end{array}$$

Here $f : \mathrm{IR}^n \to \mathrm{IR}$, $g : \mathrm{IR}^n \to \mathrm{IR}^p$, and $h : \mathrm{IR}^n \to \mathrm{IR}^q$ are all twice continuously differentiable functions (at least in a neighborhood of all points generated by our methods), and G and H are $n \times m$ column submatrices of the $n \times n$ identity matrix (with no columns in common). Hence, the constraints $G^T x \geq 0$ and $H^T x \geq 0$ represent *nonnegativity bound constraints* on certain components of x, and the notation $G^T x \perp H^T x$ signifies that $(G^T x)^T (H^T x) = 0$.

The difficulty of the problem is immediately apparent once we inquire whether the problem (MPEC) satisfies a constraint qualification, which is the key ingredient for obtaining the existence of

Lagrange multipliers in the classical nonlinear programming theory.

It turns out that the problem does not satisfy the Mangasarian Fromovitz constraint qualification (MFCQ), one of the most general constraint qualifications for nonlinear nonconvex programming (Scheel and Sholtes, 2001; Luo et al., 1996). That constraint qualification has the geometrical meaning that the linearization of the feasible set has an interior. For example, if $a = 1, b = 0$ for the case in which we have only one complementarity constraint, the linearization of the feasible set consists of all vectors $\Delta a, \Delta b$ that satisfy

$$1 + \Delta a \geq 0, \Delta b \geq 0, \Delta b = 0.$$

In turn, this condition results in the fact no feasible $\Delta a, \Delta b$ satisfies $\Delta a > 0, \Delta b > 0$,
and thus the linearization of the feasible set has no interior. Therefore, much of the theory of nonlinear programming does not apply for this class of problems, a situation that explains the keen current interest in MPECs.

In addition, connected to this problem is the difficulty that the linearization of the feasible set may not be feasible arbitrarily close to a solution of (MPEC) (Fletcher and Leyffer, 2002). This is an issue for most of the modern nonlinear programming solvers.

1.2 Origin of complementarity constraints in applications

Most applications of MPECs arise from situations in which one needs to formulate bilevel optimization problems, where an upper-level optimization problem includes as variables the primal and dual variables of a lower-level optimization problem with inequality constraints.

While the field is continuously evolving, there are currently three major sources of applications for MPECs: economics, transportation engineering, and mechanical and structural engineering.

In economics*,* such problems appear through various extensions of Stackelberg, or leader-follower, games (Lu et al., 1996; Outrata et al., 1998). In these games, the leader maximizes its utility function subject to resource constraints and subject to the constraint that each follower also chooses an optimal policy subject to its resource constraints. The follower problem is the lower-level optimization problem, whereas the leader problem is the upper-level optimization problem. Specific applications of this type, in addition to proper leader-follower games, are the determination of generalized Nash equilibrium points.

In transportation engineering, the lower level-problems are optimization formulations of the traffic assignment problem, whereas the upper-level problem is a design or optimal control problem. Specific instances of such MPECs are the continuous network design problem (Marcotte, 1986) and the toll optimization on a multicommodity network flow problem (Marcotte, 1986).

In mechanical and structural engineering*,* the lower-level problem may be a variational inequality that describes the physics of contact and friction, whereas the upper-level problem is again a design or optimal control problem (Outrata et al., 1998). In this category we have the packaging problems of membranes with rigid and flexible obstacles, the design of masonry structures, the design of elastic perfectly plastic structures, and the optimal control of robotic agents on a factory floor.

1.3 Previous methodology

The importance of the problem has led to two major approaches tuned to MPECs.

The bundle trust-region method approach (Outrata et al., 1998) applies for problems that originate in implicit MPECs, that is, problems in which the variables of the lower-level problem can be explicitly solved for and replaced in the upper-level problem, which now has a nonsmooth objective function owing to the change in the active constraints set as the parameters of the upper-level optimization problem change.

In that case the problem is approached as a nonsmooth optimization problem, and one computes a bundle of generalized gradients that are used to guide the optimization process. The key step is to compute an element of the generalized gradient, which results in an effort that increases exponentially with the number of the degenerate pairs. We say that a pair of complementary variables (a,b) is degenerate, or that the respective variables are degenerate, if they are both equal to zero.

In the disjunctive programming approach (Luo et al. 1996), (MPEC) is replaced by a subset of the set of nonlinear programs obtained by setting either $G_i^T x$ or $H_i^T x$ to 0. If we solve all these nonlinear programs, we are guaranteed to have found the solution of (MPEC). Since there are 2^m such nonlinear programs, one must choose the subset of such nonlinear programs wisely, much like in any branch-and-bound algorithm. Again, the challenge is the possible exponential explosion of the complexity of the computation. In addition, if the resulting nonlinear programs are not convex, then this explosion in complexity will not necessarily result in finding the global solution. Hence, in some sense, we have to manage an exponential explosion just to deal with a local difficulty.

2. NONLINEAR PROGRAMMING APPROACHES TO MPEC

Recently researchers have shown that nonlinear programming approaches with special safeguards avoid the difficulties of MPECs outlined above (Fletcher and Leyffer, 2002; Anitescu 2005a). Here we concentrate on the elastic mode safeguard (Anitescu 2005a; Anitescu2005b; Anitescu et al,. 2005).

2.1 Constraint qualifications

We start by defining the following active sets at a feasible point x^* of MPEC (1):

$$I_g \overset{\text{def}}{=} \{i \in \{1,2,\ldots,p\} \mid g_i(x^*) = 0\},$$

$$I_G \overset{\text{def}}{=} \{i \in \{1,2,\ldots,m\} \mid G_i^T x^* = 0\},$$

$$I_H \overset{\text{def}}{=} \{i \in \{1,2,\ldots,m\} \mid H_i^T x^* = 0\},$$

where G_i and H_i denote the i th column of G and H, respectively (in each case, a column from the identity matrix). Because x^* is feasible for (1), we have $I_G \cup I_H = \{1,2,\ldots,m\}$.

Using the active sets, we define our first notion of first-order stationarity for (1) as follows. A feasible point x^* of (MPEC) is *strongly stationary* if $d = 0$ solves the following linear program

$$\begin{aligned} \min_d \quad & \nabla f(x^*)^T d \\ \text{subject to} \quad & g(x^*) + \nabla g(x^*)^T d \geq 0, \\ & h(x^*) + \nabla h(x^*)^T d = 0, \\ & G_i^T d = 0, i \in I_G \setminus I_H, \\ & H_i^T d = 0, i \in I_H \setminus I_G, \\ & G_i^T d \geq 0,\ H_i^T d \geq 0, i \in I_G \cap I_H \end{aligned}$$

Let us introduce Lagrange multipliers and define the MPEC Lagrangian as in (Scheel and Sholtes, 2000):

$$L(x,\lambda,\mu,\tau,\nu) = f(x) - \lambda^T g(x) - \mu^T h(x) - \tau^T G^T x - \nu^T H^T x.$$

By combining the (necessary and sufficient) conditions for $d = 0$ to solve the program with the feasibility conditions for x^*, we see that x^* is strongly stationary if and only if x^* satisfies, together with some multipliers $(\lambda^*, \mu^*, \tau^*, \nu^*)$, the following conditions:

$$\nabla_x L(x^*, \lambda^*, \mu^*, \tau^*, \nu^*) = 0,$$

$$0 \leq \lambda^* \perp g(x^*) \geq 0,$$

$$h(x^*) = 0,$$

$$\tau^* \perp G^T x^* \geq 0,$$

$$\nu^* \perp H^T x^* \geq 0,$$

$$\tau_i^* \geq 0,\ i \in I_G \cap I_H,$$

$$\nu_i^* \geq 0,\ i \in I_G \cap I_H.$$

The multipliers $(\lambda^*, \mu^*, \tau^*, \nu^*)$ at a feasible point x^* are unique if MPEC-linear independence constraint qualification (MPEC-LICQ) holds, that is, if the following set of vectors is linearly independent:

$$K \overset{\text{def}}{=} \{\nabla g_i(x^*)\}_{i \in I_g} \cup \{\nabla h_i(x^*)\}_{i=1,2,\ldots,q} \cup \{G_i\}_{i \in I_G} \cup \{H_i\}_{i \in I_H}.$$

The following result, dating back to (Luo et al., 1996) but stated here in the form of (Scheel and Scholtes, 2000), shows that, under MPEC-LICQ, strong stationarity is a set of (first-order) necessary optimality conditions for the MPEC. Suppose that x^* is a local minimizer of (MPEC). If the MPEC-LICQ holds at x^*, then x^* is strongly stationary, (3) and the multiplier vector $(\lambda^*, \mu^*, \tau^*, \nu^*)$ that satisfies the stationarity conditions is unique. (4)

Our analysis also uses two weaker notions of first-order stationarity for (1) that have been studied in (5) previous works; see, for example, (Outrata et al., 1998; Scheel and Scholtes, 2000).

- A point x^* is *C-stationary* if there exist multipliers $(\lambda^*, \mu^*, \tau^*, \nu^*)$ satisfying the stationarity conditions except that the multiplier sign conditions are replaced by $\tau_i^* \nu_i^* \geq 0$, for each $i \in I_G \cap I_H$.
- A point x^* is *M-stationary* if it is C-stationary and if either $\tau_i^* \geq 0$ or $\nu_i^* \geq 0$ for each $i \in I_G \cap I_H$.

Notice that M-stationarity allows such situations as (6) $\tau_i^* < 0$ and $\mu_i^* = 0$ for some $i \in I_G \cap I_H$ but does not allow the situation $\tau_i^* < 0$ and $\mu_i^* < 0$, which is allowed by C-stationarity. In particular, strongly stationary $\Rightarrow$ M-stationary $\Rightarrow$ C-stationary.

(7)

2.2 The elastic mode

In this paper, we study a nonlinear programming formulation of (MPEC) that uses an explicit penalization of the complementarity constraint, also known as the "elastic mode." For a given penalty parameter $c \geq 0$, that formulation can be written as follows (Anitescu et al., 2005):

$$\text{PF}(c):\ \min_{x,\zeta} \quad f(x) + c\zeta + c(G^T x)^T (H^T x) \tag{8}$$

$$\text{subject to} \quad g(x) \geq -\zeta e_p, \zeta e_q \geq h(x) \geq -\zeta e_q, \tag{9}$$

$$G^T x \geq 0, H^T x \geq 0, 0 \leq \zeta. \tag{10}$$

We say that (x,ζ) is an ε*-first-order point* of PF(c) ($\varepsilon \geq 0$) if there exist multipliers $(\lambda,\mu^-,\mu^+,\tau,\nu,\pi^-,\pi^+)$ satisfying

$$
\begin{array}{rcl}
\nabla_x L_c(x,\zeta,\lambda,\mu^-,\mu^+,\tau,\nu)_\infty & \leq & \varepsilon, \\
|\, c - e_p^T\lambda - e_q^T\mu^- - e_q^T\mu^+ - \pi^- + \pi^+ \,| & \leq & \varepsilon,
\end{array}
$$

$$
\begin{array}{ccccc}
0 \leq & (\pi^-,\pi^+), & (\zeta,\overline{\zeta}-\zeta) & \geq & 0, \\
0 \leq & \lambda, & g(x)+\zeta e_p & \geq & -\varepsilon e_p, \\
0 \leq & \mu^+, & \zeta e_q - h(x) & \geq & -\varepsilon e_q, \\
0 \leq & \mu^-, & \zeta e_q + h(x) & \geq & -\varepsilon e_q, \\
0 \leq & \tau, & G^T x & \geq & 0, \\
0 \leq & \nu, & H^T x & \geq & 0,
\end{array}
$$

$$
\begin{array}{rl}
\zeta\pi^- + (\overline{\zeta}-\zeta)\pi^+ & \leq \varepsilon \\
|\,(g(x)+\zeta e_p)^T\lambda\,| & \leq \varepsilon \\
|\,(\zeta e_q - h(x))^T\mu^+\,| & \leq \varepsilon \\
|\,(\zeta e_q + h(x))^T\mu^-\,| & \leq \varepsilon \\
\tau^T G^T x & \leq \varepsilon \\
\nu^T H^T x & \leq \varepsilon
\end{array}
$$

We say that (x,ζ) is an (ε,δ) *-second-order point of* PF(c) ($\varepsilon,\delta \geq 0$) if there exist multipliers $(\lambda,\mu^-,\mu^+,\tau,\nu,\pi^-,\pi^+)$ that satisfy the above first-order conditions (so (x,ζ) is an ε-first-order point of PF(c)) and

$\tilde{u}^T\nabla^2_{(x,\zeta)(x,\zeta)} L_c(x,\zeta,\lambda,\mu^-,\mu^+,\tau,\nu)\tilde{u} \geq -C\|\tilde{u}\|^2$,

for all $\tilde{u} \in \mathrm{IR}^{n+1}$ that are simultaneously in the null space of the gradients of all active bound constraints ($G^T x \geq 0$, $H^T x \geq 0$, $0 \leq \zeta \leq \overline{\zeta}$) at (x,ζ) and in the null space of the gradients of δ-active nonbound constraints ($g(x) \geq -\zeta e_p$, $\zeta e_q \geq h(x) \geq -\zeta e_q$) at (x,ζ). Here $C \geq 0$ is an arbitrary constant independent of (x,ζ).

2.3 The elastic mode: Local convergence and rate of convergence

It is not surprising that the solution of PF(c) approaches the solution of MPEC as $c \to \infty$. But in a recent paper, we proved that, under very lax assumptions, the problem PF(c) has the same x solution as (MPEC) for sufficiently large c. We formally state this result under weaker assumptions for simplicity.

Theorem (Anitescu 2005a). Assume that at a solution x^* of (MPEC) we have that

• the Lagrange multiplier set of (MPEC) is not empty (this condition is implied by MPEC-LICQ).

• the second-order condition is satisfied at $(x^*,0)$ for $\varepsilon = \delta = 0$; and

• the data of (MPEC) are twice continuously differentiable.

Then, for sufficiently large but finite value of the penalty parameter c, we have that

1. $(x^*,0)$ is a local minimum of PF(c) at which both MFCQ and the corresponding second-order conditions hold.
2. $(x^*,0)$ is an isolated stationary point of PF(c).
3. The extension of the steepest descent to constrained optimization, if initialized sufficiently close to $(x^*,0)$, converges R-linearly to it.

The importance of this results resides in the fact that PF(c) satisfies MFCQ and can thus be solved with classical nonlinear programming algorithms. In addition, a superlinear convergence result can also be stated under some additional weak assumptions about the multipliers of (MPEC) – partial lower-level strict complementarity.

But this result does not address three important issues. First, the penalty parameter c must be adjusted, because its "sufficiently large" value is not a priori known.
Second, when the penalty parameter needs to be adjusted, we cannot afford to solve the problem PF(c) to optimality, because for any c, we need an infinite number of iterations, which is not practical.
Third, our approach does not address the issue of global convergence, that is, whether the limit point of the iterative sequence is a solution or at least a stationary point of the problem. We address these issues in the following sections.

2.4 An implementable adaptive elastic-mode algorithm: Global convergence results

We consider the following algorithm.

Algorithm Choose $c_0 > 0$, $\varepsilon_0 > 0$, $M_\varepsilon > M_c > 1$, and positive sequences $\{\delta_k\} \to 0$, $\{\omega_k\} \to 0$;

for k=0,1,2,…find an (ε_k,δ_k)-second-order point (x^k,ζ_k) of PF(c_k) with Lagrange multipliers. $(\lambda^k,\mu^{-k},\mu^{+k},\tau^k,\nu^k,\pi^{-k},\pi^{+k})$;

if $\zeta_k + (G^T x^k)^T(H^T x^k) \geq \omega_k$,

set $c_{k+1} = M_c c_k$;

else

set $c_{k+1} = c_k$;

end(if)

choose $\varepsilon_{k+1} \in (0,\varepsilon_k/M_\varepsilon]$.

end(for)

Since the algorithm does not solve the subproblems exactly, it will spend only a finite number of steps for a given penalty parameter c_k and is therefore implementable.

When stating our results we will use the following assumption.

Assumption

(a) $\{f(x^k)\}$ is bounded from below.

(b

$\{f(x^k)+c_k\zeta_k+c_k(G^Tx^k)^T(H^Tx^k)\}$

is bounded from above.

Note that the second assumption is satisfied between updates of the penalty parameter.

We then have the following result.

Theorem (Anitescu et al., 2005). Consider the sequences generated by our Algorithm. Suppose that they satisfy Assumption 1(a),(b). Then every accumulation point x^* of $\{x^k\}$ is feasible for (1). If x^* satisfies MPEC-LICQ, then the following results hold.

- x^* is M-stationary for (1).
- Suppose that $\{c_k\}$ is bounded. Then x^* is strongly stationary for (1).

Suppose that x^* satisfies $\tau^k \perp G^Tx^k$ and $\nu^k \perp H^Tx^k$ for all k. Let $S \subset \{0,1,\ldots\}$ be such that $\{x^k\}_{k\in S} \to x^*$. Then, there is a threshold c^* such that, for all $k \in S$ sufficiently large with $c_k \geq c^*$, we have $(G^Tx^k)^T(H^Tx^k)=0$.

Therefore, under the conditions outlined above, not only are we guaranteed to get to a strongly stationary point, but we also are guaranteed that the complementarity constraint will be satisfied early, even if the subproblems are solved only inexactly. This fact has been amply documented by our simulations, where we have used FilterSQP (Fletcher and Leyffer, 2002) to solve the relaxed subproblems. FilterSQP is designed to compute the approximate second-order stationary points needed in the definition of the algorithm. In addition, if the constraints $G^Tx \geq 0, H^Tx \geq 0$ are enforced as bound constraints, then it is reasonable to assume that the complementarity condition involving the Lagrange multipliers will in effect hold at the iterates.

While the Assumption (a) is acceptable for almost all nonlinear programming algorithms, however, Assumption (b) is far more problematic because it is an assumption about an outcome, as opposed to a class of problems. We next present a broad class of problems for which this assumption is not needed from the outset and for which a global convergence result can still be proved.

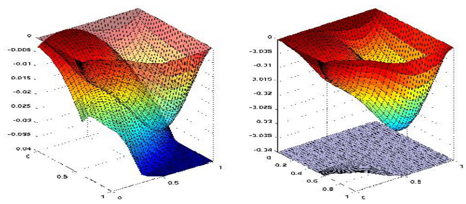


Fig. 1. Optimization of membrane with equilibrium (obstacle) constraints.

2.5 Optimization of parameterized variational inequalities

A large number of MPECs originate in the optimization of mixed P parameterized variational inequalities. Such problems include packaging problems that use membranes with obstacles as well as the Stackelberg games presented in (Outrata and al. 1998). We describe the problems as follows.

$$
\begin{array}{lll}
 & (MPEC) & \\
\min\limits_{x,y,w,z} & f(x,y,w,z) & \\
\text{sbj.to} & g(x) & \leq 0 \\
 & h(x) & = 0 \\
 & F(x,y,w,z) & = 0 \\
 & y,w & \leq 0 \\
 & (y^Tw=0)\ \ y^Tw & \leq 0
\end{array}
$$

The structural function F satisfies the mixed P partition property at any point

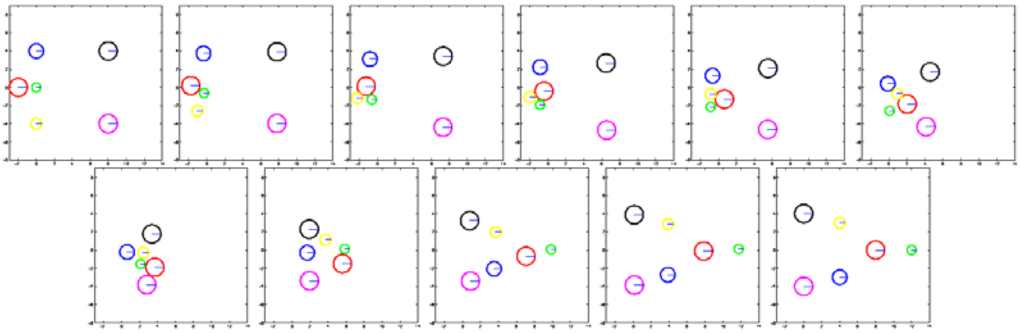
$$0 \neq (d_y,d_w,d_z) \in \mathrm{R}^{2n_c+l},$$

$$\nabla_y F d_y + \nabla_w F d_w + \nabla_z F d_z = 0 \Rightarrow$$

$$\exists i, 1 \leq i \leq n_c, \text{ such that } y_i w_i > 0.$$

For such problems we have the following result.

Theorem 4 (Anitescu 2005b). Assume that $\tilde{x}^n=(x^n,y^n,z^n,w^n,\zeta_1^n,\zeta_2^n)$ is an ε^n,δ^n second-order stationary point of (PF(c^n)), for all $n=1,2,\ldots,\infty$ and for sequences $\{c^n\},\{\varepsilon^n\},\{\delta^n\}$ that satisfy $\lim\limits_{n\to\infty} c^n=\infty, \lim\limits_{n\to\infty}\varepsilon^n=0, \lim\limits_{n\to\infty}\chi^n=0$.



Let $(x^*, y^*, z^*, w^*, \zeta_1^*, \zeta_2^*)$ be an accumulation point of this sequence that, with our main theorem, must satisfy $0 = \zeta_1^* = \zeta_2^*$. Its first component (x^*, y^*, z^*, w^*) must be a C-stationary point of (MPCC). If (x^*, y^*, z^*, w^*) satisfies (MPCC-LICQ), then (x^*, y^*, z^*, w^*) must be an M-stationary point of (MPCC). If, in addition, c_k is bounded, then the solution point is also strongly stationary.

The solution of the membrane problem with parabolic rigid obstacle is presented in Figure 1. The problem is to minimize the area of the membrane while keeping it in contact with a prescribed region. In the first subfigure we plot both the membrane and the obstacle; in the second subfigure we plot the membrane and the finite element mesh. The problem is detailed in (Outrata et al., 1998). Our algorithm has successfully solved the 18 instances of the problem (Anitescu et al., 2005; Anitescu, 2005b).

3. INITIAL INVESTIGATIONS IN CONTROL PROBLEMS

Inspired by an application of coordinated robotic control, we investigate the minimum time placement of a system of bodies that can experience contact and friction with the floor (Peng et al., 2004). The contact and friction are described by variational inequalities with a discretization of the friction cone, which make the optimization problem an MPEC. We use cylindrical bodies that we call "robots". The problem was solved with SNOPT (Gill et al., 1997), which uses a variant of our elastic-inexact algorithm.

Each pair of robots must have a minimal safe distance $ds > 0$. In our experiments, $ds = 0.01$. See Table 1 for the problems in Set A and Table 2 for computational results for Set A. The snapshots of the coordination of six robots with collision avoidance are shown in Figure 2. We include two examples, A1 and A2, of the system with only one robot to show that the computational results are very close to the theoretical results; indeed the computational results are bang-bang solutions (A1 and A2 are the only two examples that have theoretical solutions in our test examples). Convergence was obtained in all cases for the method.

The notation is as follows: *Nr* is the number of robots, *Def* the dimension of the external force(s), *Nt* the number of time steps, *K* the K-polygon approximation for circle, *Nv* the number of variables, *Ntc* the number of total constraints, *Gs* the global search solution, *LB* the analytical Lower bound of the optimal solution, *Gt* the global search time (seconds). The results are presented in Tables 1 and 2.

Table 1 Test Examples Parameters

Problem	Nr	Def	Nt	K	Nv	Ntc
A1	1	2	100	16	4385	2693
A2	1	1	100	16	4385	2693
A3	4	2	50	8	11248	8877
A4	6	2	50	4	20177	17220

Table 2 Computational Results

Problem	Gs	G t(sec)	LB
A1	9.03	5.8	8.94
A2	9.85	98.4	9.75
A3	9.60	20749.2	8.94
A4	9.71	12605.1	8.94

We note that despite the problems being large and nonconvex, the solutions are very close to the lower bound, which indicate a high-quality solution.

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