Deflating the Shifted Laplacian for the Helmholtz Equation

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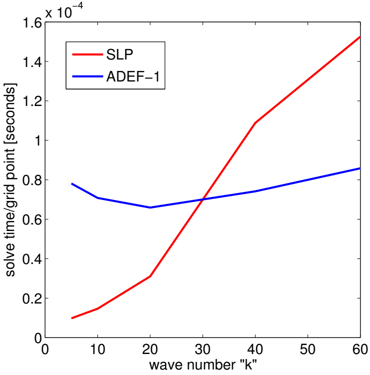
Introduction



Helmholtz Equation

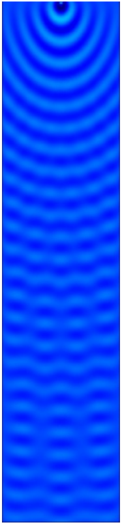
$$-\Delta \mathbf{u}(x,y) - k^2\,\mathbf{u}(x,y) = \mathbf{g}(x,y) \text{ on } \Omega$$

Dirichlet and/or Sommerfeld on $\partial\Omega$

finite differences or elements

$A\,u = f$ sparse complex symmetric

all standard solvers fail



Complex Shifted Laplace Preconditioner

preconditioning by damping

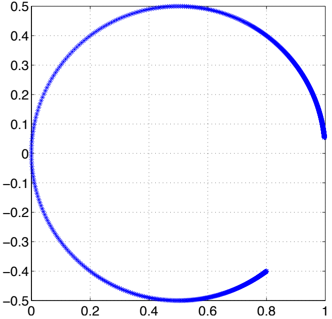
$M : -\Delta \mathbf{u} - (1 + \beta_2\, i) k^2\, \mathbf{u}$

M-solve using multigrid

$M^{-1}A$ favorable spectrum

standard in many applications

Erlangga e.a. 2006



Complex Shifted Laplace Preconditioner

Number of outer Krylov iterations

	Wavenumber					
Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**10**	17	28	44	70	13
$n = 64$	10	**17**	28	36	45	173
$n = 96$	10	17	**27**	35	43	36
$n = 128$	10	17	27	**35**	43	36
$n = 160$	10	17	27	35	**43**	25
$n = 320$	10	17	27	35	42	**80**

Complex Shifted Laplace Preconditioner

Good News

- SLP preconditioner renders spectrum favorable to Krylov

However ...

- eigenvalues rush to zero as k increases
- outer Krylov convergence limited by near-null space

Can deflation improve?

Deflation using Multigrid Vectors

Deflation perspective

- replace preconditioned system $M^{-1} A = M^{-1} b$
- by deflated preconditioned system $P^T M^{-1} A = P^T M^{-1} b$
- deflation vectors Z and Galerkin coarse grid matrix $E = Z^T A Z$
- deflation operator $P = I - A Q$ where $Q = Z E^{-1} Z^T$
- P: projection (later modified to shift to 1)
- Z: columns of the coarse to fine grid interpolation

  good approx to near-null space for $k h$ fixed

Deflation using Multigrid Vectors

Multigrid perspective

- replace smoother $I - M^{-1} A$

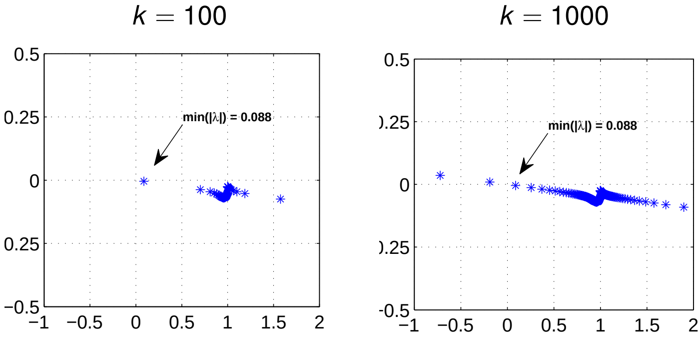
  (M complex shifted-Laplacian)

- by smoother + coarse grid solve $(I - Q A)\ (I - M^{-1} A)$

  $Q = Z E^{-1} Z^T$ coarse grid solve
  E^{-1} Galerkin coarse grid Helmholtz operator

- Fourier two-grid analysis for
  - 1D problem with Dirichlet bc
  - uniform coarsening
  - E and M inverted exactly

Spectrum Deflated Preconditioned Operator

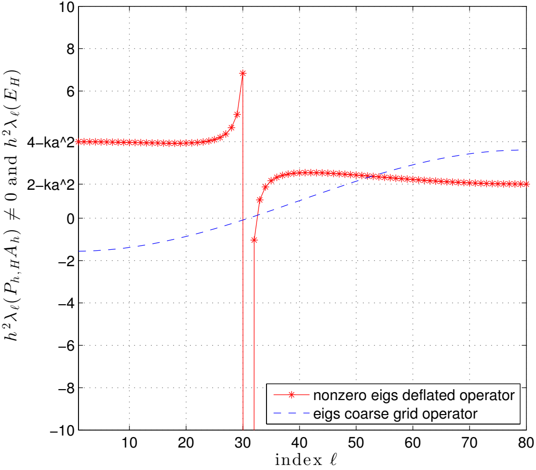


tighter clusters at low frequency

spread due to near-kernel of E

Spread due to near-kernel of E

$k = 100$



Deflation allows much larger shifts

k	$\beta_2 = .5$ PREC/PREC+DEF	$\beta_2 = 1$ PREC/PREC+DEF	$\beta_2 = 10$ PREC/PREC+DEF
10	7/3	8/4	5
20	10/5	12/6	7
40	16/8	20/8	9
80	23/8	33/9	9
160	36/13	55/14	14
320	61/19	97/20	19
640	108/33	179/33	34

Deflation using Multigrid Vectors

Multilevel Extension

- composite two-level preconditioner $P^T M^{-1} A = P^T M^{-1} b$
- deflation operator $P = I - A Q$ where $Q = Z E^{-1} Z^T$
- coarse grid Helmholtz operator $E = Z^T A Z$
- apply idea recursively to apply E
- multilevel Krylov method (Erlangga-Nabben 2009)

Convergence Outer Krylov Acceleration

Number of outer Krylov iterations with/without deflation

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**5/10**	8/17	14/28	26/44	42/70	13/14
$n = 64$	4/10	**6/17**	8/28	12/36	18/45	173/163
$n = 96$	3/10	5/17	**7/27**	9/35	12/43	36/97
$n = 128$	3/10	4/17	6/27	**7/35**	9/43	36/85
$n = 160$	3/10	4/17	5/27	6/35	**8/43**	25/82
$n = 320$	3/10	4/17	4/27	5/35	5/42	**10/80**

Less iterations and therefore speedup

(Sheikh, D.L., Ramos, Nabben and Vuik, accepted for JCP).

Numerical Results

3D problem with wedge-like contrast in wavenumber
using 20 grid points per wavelength

Wave number k	Solve Time		Iterations	
	PREC	DEF+PREC	PREC	DEF+PREC
5	0.09	0.24	9	11
10	1.07	1.94	15	12
20	16.70	18.89	32	16
30	73.82	78.04	43	21
40	1304.2	214.7	331	24
60	xx	989.5	xx	34

speedup in CPU of by a factor 6

(Sheikh, D.L., Ramos, Nabben and Vuik, accepted for JCP).

Numerical Results

2D Marmousi Problem
using 20 grid points per wavelength

Frequency f	Solve Time		Iterations	
	PREC	DEF+PREC	PREC	DEF+PREC
1	1.23	5.08	13	7
10	40.01	21.83	106	8
20	280.08	131.30	177	12
40	20232.6	3997.7	340	21

speedup in CPU of by a factor 5

Conclusions

- Rigorous Fourier spectral analysis
- less iterations than shifted-Laplacian
- faster than shifted-Laplacian solver for sufficiently large problems