Some New Results in Logical Calculi Obtained Using Automated Reasoning

Zac Ernst, Ken Harris & Branden Fitelson

Department of Philosophy
University of Wisconsin–Madison

&

AR @ MCS @ ANL

July 13, 2001

Overview of Presentation

- Some Unfinished Business from Our Last Meeting
  - Classical Propositional Calculus in the Sheffer Stroke (D)
  - Classical Propositional Calculus in C and O
- Some New Results in Other Logical Calculi
  - New Bases for LG (including new, shortest known single axioms)
  - New Bases for C5 (including new, shortest known bases)
  - New Bases for C4 (including new, shortest known single axiom)
  - New Bases for $\mathrm{RM}_{\rightarrow}$ (including new, shortest known bases)
  - Miscellaneous Results for $\mathrm{E}_{\rightarrow}$, $\mathrm{R}_{\rightarrow}$, $\mathrm{RM}_{\rightarrow}$, and $\text{Ł}_{\rightarrow}$
- Discussion of AR Techniques Used and Developed ...

Sheffer Stroke Axiomatizations of Sentential Logic: I

- At our last meeting, we reported some investigations into Sheffer Stroke axiomatizations of the classical propositional calculus.
- In these systems, the only rule of inference is the following, somewhat odd, detachment rule for D:

(D-Rule) From $DpDqr$ and p, infer r.

- Four 23-symbol single axioms have been known for many years.[a]

(N)	$DDpDqrDDtDttDDsqDDpsDps$
(Ł$_1$)	$DDpDqrDDsDssDDsqDDpsDps$
(Ł$_2$)	$DDpDqrDDpDrpDDsqDDpsDps$
(W)	$DDpDqrDDDsrDDpsDpsDpDpq$

[a] These axioms are discussed in [11], [5, pp. 179–196], and [22, pp. 37–39].

Sheffer Stroke Axiomatizations of Sentential Logic: II

- As we reported last time, Ken and I were able to find many (60+) new 23-symbol single axioms for this system, including:

$$DDpDqrDDpDqrDDsrDDrsDps$$

- Last year, we began to look for shorter single axioms. But, at that time, our methods for generating and filtering candidates were too inefficient to exhaustively search candidates of length 19 or 21.
- Since then, Zac Ernst joined the team. Zac's fast `C` programs for generating and filtering formulas have allowed us to exhaustively search all formulas up through size 21 (in several systems).
- We can now report that *the shortest single axioms for these Sheffer Stroke systems contain 23 symbols*. Similar negative results for other systems have been established, and will be reported below.

Single Axioms for C-O Axiomatizations of Sentential Logic

- Meredith [9, 8] reports two 19-symbol single axioms for classical sentential logic (using only the rule of condensed detachment) in terms of implication C and the constant O (semantically, O is "The False"):

$$CCCCCpqCrOstCCtpCrp$$
$$CCCpqCCOrsCCspCtCup$$

- As Ted pointed out after last year's workshop, Meredith [9, page 156] claims to have "almost completed a proof that no single axiom of (C,O) can contain less than 19 letters." As far as we know (is this right Ted?), no such proof was ever completed (that is, until now ...).

- We have performed an exhaustive search/elimination of all (C,O) theorems with fewer than 19 symbols. We have proven Meredith's conjecture: *no single axiom of (C,O) can contain less than 19 letters.*[a]

[a] The elimination of some (C,O) candidates relied on matrices generated using *local search* techniques (as described by Ted in his [19, 21]). Local search is very powerful in the context of implicational logics. Ken and Zac can say more about this.

New Results I: The Left Group Calculus

- The left group calculus (LG) is a subsystem of the equivalential calculus. The following 5-axiom basis for LG (assuming only the rule of condensed detachment) was reported by Kalman [3]:

(L_1) $EEEpEEqqprr$

(L_2) $EEEEEpqEprEqrss$

(L_3) $EEEEEEpqEprsEEqrstt$

(L_4) $EEEEpqrsEEEptrEEqts$

(L_5) $EEEpEEqprEEsptEEEEpqsrt$

- Using many clever automated reasoning techniques, McCune [6] later proved a great many results pertaining to LG. Among other things, McCune showed that axioms L_1, L_4, and L_5 are dependent, and that the following 27-symbol (7-variable) formula is a single axiom for LG:

$$EEEEpqrEEstEEEutEusvErEEqpv$$

New Results I: The Left Group Calculus (Cont'd)

- McCune [6] was also able to do exhaustive searches for single axioms of length 19 and shorter. He found no single axioms at these lengths.
- At that time, McCune was unable to do an exhaustive search of 23-symbol formulas (as candidate single axioms for LG).
- Recently, (using McCune's [6] techniques, and a few others) we were able to complete an exhaustive search of 23-symbol LG theorems, which yielded the following six new 23-symbol (5-variable) LG single axioms.[a]

$$EEEpqrEEEspEEEtuEqsEutr$$
$$EEEpqEErpsEEtuEEutEErqs$$
$$EEEEpqEprEErqsEEtuEEuts$$
$$EEEEpqEprEErqsEEutEEtus$$
$$EEEEEEpqErsEqpEEtsEtruu$$
$$EEEEpqEEErsEtpEsruEEqtu$$

[a] It remains open whether these are *shortest* LG single axioms. Indeed, there is even an 11-symbol formula, $EEpqEEprEqr$, which has yet to be formally ruled-out!

New Results II: New Bases for C5

- In their classic paper [4], Lemmon, Meredith, Meredith, Prior, and Thomas present several axiomatizations (assuming only the rule of condensed detachment) of the system C5, which is the strict implicational fragment of the modal logic S5.

- Bases for C5 containing 4, 3, 2, and a single axiom are presented in [4]. The following 2-basis is the shortest of these bases. It contains 20 symbols, 5-variables, and 9 occurrences of the connective C.

$$Cpp$$
$$CCCCpqrqCCqsCtCps$$

- The following 21-symbol (6-variable, 10-C) single axiom (due to C.A. Meredith) for C5 is also reported in [4]:

$$CCCCCttpqCrsCCspCuCrp$$

New Results II: New Bases for C5 (Cont'd)

- We (really, Zac) searched both for new (hopefully, shorter than previously known) single axioms for C5 and for new 2-bases for C5.
- Zac discovered the following new 2-basis for C5, which (to the best of our knowledge) is shorter than any previously known basis (it has 18 symbols, 4 variables, and 8 occurrences of C):

$$Cpp$$
$$CCpqCCCCqrsrCpr$$

- Moreover, Zac discovered the following new 21-symbol (6-variable, 10-C) single axiom for C5 (as well as 5 others, not given here):

$$CCCCpqrCCssqCCqtCuCpt$$

- We have formally ruled-out *all* shorter theorems of C5. Therefore, *no formula with fewer than 21 symbols is a single axiom for C5.*

New Results III: New Bases for C4

- C4 is the strict-implicational fragment of the modal logic S4 (and several other modal logics in the neighborhood of S4 — see Ted's [20]).
- As far as we know, the shortest known basis for C4 is due to Ulrich (see Ted's [20]), and is the following 25-symbol, 11-C, 3-axiom basis:

$$Cpp$$
$$CCpqCrCpq$$
$$CCpCqrCCpqCpr$$

- Anderson & Belnap [1, p. 89] state the finding of a (short) single axiom for C4 as an open problem (as far as we know, this has *remained* open). The following is a 21-symbol (6-variable, 10-C) single axiom for C4:

$$CCpCCqCrrCpsCCstCuCpt$$

- We have also searched for short, 2-axiom bases for C4, of the form:

$$Cpp$$
$$X$$

There are no such 2-bases for C4 in which X has fewer than 17 symbols.

New Results IV: New Bases for $\mathrm{RM}_{\rightarrow}$

- The "classical" relevance logic R-Mingle (RM) was first carefully studied by Dunn in the late 60's (*e.g.*, in [2]). Interestingly, the implicational fragment of R-Mingle ($\mathrm{RM}_{\rightarrow}$) has an older history.
- $\mathrm{RM}_{\rightarrow}$ was studied (albeit, unwittingly!) by Sobociński in the early 50's. Sobociński [18] discusses a two-designated-value-variant of Łukasiewicz's three-valued implication-negation logic (I'll call Sobociński's logic **S**). Sobociński leaves the axiomatization of $\mathbf{S}_{\rightarrow}$ as an open problem.
- Rose [15, 16] solved Sobociński's open problem, but his axiomatizations of $\mathbf{S}_{\rightarrow}$ are very complicated and highly redundant (see Parks' [12]).
- Meyer and Parks [10, 13] report an independent 4-axiom basis for $\mathbf{S}_{\rightarrow}$. They also show that $\mathbf{S}_{\rightarrow} = \mathrm{RM}_{\rightarrow}$, thus providing an independent 4-basis for $\mathrm{RM}_{\rightarrow}$. Meyer and Parks show that $\mathrm{RM}_{\rightarrow}$ can be axiomatized by adding the following "unintelligible" 21-symbol formula to $\mathrm{R}_{\rightarrow}$:

$$CCCCCpqqprCCCCCqppqrr$$

New Results IV: New Bases for $\mathrm{RM}_{\rightarrow}$ (Cont'd)

- In other words, the following is a 5-basis for $\mathrm{RM}_{\rightarrow}$:

$$Cpp \qquad CpCCpqq$$
$$CCpqCCrpCrq \qquad CCpCpqCpq$$
$$CCCCCpqqprCCCCCqppqrr$$

- The reflexivity axiom Cpp is dependent in the above 5-basis. The remaining (independent) 4-basis is the Meyer-Parks basis for $\mathrm{RM}_{\rightarrow}$.
- After much effort (and, with valuable assistance from Bob and Larry), we (Zac) discovered the following 13-symbol replacement for Parks' 21-symbol formula (we've also shown that there are none shorter):

$$CCCCCpqrCqprr^{\mathrm{a}}$$

- The contraction axiom $CCpCpqCpq$ is dependent in our new 4-basis. The remaining (independent) *3*-basis for $\mathrm{RM}_{\rightarrow}$ contains 31 symbols and 14 C's (the Meyer-Parks basis has 4 axioms, 48 symbols, and 22 C's).

[a] The alternative 13-symbol formula $CCCpCCCqprqrr$ will also serve this purpose.

New Results V: Miscellaneous Other Results

- It was shown by Rezuş [14] that the systems $E_{\rightarrow}$, $R_{\rightarrow}$, and Ł$_{\rightarrow}$ have single axioms. However, applying the methods of [14] yields very long, *in*organic single axioms. As far as we know, these axioms have never been explicitly written down. Here is a 69-symbol (17-variable!) single axiom for the implicational fragment of Łukasiewicz's infinite-valued logic Ł$_{\rightarrow}$, generated by Ken, using his own variations on and simplifications of the methods discussed in [14]:

Ł$_{\rightarrow}$: CCCfCgfCCCCCCCCCcdCCecCedCCaCbazzCCCCxyyCCyxxwwCCCCtuCutCutssCCqCrqpp

- Single axioms of comparable length (*i.e.*, containing fewer than 100 symbols) can also be generated for the relevance logics $E_{\rightarrow}$ and $R_{\rightarrow}$ (omitted). Here's what we know about the shortest single axioms for the systems $E_{\rightarrow}$, $R_{\rightarrow}$, Ł$_{\rightarrow}$, and $RM_{\rightarrow}$:
  - The shortest single axiom for $E_{\rightarrow}$ has between 23 and 100 symbols.
  - The shortest single axiom for $R_{\rightarrow}$ has between 23 and 100 symbols.
  - The shortest single axiom for Ł$_{\rightarrow}$ has at most 69 symbols.
  - The shortest single axiom for $RM_{\rightarrow}$ (if there is one[a]) has at least 23 symbols.

[a] Rezuş's work does *not* apply to $RM_{\rightarrow}$, so whether $RM_{\rightarrow}$ has a single axiom remains open.

The Indispensability of Automated Reasoning Techniques

- The use of automated reasoning in the research discussed above was *essential*. In particular, McCune's automated reasoning program OTTER [7] was indispensable in our investigations. Without OTTER, and without the vast array of powerful automated reasoning strategies developed by the Argonne team (Larry, Bob, Bill, Ross, *et al.*) over the years (many of which are discussed in Larry's recent book [23]), none of the results reported here would have been established (by us!).
- Aside from OTTER, the model-finding programs SEM [24] and MAGIC [17] were invaluable for producing useful logical matrices for the purpose of eliminating almost all (and, sometimes, all!) candidate formulas.
- Finally, the fast `C`-programs written by Zac and Ken for (1) doing formula generation and filtering, and (2) performing local search (as described by Ted in [19, 21]) to find the models needed to eliminate a few tough candidates were the key to finishing these problems in finite time.

References

[1] Alan Ross Anderson and Nuel D. Belnap, Jr., *Entailment*, Princeton University Press, Princeton, N. J., 1975, Volume I: The logic of relevance and necessity.

[2] J. Michael Dunn, *Algebraic completeness results for R-mingle and its extensions*, J. Symbolic Logic **35** (1970), 1–13.

[3] J.A. Kalman, *Axiomatizations of logics with values in groups*, Journal of the London Math. Society **2** (1975), 193–199.

[4] E. J. Lemmon, C. A. Meredith, D. Meredith, A. N. Prior, and I. Thomas, *Calculi of pure strict implication*, Philosophy Department, Canterbury University College, Christchurch, New Zealand, 1957.

[5] J. Łukasiewicz, *Selected Works*, Noth Holland, 1970.

[6] William W. McCune, *Automated discovery of new axiomatizations of the left group and right group calculi*, J. Automat. Reason. **9** (1992), no. 1, 1–24.

[7] ———, *Otter 3.0 Reference Manual and Guide*, Tech. Report ANL-94/6, Argonne National Laboratory, Argonne, IL, 1994.

[8] C. A. Meredith and A. N. Prior, *Notes on the axiomatics of the propositional*

calculus, Notre Dame J. Formal Logic **4** (1963), 171–187.

[9] Carew A. Meredith, *Single axioms for the systems* (C, N), (C, O) *and* (A, N) *of the two-valued propositional calculus*, J. Computing Systems **1** (1953), 155–164.

[10] Robert K. Meyer and Zane Parks, *Independent axioms for the implicational fragment of Sobociński's three-valued logic*, Z. Math. Logik Grundlagen Math. **18** (1972), 291–295.

[11] J.G. Nicod, *A reduction in the number of primitive propositions of logic*, Proc. Camb. Phil. Soc. **19** (1917), 32–41.

[12] Zane Parks, *Dependence of some axioms of Rose*, Z. Math. Logik Grundlagen Math. **18** (1972), 189–192.

[13] ______, *A note on R-Mingle and Sobociński's three-valued logic*, Notre Dame J. Formal Logic **13** (1972), 227–228.

[14] Adrian Rezuş, *On a theorem of Tarski*, Libertas Math. **2** (1982), 63–97.

[15] Alan Rose, *A formalization of Sobociński's three-valued implicational propositional calculus*, J. Computing Systems **1** (1953), 165–168.

[16] ______, *An alternative formalisation of Sobociński's three-valued implicational*

propositional calculus, Z. Math. Logik. Grundlagen Math. **2** (1956), 166–172.

[17] J. Slaney, MaGIC, *matrix generator for implication connectives: Release 2.1 notes and guide, technical report tr-arp-11-95*, Tech. report, Automated Reasoning Project, Australian National University, 1995.

[18] Bolesław Sobociński, *Axiomatization of a partial system of three-value calculus of propositions*, J. Computing Systems **1** (1952), 23–55.

[19] Dolph Ulrich, *Local search*, mimeograph of unpublished lecture notes (obtained in private correspondence of 12 March 2001).

[20] ———, *Strict implication in a sequence of extensions of S4*, Z. Math. Logik Grundlag. Math. **27** (1981), no. 3, 201–212.

[21] ———, *A legacy recalled and a tradition continued*, to appear in our upcoming special issue of the *Journal of Automated Reasoning*, 2001.

[22] M. Wajsberg, *Logical Works*, Polish Academy of Sciences, 1977.

[23] L. Wos, *A fascinating country in the world of computing: Your guide to automated reasoning*, World Scientific, 1999.

[24] J. Zhang and H. Zhang, *SEM: A system for enumerating models*, Proceedings of the International Joint Conference on Artificial Intelligence, 1995.