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## Stat 310, Part II, Optimization. Homework 5.

1. Consider the constrained optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f(x) \\ s.t. \quad & c_E(x) = 0 \\ & c_I(x) \geq 0 \end{aligned}$$

   Assume that the point $x^*$ is the solution of this problem, at which the linear independent constraint qualification (LICQ) holds. Using the necessary optimality conditions for problems with equality constraints applied to the equivalent formulation

$$\begin{aligned} \min_{x\in\mathbb{R}^n, z\in\mathbb{R}^{n_I}}, \quad & f(x) \\ s.t. \quad & c_E(x) = 0 \\ & \left[c_I(x)\right]_j - z_j^2 = 0 \quad j = 1,2,\ldots n_I \end{aligned}$$

   prove that the first and second-order necessary optimality conditions for problems with inequality constraints (also called the Karush-Kuhn-Tucker conditions, or KKT) must hold at $x^*$ (you can assume f,c as regular as needed).

2. Let $A \in \mathbb{R}^{m\times n}$ have full row rank. Let $Q \in \mathbb{R}^{n\times n}$ be such that $u \neq 0, Au = 0 \Rightarrow u^T Q u > 0$. Prove that
   a. There exists $c > 0$ such that $Q + cA^T A$ is positive definite.
   b. The matrix $\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix}$ is not singular.
   c. Assume that you have an equality-constrained problem for which the LICQ and the second-order sufficient conditions hold. Discuss the implications of a), for the existence of an augmented Lagrangian and of b) for the local well-posedness of the system of nonlinear equations that is formed by the first-order optimality conditions of the problem.

3. (Variation of 19.7). Program the interior-point algorithm 19.1 and apply it to the problem (18.66)-(18.69), with the additional constraints $x_2 \geq 0, x_3 \geq 0$ (note that the solution does not change). This algorithm is not guaranteed to converge arbitrarily far from the solution so try to start it sufficiently close to the solution (experiment first with the point suggested by the book).