10.1 TYPES OF CONSTRAINED OPTIMIZATION ALGORITHMS

Quadratic Programming Problems

- Algorithms for such problems are interested to explore because
  - 1. Their structure can be efficiently exploited.
  - 2. They form the basis for other algorithms, such as augmented Lagrangian and Sequential quadratic programming problems.

$$
\begin{aligned}
\min_{x} \quad & q(x) = \tfrac{1}{2}x^T G x + x^T c \\
\text{subject to} \quad & a_i^T x = b_i, \qquad i \in \mathcal{E}, \\
& a_i^T x \geq b_i, \qquad i \in \mathcal{I},
\end{aligned}
$$

CHICAGO

Penalty Methods

Solve with unconstrained optimization

- Idea: Replace the constraints by a penalty term.
- Inexact penalties: parameter driven to infinity to recover solution. Example:

$$x^* = \arg\min f(x) \text{ subject to } c(x) = 0 \Leftrightarrow$$

$$x^\mu = \arg\min f(x) + \frac{\mu}{2}\sum_{i\in\mathcal{E}} c_i^2(x);\ x^* = \lim_{\mu\to\infty} x^\mu = x^*$$

- Exact but nonsmooth penalty – the penalty parameter can stay finite.

$$x^* = \arg\min f(x) \text{ subject to } \mathrm{c}(x) = 0 \Leftrightarrow x^* = \arg\min f(x) + \mu\sum_{i\in\mathcal{E}} |c_i(x)|;\ \mu \ge \mu_0$$

Augmented Lagrangian Methods

- Mix the Lagrangian point of view with a penalty point of view.

$$x^* = \arg\min f(x) \text{ subject to } \mathrm{c}(x) = 0 \Leftrightarrow$$

$$x^{\mu,\lambda} = \arg\min f(x) - \sum_{i\in\mathcal{E}} \lambda_i c_i(x) + \frac{\mu}{2}\sum_{i\in\mathcal{E}} c_i^2(x) \Rightarrow$$

$$x^* = \lim_{\lambda\to\lambda^*} x^{\mu,\lambda} \text{ for some } \mu \geq \mu_0 > 0$$

CHICAGO

Sequential Quadratic Programming Algorithms

- Solve successively Quadratic Programs.

$$\begin{aligned} \min_p \quad & \frac{1}{2} p^T B_k p + \nabla f(x_k) \\ \text{subject to} \quad & \nabla c_i(x_k) d + c_i(x_k) = 0 \quad i \in \mathcal{E} \\ & \nabla c_i(x_k) d + c_i(x_k) \geq 0 \quad i \in \mathcal{I} \end{aligned}$$

- It is the analogous of Newton's method for the case of constraints if

$$B_k = \nabla_{xx}^2 \mathcal{L}(x_k, \lambda_k)$$

- But how do you solve the subproblem? It is possible with extensions of simplex which I do not cover.
- An option is BFGS which makes it convex.

Interior Point Methods

- Reduce the inequality constraints with a barrier

$$
\begin{aligned}
\min_{x,s} \quad & f(x) - \mu \sum_{i=1}^{m} \log s_i \\
\text{subject to} \quad & c_i(x) = 0 \qquad i \in \mathcal{E} \\
& c_i(x) - s_i = 0 \qquad i \in \mathcal{I}
\end{aligned}
$$

- An alternative, is use to use a penalty as well:

$$
\min_x f(x) - \mu \sum_{i \in \mathcal{I}} \log s_i + \frac{1}{2\mu} \sum_{i \in \mathcal{I}} \left(c_i(x) - s\right)^2 + \frac{1}{2\mu} \sum_{i \in \mathcal{E}} \left(c_i(x)\right)^2
$$

- And I can solve it as a sequence of unconstrained problems!

10.2 MERIT FUNCTIONS AND FILTERS

Feasible algorithms

- If I can afford to maintain feasibility at all steps, then I just monitor decrease in objective function.
- I accept a point if I have enough descent.
- But this works only for very particular constraints, such as linear constraints or bound constraints (and we will use it).
- Algorithms that do that are called feasible algorithms.

Infeasible algorithms

- But, sometimes it is VERY HARD to enforce feasibility at all steps (e.g. nonlinear equality constraints).
- And I need feasibility only in the limit; so there is benefit to allow algorithms to move on the outside of the feasible set.
- But then, how do I measure progress since I have two, apparently contradictory requirements:
  - Reduce infeasibility (e.g. $\sum_{i\in\mathcal{E}}\left|c_i(x)\right|+\sum_{i\in\mathcal{I}}\max\left\{-c_i(x),0\right\}$)
  - Reduce objective function.
  - It has a multiobjective optimization nature!

10.2.1 MERIT FUNCTIONS

Merit function

- One idea also from multiobjective optimization: minimize a weighted combination of the 2 criteria.

$$\phi(x) = w_1 f(x) + w_2 \left[\sum_{i \in \mathcal{E}} |c_i(x)| + \sum_{i \in \mathcal{I}} \max\{-c_i(x), 0\} \right]; \quad w_1, w_2 > 0$$

- But I can scale it so that the weight of the objective is 1.
- In that case, the weight of the infeasibility measure is called "penalty parameter".
- I can monitor progress by ensuring that $\phi(x)$ decreases, as in unconstrained optimization.

CHICAGO

Nonsmooth Penalty Merit Functions

$$\phi_1(x; \mu) = f(x) + \mu \sum_{i \in \mathcal{E}} |c_i(x)| + \mu \sum_{i \in \mathcal{I}} [c_i(x)]^-, \qquad [z]^- = \max\{0, -z\}.$$

Penalty parameter

- It is called the l1 merit function.
- Sometimes, they can be even EXACT.

Definition 15.1 (Exact Merit Function).
A merit function $\phi(x; \mu)$ is exact if there is a positive scalar μ^* such that for any $\mu > \mu^*$, any local solution of the nonlinear programming problem (15.1) is a local minimizer of $\phi(x; \mu)$.

We show in Theorem 17.3 that, under certain assumptions, the ℓ_1 merit function $\phi_1(x; \mu)$ is exact and that the threshold value μ^* is given by

$$\mu^* = \max\{|\lambda_i^*|, \ i \in \mathcal{E} \cup \mathcal{I}\},$$

Smooth and Exact Penalty Functions

- Excellent convergence properties, but very expensive to compute.
- Fletcher' s augmented Lagrangian:

$$\phi_F(x;\mu) = f(x) - \lambda(x)^T c(x) + \tfrac{1}{2}\mu \sum c_i(x)^2,$$

$$\lambda(x) = [A(x)A(x)^T]^{-1} A(x) \nabla f(x).$$

- It is both smooth and exact, but perhaps impractical due to the linear solve.

Augmented Lagrangian

- Smooth, but inexact. $$\phi(x) = f(x) - \sum_{i \in \mathcal{E}} \lambda_i c_i(x) + \frac{\mu}{2} \sum_{i \in \mathcal{E}} c_i^2(x) \Rightarrow$$

- An update of the Lagrange Multiplier is needed.
- We will not use it, except with Augmented Lagrangian methods themselves.

Line-search (Armijo) for Nonsmooth Merit Functions

$$\phi_1(x;\mu) = f(x) + \mu \sum_{i\in\mathcal{E}} |c_i(x)| + \mu \sum_{i\in\mathcal{I}} [c_i(x)]^-,$$

- How do we carry out the "progress search"?
- That is the line search or the sufficient reduction in trust region?
- In the unconstrained case, we had

- But we cannot use this anymore, since the function is not differentiable.

$$f(x_k) - f(x_k + \beta^m d_k) \geq -\rho\beta^m \nabla f(x_k)^T d_k; \quad 0 < \beta < 1, 0 < \rho < 0.5$$

Directional Derivatives of Nonsmooth Merit Function

$$\phi_1(x;\mu) = f(x) + \mu \sum_{i\in\mathcal{E}} |c_i(x)| + \mu \sum_{i\in\mathcal{I}} [c_i(x)]^-,$$

- Nevertheless, the function has a directional derivative (follows from properties of max function). EXPAND

$$D(\phi(x,\mu);p) = \lim_{t\to 0, t>0} \frac{\phi(x+tp,\mu)-\phi(x,\mu)}{t}; \quad D(\max\{f_1,f_2\},p) = \max\{\nabla f_1 p, \nabla f_1 p\}$$

- Line Search: $\phi(x_k,\mu)-\phi(x_k+\beta^m p_k,\mu) \geq -\rho\beta^m D(\phi(x_k,\mu),p_k);$
- Trust Region $\phi(x_k,\mu)-\phi(x_k+\beta^m p_k,\mu) \geq -\eta_1(m(0)-m(p_k));$

  $0<\eta_1<0.5$

And …. How do I choose the penalty parameter?

- VERY tricky issue, highly dependent on the penalty function used.
- For the l1 function, guideline is:

$$\mu^* = \max\{|\lambda_i^*|, \ i \in \mathcal{E} \cup \mathcal{I}\},$$

- But almost always adaptive. Criterion: If optimality gets ahead of feasibility, make penalty parameter more stringent.
- E.g l1 function: the max of current value of multipliers plus safety factor (EXPAND)
  –

10.2.2 FILTER APPROACHES

Principles of filters

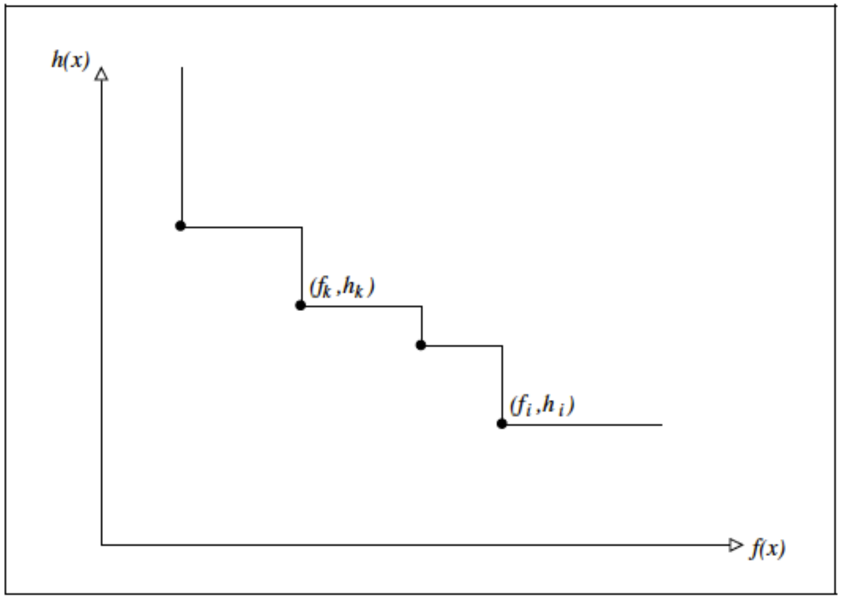
- Originates in the multiobjective optimization philosophy: objective and infeasibility

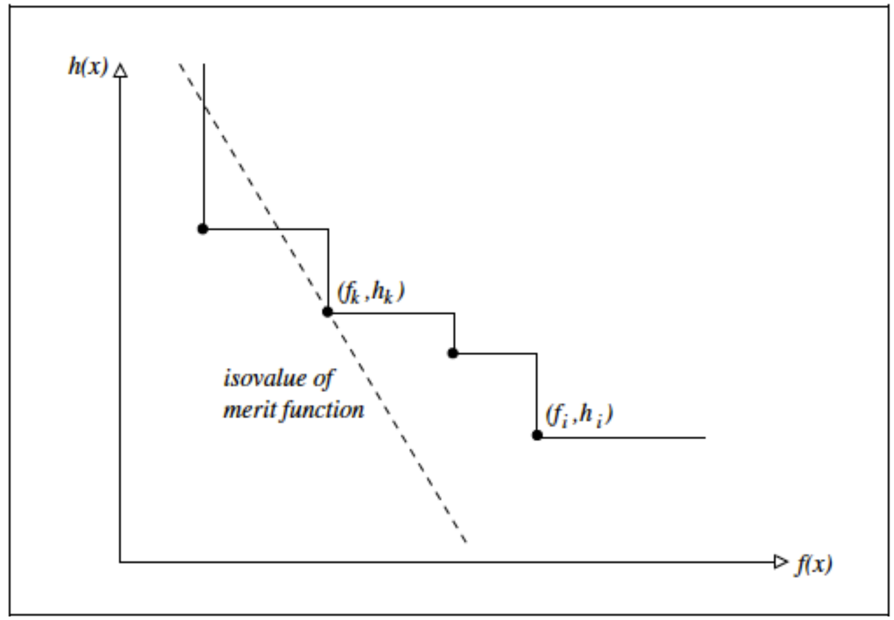
$$h(x) = \sum_{i \in \mathcal{E}} |c_i(x)| + \sum_{i \in \mathcal{I}} [c_i(x)]^-,$$

- The problem

$$\min_x f(x) \quad \text{and} \quad \min_x \; h(x).$$

The Filter approach





Definition 15.2.

(a) *A pair* (f_k, h_k) *is said to dominate another pair* (f_l, h_l) *if both* $f_k \le f_l$ *and* $h_k \le h_l$.

(b) *A* filter *is a list of pairs* (f_l, h_l) *such that no pair dominates any other.*

(c) *An iterate* x_k *is said to be acceptable to the filter if* (f_k, h_k) *is not dominated by any pair in the filter.*

Some Refinements

- Like in the line search approach, I cannot accept EVERY decrease since I may never converge.
- Modification:

A trial iterate x^+ is acceptable to the filter if, for all pairs (f_j, h_j) in the filter, we have that

$$f(x^+) \le f_j - \beta h_j \qquad \text{or} \qquad h(x^+) \le h_j - \beta h_j, \qquad \beta \sim 10^{-5} \quad (15.33)$$

CHICAGO

10.3 MARATOS EFFECT AND CURVILINEAR SEARCH

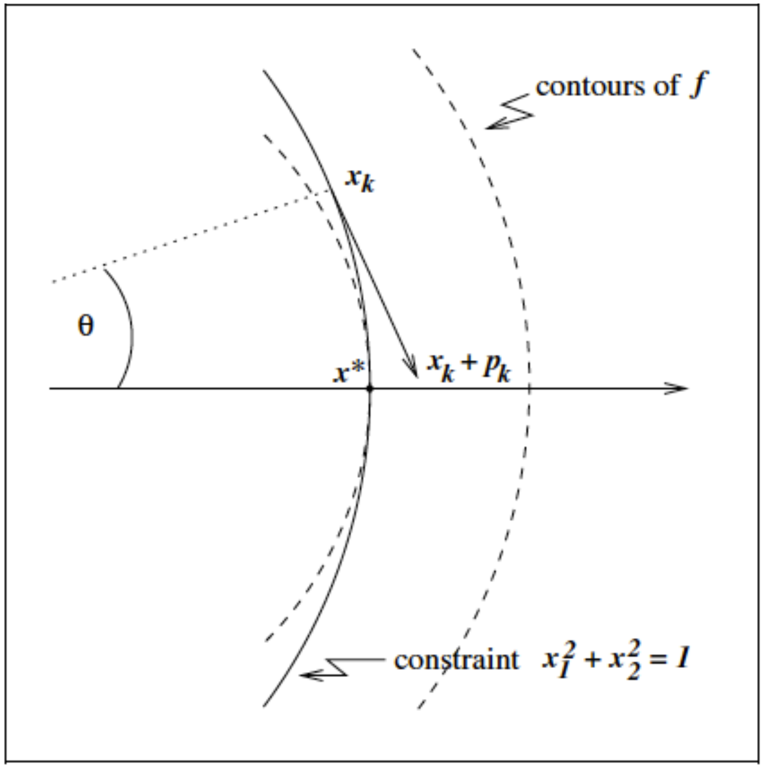
CHICAGO

Unfortunately, the Newton step may not be compatible with penalty

- This is called the Maratos effect.
- Problem:

$$\min\ f(x_1, x_2) = 2(x_1^2 + x_2^2 - 1) - x_1,$$

$$x_1^2 + x_2^2 - 1 = 0.$$



- Note: the closest point on search direction (Newton) will be rejected !
- So fast convergence does not occur

CHICAGO

Solutions?

- Use Fletcher's function that does not suffer from this problem.
- Following a step: $A_k p_k + c(x_k) = 0.$
- Use a correction that satisfies

$$A_k \hat{p}_k + c(x_k + p_k) = 0.$$

$$\hat{p}_k = -A_k^T (A_k A_k^T)^{-1} c(x_k + p_k),$$

- Followed by the update or line search:

$$x_k + p_k + \hat{p}_k \qquad x_k + \tau p_k + \tau^2 \hat{p}_k$$

- Since $c(x_k + p_k + \hat{p}_k) = O\left(\|x_k - x^*\|^3\right)$ compared to $c(x_k + p_k) = O\left(\|x_k - x^*\|^2\right)$ corrected Newton step is likelier to be accepted.