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On the Dynamic Stability of Electricity Markets

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Abstract In this work, we present new insights into the dynamic stability of electricity markets. We discuss how short forecast horizons, limited coordination, and physical ramping constraints can give rise to high price volatility. Using concepts of market efficiency, Lyapunov stability, and predictive control, we construct a framework to design and evaluate stabilizing market clearing procedures. A numerical case study is used to illustrate the developments.

Keywords Dynamics · Markets · Electricity · Efficiency · Game-Theory · Stability

1 Introduction

Stability (volatility) of wholesale electricity markets is associated with strong fluctuations of prices. Extreme price fluctuations reflect periods of scarcity induced by physical generation and transmission constraints from which a small subset of market participants benefit [7]. Consequently, controlling price fluctuations can lead to a more homogenous distribution of the social welfare. In addition, it can reduce speculation and thus incentivize investment and market participation [2].

There exist several evidences of extreme price volatility in operational markets [27, 1,19]. Broadly speaking, price fluctuations are observed under high demand conditions where physical capacity becomes constrained. For instance, volatility is particularly prevailing in spot (real-time) markets where fast ramping is needed to respond to unexpected demand fluctuations and/or contingencies. The drastic difference in price

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volatility in day-ahead and spot markets at the Illinois hub of the MidwestISO region for year 2010 can be observed in Figure 1.

Several studies have tried to understand the emergence of market volatility from *static* network effects such as transmission congestion [26, 12, 5]. In these studies, it has been observed that congestion leads to locational scarcity from which a small subset of suppliers benefit. This results in an uneven spread of welfare across the network and gives rise to the notion of locational marginal prices. Limited transmission capacity introduces high sensitivity of the system to demand changes which ultimately manifests in large variations of prices [7]. For instance, demand can increase slightly between time periods and make a transmission line congested, leading to a strong price change. In addition, the lack of sufficient physical transmission capacity provides opportunities for market players to raise their bids to high values.

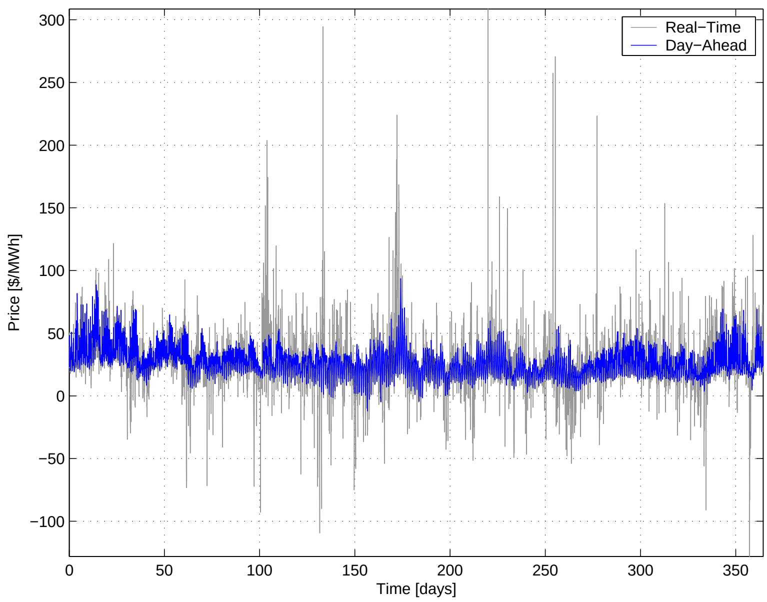

Fig. 1 Volatility of day-ahead and real-time markets at the Illinois Hub of MidwestISO in 2010.

The transmission congestion viewpoint of market volatility, however, is not sufficient to entirely explain the emergence of price fluctuations. In particular, volatility can arise from unexpected variations of demand from the forecast that require the use of ramping units (e.g., peaking units) to respond to the mismatch in real-time. Meyn and coworkers [8, 28] have observed that ramping leads to price volatility because it introduces market friction. In particular, they show that in the presence of friction, prices fluctuate with no tendency to convergence to the marginal cost. This indicates that ramping constraints add a *dynamic* dimension to the market volatility problem since, unlike transmission congestion, ramping effects propagate forward in time and can affect stability in the long-term. Kannan and Zavala [18] observed that ramping issues can arise from insufficient foresight (e.g., look-ahead) of clearing procedures and/or from an insufficient time resolution (i.e., how often the market is cleared). In particular, short foresight horizons can lead to more frequent ramp saturation and price volatility. This is important since it implies that the market design itself (i.e., clearing formulation) and not only the physical system capacity contribute to market volatility.

The questions that we address in this paper are: How can we formalize the analysis of market stability? How can we use such a construct to modify market clearing procedures? To address the market analysis question, we propose a control-theoretic framework based on predictive control. We demonstrate that predictive control provides a conceptual framework that can capture general mechanistic effects (e.g., physical constraints), decision-making behavior (e.g., competitive, strategic), and uncertainty handling (e.g., forecast errors). This framework can be used to understand the impact of different market clearing procedures and physical constraints on market behavior that can ultimately lead to more robust market designs capable of sustaining high demand and supply fluctuations, contingencies, and so on. Unfortunately, existing predictive control theory is limited to systems with natural steady-state (e.g., equilibrium) [20] or strong periodicity [14] conditions which do not exist in most economic systems. To address this issue, we construct a market-specific Lyapunov function that can be used to anticipate and monitor market stability. This function is constructed using a clearing efficiency metric that compares the social welfare under constrained and unconstrained conditions. In other words, it assesses the effect of physical constraints on price behavior. The efficiency notion used in this work is borrowed from [2, 7, 23] and reflects the ability of the ISO to keep prices stable (i.e., closer to a given reference) and thus more predictable. We use our framework to explain how volatility arises from short foresight horizons in clearing procedures, from forecast errors, and from limited coordination between the ISO and the players. In addition, we propose the incorporation of stabilizing constraints to market clearing procedures to mitigate volatility.

The paper is structured as follows. In Section 2 we present a simplified market structure to develop the control concepts. In Section 4 we discuss implementation issues of market clearing procedures arising from incomplete bidding. In Section 3 we analyze the numerical stability properties of the market game. In Section 5 we derive a framework to analyze dynamic stability properties. In Section 6 we present a numerical case study. Finally, in Section 7 we provide concluding remarks and recommendations for future extensions.

2 Market Structure

We first define the market structure under consideration and discuss the underlying modeling assumptions. We highlight that the market structure has been simplified in order to illustrate the control concepts. To avoid confusion, we discuss extensions to more general market models in Section 7.

2.1 Suppliers

We consider a supply-function equilibrium market structure under a competitive setting similar to those proposed in [15,18]. Here, the supplier decisions are the parameters a_t^i, b_t^i of the affine supply function:

$$q_t^i(p_t, b_t^i, a_t^i) = b_t^i \cdot (p_t - a_t^i). \tag{1}$$

Here, $q_t^i \geq 0$ is the production quantity of supplier $i \in \mathcal{S} := \{1..S\}$ at time t; $p_t \geq 0$ is the price at time t, and a_t^i, b_t^i are the bidding coefficients at time t for supplier i. We assume that the supply function is non-decreasing in p_t. Consequently, we impose the requirement that $b_t^i \geq 0$. We will restrict the intercept parameter a_t^i to be zero. This is necessary in order to avoid multiplicity of solutions. The supply function can also be expressed in inverse form as

$$p_t(q_t^i, b_t^i) = \frac{1}{b_t^i} q_t^i. \tag{2}$$

The consumer demands will be assumed to be *inelastic* d_t^j, $j \in \mathcal{C} := \{1..C\}$ at time t. In addition, we ignore forward contracts as those arising in two-settlement markets [17].

The supplier decision-making problem can be posed as follows. Starting at current time k, and given the price signals p_t over the future horizon $\mathcal{T}_k = \{k..k+T\}$, where T is the horizon length and $\hat{q}_k^i, \hat{b}_k^i$ are the current states at time k for the supplier, find the bidding parameters trajectories b_t^i, $t \in \mathcal{T}_k$, that maximize the future profit (revenue minus production cost). The suppliers $i \in \mathcal{S}$ solve the following problem:

$$\max_{b_t^i, q_t^i} \sum_{t \in \mathcal{T}_k} \phi_t^i := \sum_{t \in \mathcal{T}_k} \left(p_t \cdot q_t^i - c_t^i(q_t^i) \right) \tag{3a}$$

$$\text{s.t. } q_t^i = b_t^i \cdot p_t, \; t \in \mathcal{T}_k \tag{3b}$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i, \; t \in \mathcal{T}_k \tag{3c}$$

$$b_t^i \geq 0, \; t \in \mathcal{T}_k \tag{3d}$$

$$q_k^i = \hat{q}_k^i, \; b_k^i = \hat{b}_k^i, \tag{3e}$$

where $\underline{q}^i, \overline{q}^i \geq 0$ are the lower and upper production limits, respectively. The accumulated future profit is denoted by $\sum_{t \in \mathcal{T}_k} \phi_t^i$. The marginal cost function is assumed to have the form

$$c_t^i(q_t^i) = h_t^i \cdot q_t^i + \frac{1}{2} g_t^i \cdot (q_t^i)^2. \tag{4}$$

We make the common assumption that $g_t^i > 0$ so the production cost is strongly convex in q_t^i [25]. Consequently, we have that the supplier problem is strongly convex in the space of q_t^i. We also note that the case where $p_t = 0$ has a feasible solution only if $b_t^i = q_t^i = 0$ is admissible (i.e., the minimum capacity must be $\underline{q}^i = 0$). We summarize the problem properties in the following statement.

Property 1 If $g_t^i > 0$, problem (3) is strongly convex. If $p_t > 0$, the problem has a feasible solution for any $\underline{q}^i, \overline{q}^i \geq 0$. If $p_t = 0$, the problem admits a solution only if $\underline{q}^i = 0$.

We can pose this problem entirely in terms of the prices p_t and the supply function parameters b_t^i by substituting (3b) into (3a) and (3c). In addition, we interpret the bidding parameters b_t^i as the suppliers *states*. These modifications lead to the following equivalent formulation in state-space form:

$$\max_{b_t^i, \Delta b_t^i} \sum_{t \in \mathcal{T}_k} \left(p_t \cdot b_t^i \cdot p_t - c_t^i \left(b_t^i \cdot p_t \right) \right) \tag{5a}$$

$$\text{s.t. } b_{t+1}^i = b_t^i + \Delta b_t^i, \; t \in \mathcal{T}_k^- \tag{5b}$$

$$\underline{q}^i \leq b_t^i \cdot p_t \leq \overline{q}^i, \; t \in \mathcal{T}_k \tag{5c}$$

$$b_t^i \geq 0, \; t \in \mathcal{T}_k \tag{5d}$$

$$b_k^i = \hat{b}_k^i, \tag{5e}$$

where $\mathcal{T}_k^- := \mathcal{T}_k \setminus \{k + T\}$. The bidding increments Δb_t^i are interpreted as the control actions of the supplier. Note that these are unconstrained. A direct consequence of this is that the feasible set of the problem is invariant to the initial states $\hat{b}_k^i$. In addition, the feasible set is invariant to the price signals p_t since it is always possible to find $b_t^i \geq 0$ mapping any p_t to a feasible quantity q_t^i. Consequently, we denote the feasible set of this problem as Ω^i.

2.2 ISO Market Clearing

The independent system operator (ISO) receives the bidding parameters $b_t^i, \; i \in \mathcal{S}, t \in \mathcal{T}_k$ and physical information about the generators (ramps and capacity limits) and clears the market by determining the generation quantities and prices that balance total supply and demand. The interaction between the ISO and the suppliers results in a game whose solution is a dynamic equilibrium denoted as $q_{t*}^i, p_{t*}, b_{t*}^i, \; i \in \mathcal{S}, t \in \mathcal{T}_k$.

The main objectives of the ISO include, among others, to maximize social welfare and to keep clearing prices well-behaved [23]. This can be done, for instance, by ensuring that spot and day-ahead prices converge or by keeping prices close to the marginal costs which are more stable. In this work, market stability will be measured as the distance of the clearing prices from a *more stable reference* in the presence of dynamic fluctuations of demands and renewable supply and physical constraints. In other words, we will not seek to keep prices constant since this is impossible (even in the absence of constraints) because of the inherent dynamic variations of the inelastic demand. To define our reference, we propose to use a basic concept of *efficiency* which will measure the effect of physical constraints on social welfare and price behavior.

To establish our efficiency metric, we first define an *ideal unconstrained* market clearing problem that does not account for ramp constraints. This problem can be stated as follows. Given supply function states b_t^i, solve [4]:

$$\min_{q_t^i} \quad \sum_{t\in\mathcal{T}_k} \bar{\varphi}_t := \sum_{t\in\mathcal{T}_k}\sum_{i\in\mathcal{S}} \int_0^{q_t^i} p_t(q,b_t^i)dq \tag{6a}$$

s.t.

$$\sum_{i\in\mathcal{S}} q_t^i \geq \sum_{j\in\mathcal{C}} d_t^j,\ t\in\mathcal{T}_k \tag{6b}$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i,\ i\in\mathcal{S}, t\in\mathcal{T}_k, \tag{6c}$$

where

$$\int_0^{q_t^i} p_t(q,b_t^i)dq = \frac{1}{2\,b_t^i}(q_t^i)^2. \tag{7}$$

Since the demands are fixed, the objective function is the *negative social welfare* [15], denoted as $\sum_{t\in\mathcal{T}_k}\bar{\varphi}_t$. We have that $\bar{\varphi}_t \geq 0$ since $q_t^i, b_t^i \geq 0$. The multipliers for the constraint (6b) are the prices $\bar{p}_t \geq 0$. Note that the feasible set of this problem is not affected by the bidding parameters, since they enter only in the objective function. In addition, in this ideal unconstrained formulation, we assume that the generators can move infinitely fast between production levels. This assumption decouples the problem in time. Hence, the feasible set of this problem is invariant to the current state of the generators $\hat{q}_k^i$. Consequently, in the absence of ramp constraints, the effect of physical constraints on prices is only instantaneous and does not propagate forward in time.

The unconstrained market clearing problem is strongly convex for fixed $b_t^i > 0$. The case where $b_t^i = 0$ only has a feasible solution if $q_t^i = 0$ is admissible (i.e., $\underline{q}^i = 0$). This can be seen from the optimality condition (34a) in the Appendix. In this case, it is possible to eliminate q_t^i from the formulation by fixing its value to zero. The problem always has a feasible solution as long as the demand is reachable. This can be achieved if the demand satisfies $\sum_{i\in\mathcal{S}}\underline{q}^i \leq \sum_{j\in\mathcal{C}} d_t^j \leq \sum_{i\in\mathcal{S}}\overline{q}^i,\ t\in\mathcal{T}_k$. We summarize these properties in the following statement.

Property 2 If $b_t^i > 0$, problem (6) is strongly convex. The problem has a feasible solution if $\sum_{i\in\mathcal{S}}\underline{q}^i \leq \sum_{j\in\mathcal{C}} d_t^j \leq \sum_{i\in\mathcal{S}}\overline{q}^i$ holds. If $b_t^i > 0$, feasibility holds for any $\underline{q}^i, \overline{q}^i \geq 0$. If $b_t^i = 0$, the problem admits a solution only if $\underline{q}^i = 0$.

For our analysis, we note that having infinitely fast dynamics in the generators is equivalent to assume that their ramp capacities are equal to the distance between the maximum and minimum generation capacities $\overline{q}^i$ and $\underline{q}^i$, respectively. Thus, we can pose (6) in the following equivalent state-space form:

$$\min_{q_t^i, \Delta q_t^i} \quad \sum_{t \in \mathcal{T}_k} \bar{\varphi}_t := \sum_{t \in \mathcal{T}_k} \sum_{i \in \mathcal{S}} \int_0^{q_t^i} p_t(q, b_t^i) dq \tag{8a}$$

s.t.

$$q_{t+1}^i = q_t^i + \Delta q_t^i, \ i \in \mathcal{S}, \ t \in \mathcal{T}_k^- \tag{8b}$$

$$\sum_{i \in \mathcal{S}} q_t^i \geq \sum_{j \in \mathcal{C}} d_t^j, \ t \in \mathcal{T}_k \tag{8c}$$

$$-(\overline{q}^i - \underline{q}^i) \leq \Delta q_t^i \leq (\overline{q}^i - \underline{q}^i), \ i \in \mathcal{S}, t \in \mathcal{T}_k^- \tag{8d}$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i, \ i \in \mathcal{S}, t \in \mathcal{T}_k \tag{8e}$$

$$q_k^i = \hat{q}_k^i, \ i \in \mathcal{S}. \tag{8f}$$

The variables Δq_t^i are the generation ramp increments that are bounded by $\pm(\overline{q}^i - \underline{q}^i)$, the maximum generation ramp that is physically possible. We will see in the following proposition that it is possible to drop the dynamic constraints (8b). Hence, the feasible set is invariant to the initial state of the suppliers $\hat{q}_k^i$. Accordingly, the feasible set will be denoted as $\Omega_{UNC}^{ISO}(\hat{q}_k^i)$ or Ω_{UNC}^{ISO}.

Proposition 1 *Problems* (6) *and* (8) *are equivalent.*

Proof: The unconstrained problem (6) generates optimal trajectories $\{q_t^i\}, \ i \in \mathcal{S}$. Since $\underline{q}^i \leq q_t^i \leq \overline{q}^i, \ t \in \mathcal{T}_k$, we have $-(\overline{q}^i - \underline{q}^i) \leq q_{t+1}^i - q_t^i \leq (\overline{q}^i - \underline{q}^i), t \in \mathcal{T}_k^-$. Moreover, this trajectory is invariant to the initial states $\hat{q}_k^i$ since $\underline{q}^i \leq \hat{q}_k^i \leq \overline{q}^i$. For problem (8), since the ramp increments Δq_t^i are bounded by $\pm(\overline{q}^i - \underline{q}^i)$, the optimal trajectories of (6) can be reached from any initial condition $\hat{q}_i^k$. This is equivalent to removing the variables Δq_t^i, dynamic constraints (8b), and initial conditions (8f). □

The prices resulting from the ideal clearing problem are the multipliers of the clearing condition (8c) and will be denoted as $\bar{p}_t$. Since these *ideal price signals* are not affected by volatility effects introduced by ramping constraints, they can be used as a reference in assessing the volatility of the actual physically constrained market. We also note that, in the absence of the capacity constraints (8e), the prices balance the suppliers marginal costs. In this case, the optimality conditions for the ISO (34) reduce to,

$$p_t = \frac{1}{b_t^i} q_t^i, \ t \in \mathcal{T}_k, \ i \in \mathcal{S} \tag{9a}$$

$$0 \leq p_t \perp \sum_{i \in \mathcal{S}} q_t^i - \sum_{j \in \mathcal{C}} d_t^j \geq 0, \ t \in \mathcal{T}_k. \tag{9b}$$

If supply matches demand, the price is given by,

$$p_t = \frac{\sum_{j \in \mathcal{C}} d_t^j}{\sum_{i \in \mathcal{S}} b_t^i}, \ t \in \mathcal{T}_k. \tag{10}$$

From (9a) we can see that the price balances the marginal costs for the suppliers. In the presence of capacity constraints, however, the bound multipliers $\underline{\nu}_t^i, \overline{\nu}_t^i$ in (34a) can become non-zero, leading to non-smooth changes in the price away from the marginal costs.

The solution of the ideal ISO clearing problem with no ramp or capacity constraints represents the *maximum possible performance of the system* and leads to the most *stable* price signals given by those balancing the marginal costs. As can be seen, physical constraints induce market friction and volatility into the price signals that make prices drift away from the suppliers marginal costs [7,8]. Also note that, even in the absence of physical constraints, prices exhibit dynamics induced by the demand dynamics. Consequently, there is no steady-state equilibrium for the prices.

We now consider the *constrained* market clearing problem:

$$\min_{q_t^i, \Delta q_t^i} \sum_{t \in \mathcal{T}_k} \varphi_t := \sum_{t \in \mathcal{T}_k} \sum_{i \in \mathcal{S}} \int_0^{q_t^i} p_t(q, b_t^i) dq \tag{11a}$$

$$\text{s.t. } q_{t+1}^i = q_t^i + \Delta q_t^i, \; i \in \mathcal{S}, t \in \mathcal{T}_k^- \tag{11b}$$

$$\sum_{i \in \mathcal{S}} q_t^i \geq \sum_{j \in \mathcal{C}} d_t^j, \; t \in \mathcal{T}_k \tag{11c}$$

$$-\underline{r}^i \leq \Delta q_t^i \leq \overline{r}^i, \; i \in \mathcal{S}, t \in \mathcal{T}_k^- \tag{11d}$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i, \; i \in \mathcal{S}, t \in \mathcal{T}_k \tag{11e}$$

$$q_k^i = \hat{q}_k^i, \; i \in \mathcal{S}. \tag{11f}$$

The multipliers for the constraint (11c) are the prices $p_t \geq 0$. In this formulation, the ramps are bounded by $\underline{r}^i, \overline{r}^i \leq (\overline{q}^i - \underline{q}^i)$, respectively. This constrains the dynamic response of the generators. As before, we note that the bidding parameters b_t^i enter only the cost function and thus do not affect the feasible set. In this case, however, the dynamic constraints introduce time coupling because the ramp constraints might become active. Consequently, the feasible set does depend on the initial conditions $\hat{q}_k^i$. Accordingly, the feasible set of this problem will be denoted as $\Omega^{ISO}(\hat{q}_k^i)$.

The constrained social welfare is denoted as $\sum_{t \in \mathcal{T}_k} \varphi_t$ with $\varphi_t \geq 0$ since $b_t^i, q_t^i \geq 0$. It is easy to prove that $\sum_{t \in \mathcal{T}_k} \varphi_t \geq \sum_{t \in \mathcal{T}_k} \bar{\varphi}_t$ since $\Omega^{ISO}(\hat{q}_k^i) \subseteq \Omega_{UNC}^{ISO}(\hat{q}_k^i)$. In other words, the performance of the constrained clearing problem is bounded by that of the unconstrained counterpart. It is not obvious, however, that $\varphi_t \geq \bar{\varphi}_t$ holds point wise since the constrained problem (11) exhibits time coupling. We prove this in a different way in the following proposition.

Proposition 2 *For fixed $b_t^i > 0$, the point social welfare φ_t evaluated at a solution of problem* (11) *and $\bar{\varphi}_t$ evaluated at a solution of* (6) *satisfy $\varphi_t \geq \bar{\varphi}_t$, $t \in \mathcal{T}_k$. Moreover, equality holds only if the ramp constraints are non binding.*

Proof: At $t = k$ we have that $\varphi_k = \bar{\varphi}_k$ since the initial conditions $\hat{q}_k^i$ are fixed. The unconstrained cost $\bar{\varphi}_{k+1}$ is invariant to the state of the current time step since there are no ramp constraints. Consequently, there does not exist a feasible combination of increments Δq_k^i that can reach a feasible state q_{k+1}^i satisfying $\varphi_{k+1} < \bar{\varphi}_{k+1}$. Using induction over $t = k, ..., k+T$, we have that $\varphi_t \geq \bar{\varphi}_t$ point wise with equality if and only if the ramp constraints are non-binding. □

We now formally define the *point efficiency* η_t as

$$\eta_t := \frac{\bar{\varphi}_t}{\varphi_t} = 1 - \frac{\varphi_t - \bar{\varphi}_t}{\varphi_t}, \quad t \in \mathcal{T}_k. \tag{12}$$

By definition and from Proposition 2, we have that $\eta_t \in [0, 1]$. The case where $\eta_t = 1$ is achieved if and only if $\varphi_t = \bar{\varphi}_t$. The case where $\eta_t = 0$ occurs if and only if the constrained social welfare diverges to infinity. This case can occur, for instance, when the demands cannot be met given the current states the generators and the ramping constraints. This implies that the constrained prices p_t diverge (i.e., a small change in demand leads to large changes in price). Based on the definition of efficiency, we can see that maximizing efficiency is equivalent to minimizing the negative social welfare. While this is an obvious result from a modeling point of view, we will see that adding the efficiency definition in the market clearing problem (11) is advantageous if one seeks to stabilize prices.

In the following, by market stability we will imply *price stability* which we measure in terms of the distance between the prices of the constrained and unconstrained market clearing problems $|p_t - \bar{p}_t|$. In other words, our hope is that, by keeping η_t stable or bounded by keeping the costs close to the costs of the unconstrained problem, we can keep the distance $|p_t - \bar{p}_t|$ stable as well. This strategy is motivated from the fact that the prices of the problems defining φ_t and $\bar{\varphi}_t$ coincide when the ramp constraints are non-binding. In the following section we will also see that, if the solution of the game is stable, then the price difference can be bounded above by the cost difference. Note that it is also possible to define the reference price $\bar{p}_t$ as that given by the clearing problem with no ramping and capacity constraints (the price that balances the marginal costs). While this can significantly affect the actual magnitude of efficiency (because capacity constraints are often active), it does not affect the generality of our results.

We highlight the relevance of using the proposed efficiency metric as an indirect way to stabilize prices since prices are derived quantities (dual variables) of the clearing procedure problem. Formulating clearing procedures to shape dual variables is in general extremely difficult so we prefer to do this implicitly. We also highlight that maximizing the proposed efficiency metric seeks to stabilize the prices by keeping the social welfare close to that of the unconstrained counterpart (i.e., it serves as a measure of the effect of physical constraints) and does not necessarily have a direct connection with other efficiency metrics used in economic studies such as allocative and productive efficiencies [16, 3]. The proposed efficiency metric was derived based on the observation that physical constraints (capacity, ramping, congestion) introduce market friction and thus lead to volatility [8]. Our metric implicitly tries to quantify the effect of physical constraints. In this sense, our efficiency notion is related to that used in [23, 2] in which efficiency implies that prices stay close to a given reference trajectory and predictable. In actual markets, for instance, ISOs use the deviation of spot markets from day-ahead markets as an efficiency metric [23]. The idea is that, if the deviation is high, speculation can arise leading to low market participation and other undesired effects. In addition, large price deviations might point toward manipulation practices [7].

Our analysis does not considered network constraints in order to simplify the presentation. However, the definition of efficiency can account for other physical constraints limiting performance with respect to the unconstrained clearing problem (6).

3 Game Numerical Stability

In this section, we provide conditions guaranteeing that the distance in performance between the constrained and unconstrained games can be bounded by the magnitude of the ramp limits. In addition, we establish an upper bound on the distance of the prices as a function of the distance in cost. The analysis also allows us to bound the equilibrium solution of the game under parametric perturbations of exogenous factors such as demands or renewable generation. This will be necessary in our dynamic stability results of Section 5.

Using the equivalence between (8) and (6), we have that the only difference between problems (8) and (11) are the ramp lower and upper bounds. Consequently, problem (11) results from parameter embedding of (8) with perturbations on the ramp limits of the form $\overline{r}^i - (\overline{q}^i - \underline{q}^i)$ and $\underline{r}^i - (\overline{q}^i - \underline{q}^i), i \in \mathcal{S}$. Using this observation we can analyze stability for the solution of the unconstrained game under perturbations of the ramp limits and other parameters such as demands.

A way to establish *numerical* stability of the game equilibrium solution is to ensure that the mapping matrix of the variational inequality resulting from coupling the optimality conditions of the ISO (11) and suppliers (3) problems is nonsingular at a given solution [24]. By coupling the block KKT systems (32) and (37) derived in the Appendix at a given solution of the game $q^s_*, \nu^s_*, b_*, q^I_*, p_*, \nu^I_*$, we obtain

$$\begin{bmatrix} G & A^s_\nu & 0 & 0 & A^s_p & 0 \\ A^s_\nu & 0 & 0 & 0 & 0 & 0 \\ I & 0 & P & 0 & BH & 0 \\ 0 & 0 & -B^{-2} & B^{-1} & {A^I_p}^T & {A^I_\nu}^T \\ 0 & 0 & 0 & {A^I_p}^T & 0 & 0 \\ 0 & 0 & 0 & {A^I_\nu}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} q^s \\ \nu^s \\ b \\ q^I \\ p \\ \nu^I \end{bmatrix} = \begin{bmatrix} r_q \\ r^s_\nu \\ r_b \\ r^I_q \\ r_p \\ r^I_\nu \end{bmatrix}$$

Lemma 1 *Consider the following block matrix,*

$$J = \begin{bmatrix} G & A^s_\nu & 0 & 0 & A^s_p & 0 \\ A^s_\nu & 0 & 0 & 0 & 0 & 0 \\ I & 0 & P & 0 & BH & 0 \\ 0 & 0 & -B^{-2} & B^{-1} & {A^I_p}^T & {A^I_\nu}^T \\ 0 & 0 & 0 & {A^I_p}^T & 0 & 0 \\ 0 & 0 & 0 & {A^I_\nu}^T & 0 & 0 \end{bmatrix}$$

If the following assumptions hold,

[A1] The matrices G, P, B are square and diagonal and have the same dimensions.
[A2] The matrix $[A^I_p\ A^I_\nu]$ has full row rank.
[A3] $\lambda_{min}(G)$,$\lambda_{min}(P)$, and $\lambda_{min}(B)$ are bounded away from zero.

Then, the matrix J is nonsingular.

Proof: From assumption [A1] we have that G and P are invertible. We define

$$\begin{bmatrix} \bar{G} & 0 \\ C & P \end{bmatrix} = \begin{bmatrix} G & A_\nu^s & 0 \\ A_\nu^s & 0 & 0 \\ I & 0 & P \end{bmatrix}$$

with $C = [I\ 0]$ and

$$\bar{G} = \begin{bmatrix} G & A_\nu^s \\ A_\nu^s & 0 \end{bmatrix}.$$

We have the following algebraic relationship,

$$\begin{bmatrix} \bar{G} & 0 \\ C & P \end{bmatrix}^{-1} = \begin{bmatrix} \bar{G}^{-1} & 0 \\ -P^{-1}C\bar{G}^{-1} & P^{-1} \end{bmatrix}.$$

We construct the Schur complement of the upper 2×2 block of blocks, which we denote by S

$$S = \begin{bmatrix} B^{-1} & {A_p^I}^T & {A_\nu^I}^T \\ A_p^I & 0 & 0 \\ A_\nu^I & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -B^{-2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{G}^{-1} & 0 \\ -P^{-1}C\bar{G}^{-1} & P^{-1} \end{bmatrix} \begin{bmatrix} 0 & A_p^s & 0 \\ 0 & BH & 0 \end{bmatrix}$$
$$= \begin{bmatrix} B^{-1} & {A_p^I}^T & {A_\nu^I}^T \\ A_p^I & 0 & 0 \\ A_\nu^I & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \left[B^{-2}P^{-1}C\bar{G}^{-1}A_p^s - B^{-1}P^{-1}H\right] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} B^{-1} & \left({A_p^I}^T - B^{-2}P^{-1}C\bar{G}^{-1}A_p^s + B^{-1}P^{-1}H\right) & {A_\nu^I}^T \\ A_p^I & 0 & 0 \\ A_\nu^I & 0 & 0 \end{bmatrix}$$

From the properties of Schur complements (and of determinants) it follows that J is invertible if and only if S is invertible. Using the Schur complement argument again, we obtain that since B (Assumption [A1]) is invertible, then S is invertible if and only if the Schur complement

$$T = \begin{bmatrix} A_p^I \\ A_\nu^I \end{bmatrix} B \begin{bmatrix} A_p^I \\ A_\nu^I \end{bmatrix}^T + \begin{bmatrix} A_p^I \\ A_\nu^I \end{bmatrix} \left[(-B^{-1}P^{-1}C\bar{G}^{-1}A_p^s + P^{-1}H)\ 0 \right]$$

is invertible. From [A1] and [A2] it follows that the first matrix has eigenvalues bounded away from 0. Since the matrices $A_p^I, A_\nu^I, \bar{G}^{-1}, A, H$ are fixed it follows that since $P^{-1}B^{-1}$ and P^{-1} are sufficiently small from [A3], then the matrix T is positive definite (even if not symmetric) and thus invertible. The conclusion follows. $\Box$.

We now establish conditions for numerical stability of the game equilibrium solution created by the coupled solution of (3) and (11).

Theorem 1 *Let J be the KKT matrix of the game* (3) *and* (11)*, reduced after the elimination of the inactive variables, at a given solution. Then, if at solution of the game each of the optimization problems satisfies the linear independence constraint qualification (LICQ) and the prices p_{t*}, $t \in \mathcal{T}$ and the production q^i_{t*}, $t \in \mathcal{T}$, $i \in \mathcal{S}$ values are bounded away from zero, then J is invertible.*

Proof: We show that the assumptions of Lemma 1 are satisfied. $\bar{G}$ is nonsingular from the assumption that LICQ holds for (3) and the fact that its objective function is strongly convex which makes $\lambda_{min}(G) > 0$ hold. Assumption [A1] follows from the problem structure. Assumption [A2] follows from the fact that the ISO problem (11) satisfies LICQ. Finally, [A3] follows as long as $p_{t*} \cdot b^i_{t*}$ and p_{t*} are sufficiently bounded away from zero. From (3b) the first is equivalent with q^i_{t*} being sufficiently bounded away from zero. The result follows. □

We note that our stability condition is necessary but not sufficient for the stability of the solution of the resulting variational inequality [11,24], at least not in the general case. The typical result for stability involves a P-property, which is a more general case and more difficult to prove. Nevertheless, in the case where strict complementarity holds, the variational inequality defining the optimality conditions is locally equivalent with a nonlinear equation (after the elimination of the inactive constraints), and the nonsingularity of the reduced Jacobian as stated in Theorem 1 is sufficient for local stability of the solution [11].

The stability result gives us an interesting insight, which is that among the cases for which we can guarantee numerical stability of the game are the cases where the prices and production levels are *sufficiently high*. We highlight that degeneracies might occur, for instance, if ramp down and lower capacity constraints lead to excess supply and thus drive the price to zero. In this case, the solution of the game will be highly sensitive to parametric perturbations. Assuming stability of the solution of the underlying variational inequality given by the game, we can establish the following result that bounds the distance between the solution of the unconstrained and constrained games. To simplify notation, we define $\varphi_* = \sum_{t \in \mathcal{T}_k} \varphi_{t,*}$, $p_*^T = [p_{k,*} .... p_{k+T,*}]^T$, and $\eta_*^T = [\eta_{k,*}, ..., \eta_{k+T,*}]$ as the solution of the constrained game over horizon $t \in \mathcal{T}_k$. Similarly, we define the solution for the unconstrained game as $\bar{\varphi}_*$, $\bar{p}_*$, and $\bar{\eta}_* = 1$.

Theorem 2 *Assume that a solution of the unconstrained game $\bar{\varphi}_*, \bar{p}_*$ given by* (5) *and* (6) *is stable. Then, there there exist Lipschitz constants $L_\varphi, L_p \geq 0$ such that the solution of the constrained game φ_*, p_* given by* (5) *and* (11) *satisfies,*

$$|\varphi_* - \bar{\varphi}_*| \leq L_\varphi \sum_{i \in \mathcal{S}} \left(|\overline{r}^i - (\overline{q}^i - \underline{q}^i)| + |\underline{r}^i - (\overline{q}^i - \underline{q}^i)| \right)$$

$$\|p_* - \bar{p}_*\| \leq L_p \sum_{i \in \mathcal{S}} \left(|\overline{r}^i - (\overline{q}^i - \underline{q}^i)| + |\underline{r}^i - (\overline{q}^i - \underline{q}^i)| \right).$$

Proof: The result is immediate from stability of the equilibrium solution which leads to local invertibility [24] and from the fact that the constrained game is a parametric embedding of the unconstrained counterpart. □

Since the cost difference is bounded so is the efficiency difference. The above result is of relevance since it establishes an upper bound for the price difference $\|p_* - \bar{p}_*\|$ as a function of the ramp limits. In addition, it gives the asymptotic result $p_* \to \bar{p}_*$ and $\varphi_* \to \bar{\varphi}_*$ as the ramps are relaxed. To relate the price difference to the cost difference and efficiency we note that, if the solution of the game is stable, then we have that there exists constant $C_p \geq 0$ such that,

$$\|p_* - \bar{p}_*\| \leq C_p \|y_* - \bar{y}_*\|, \tag{13}$$

where $y_*, \bar{y}_*$ are the primal solutions of the constrained and unconstrained games, respectively. This bound can be obtained by perturbing the feasible region of the unconstrained problem by a factor $O(\|y_* - \bar{y}_*\|)$ corresponding to the solution y_* and then exploit the Lipschitz continuity property of the primal and dual variables with respect to the perturbation resulting from stability. Also, if stability holds, we have that the following quadratic growth condition [6] holds,

$$\|y_* - \bar{y}_*\| \leq C_\varphi \sqrt{|\varphi_* - \bar{\varphi}_*|} \tag{14}$$

and,

$$\|p_* - \bar{p}_*\| \leq C_p C_\varphi \sqrt{|\varphi_* - \bar{\varphi}_*|}. \tag{15}$$

Thus as, can be seen, as the cost difference decreases the efficiency increases and the price difference decreases. Consequently, maximizing efficiency is a consistent way of stabilizing the price difference.

4 Market Implementation Issues

To represent the game given by (3) and (11) in abstract form, we define the market states x_k as the set of quantities q_k^i and prices p_k and define the aggregated vector over the set $\mathcal{T}_k$ as $x_{\mathcal{T}_k} := \{x_k, ..., x_{k+T}\}$. The controls u_k are defined as the set of ramps for all suppliers $\Delta q_k^i, i \in \mathcal{S}$ with $u_{\mathcal{T}_k} = \{u_k, ..., u_{k+T-1}\}$. The bidding increments Δb_k^i are interpreted as the supplier controls and are denoted as w_k^i, and we define $w_k := \{w_k^1, ..., w_k^S\}$. We define the disaggregated supplier vectors $w_{\mathcal{T}_k}^i, i \in \mathcal{S}$, and the total aggregated vector $w_{\mathcal{T}_k}$. The bidding states b_k^i are interpreted as the supplier states z_k with aggregated vector $z_{\mathcal{T}_k}$. We include the problem data over the horizon (e.g., the demands) in the aggregated vector $m_{\mathcal{T}_k}$. We define the abstract dynamic system as

$$(x_{k+1}, z_{k+1}) = \psi_k(x_k, z_k, u_k, w_k), \; \forall\, k \geq 0. \tag{16}$$

We can eliminate the states x_k, z_k by forward propagation of (16). With this, we can express the supplier and market clearing problem entirely in terms of the controls and initial state conditions. We thus have the supplier problem,

$$\min_{w_{\mathcal{T}_k}} \sum_{t \in \mathcal{T}_k} \phi_t^i(w_t^i, u_t) \tag{17a}$$

$$\text{s.t. } w_{\mathcal{T}_k}^i \in \Omega^i, \tag{17b}$$

for $i \in \mathcal{S}$ and the constrained market clearing problem,

$$\min_{u_{\mathcal{T}_k}} \sum_{t\in\mathcal{T}_k} \varphi_t(u_t, w_t) \tag{18a}$$

$$\text{s.t. } u_{\mathcal{T}_k} \in \Omega^{ISO}(x_k, m_{\mathcal{T}_k}). \tag{18b}$$

Since the decisions of the players do not affect each others feasible sets, the resulting game is a pure Nash equilibrium problem [10].

For implementation, the game given by (17) and (18) can be solved over a receding horizon: At time k we use the forecast data $m_{\mathcal{T}_k}$ (e.g., demands $d_t^j,\ t \in \mathcal{T}_k = \{k..k+T\}$) and the current states x_k, z_k. We solve the game (17) and (18) over the horizon $\mathcal{T}_k$ to obtain $u^*_{\mathcal{T}_k}, w^*_{\mathcal{T}_k}$. From these sequences, we extract only the first actions $u_k \leftarrow u^*_k, w_k \leftarrow w^*_k$. The system will evolve from its current state x_k, z_k into the states x_{k+1}, z_{k+1} according to the model (16). In the nominal case (no forecast errors in the data $m_{\mathcal{T}_k}$), the state will evolve as predicted. At the next step $k+1$, we introduce feedback in the market by shifting the horizon of the game to obtain $\mathcal{T}_{k+1} \leftarrow \{k+1..k+T+1\}$ and use the new state x_{k+1}, z_{k+1} as initial conditions. The new data $m_{\mathcal{T}_{k+1}}$ is forecast and the game problem is solved to obtain the new decisions u_{k+1}, w_{k+1}. This approach generates the feedback law $(u_k, w_k) = h(x_k, z_k, m_{\mathcal{T}_k})$.

A key observation that we make in this work is that existing market clearing procedures solve the game *incompletely* by iterating once between the suppliers and the ISO in a distributed manner [23,3]. Here, each supplier guesses the ISO decisions (e.g., prices) in coming up with their bids. This guess is denoted by $u_{\mathcal{T}_k,\ell}$, where ℓ is an iteration counter. The suppliers compute bidding parameters $w_{\mathcal{T}_k,\ell}$ by solving (17). These are sent to the ISO to solve the market clearing problem (18) to compute the clearing prices $u_{\mathcal{T}_k,\ell+1}$. This can be interpreted as a hybrid Jacobi/Gauss-Seidel iteration.

The iterate $u_{\mathcal{T}_k,\ell+1}, w_{\mathcal{T}_k,\ell}$ is feasible but not optimal for the game (it is not an equilibrium solution). In other words, this iteration is a *limited coordination* or incomplete gaming solution. Feasibility follows since the suppliers decisions $w_{\mathcal{T}_k}$ do not enter the feasible set $\Omega^{ISO}(\cdot)$ and since the supplier problems always have a feasible solution for any decisions of the ISO $u_{\mathcal{T}_k}$. A key observation is that *the resulting solution error generated at each step* is propagated forward in time through the initial states and thus introduces additional dynamics into the market that can further destabilize the market. For instance, the suboptimal solution obtained at time k might place the generators at a future state $k+1$ from which the future demands at times $k+1...k+1+T$ cannot be reached, thus making the game at time $k+1$ infeasible.

We highlight that the limited coordination between ISO and suppliers can introduce high levels of speculation; particularly if the prices are highly volatile and thus cannot be predicted accurately or if residual demands cannot be anticipated correctly (e.g., renewable supply or residual demands are highly uncertain). This can result in spurious bids from the suppliers that can introduce further price volatility into the market. Performing extra *coordinating* iterations can help to ameliorate this problem; particularly in periods of high uncertainty. Effectively, this is a way of mitigating uncertainty.

5 Dynamic Stability Analysis

In this section, we merge the previous results and create a framework to design clearing strategies capable of ensuring market stability. Traditional control-theoretic stability analysis concepts, however, are not directly applicable in our context because the market is inherently dynamic and does not exhibit a natural steady-state. While it is possible to design market clearing procedures (these can be viewed as *market controllers*) that artificially introduce equilibria (i.e., by enforcing periodicity in some form), this strategy can constrain and degrade market performance and participation. New stability analysis concepts are thus needed to enable a systematic design, analysis, and implementation of *robust* market clearing procedures that can sustain intentional or unintentional market manipulation and strong dynamic variations of demands and renewable supply. In this section, we take a first step toward this goal by making use of a Lyapunov stability framework.

We can express efficiency as an implicit function of the states of the form $\eta_k(x_k, z_k)$ or η_k for short-hand notation. Here, we use the following definition of market stability.

Definition 1 The market system defined by the game (17) and (18) is said to be *stable* if, given $\eta_0 \in \Omega^\eta(\epsilon) := \{\eta \,|\, \eta \geq \epsilon\}$ with $\epsilon \in [0, 1]$, there exist feasible sequences u_k, w_k over $k = 0..\infty$ such that $\eta_k \in \Omega^\eta(\epsilon)$.

Here, ϵ is an *efficiency threshold value* that is selected to keep the price distance $|p_t - \bar{p}_t|$ bounded. In other words, to ensure market stability, we want the efficiency to stay within the domain $[\epsilon, 1]$. This will guarantee that the price distances to the reference trajectories of the ideal market will remain bounded. We note that efficiency is a state derived from the system physical states.

To construct a Lyapunov function in terms of the efficiency, we now define the *summarizing market state*:

$$\delta_{k+1} := (1 - (\eta_{k+1} - \epsilon)) \cdot \delta_k, \; k = 0..\infty, \tag{19}$$

with initial conditions $\delta_0 \geq \alpha > 0$. If $\eta_k(\cdot, \cdot) \geq \epsilon$, $k = 0..\infty$, then for any $\alpha > 0$ such that $\delta_0 \geq \alpha$ there exists $\kappa \geq 0$ such that $\delta_k \to \kappa$ for all $k = 0..\infty$. In other words, the summarizing market state has a stable origin. Stability of this origin implies market stability in the sense of Definition 1. On the other hand, if at any step we have $\eta_k(\cdot, \cdot) < \epsilon$, the summarizing market state will increase. Subsequent violations of the efficiency threshold will make the summarizing state diverge from the origin. Using this construct, we are able to use traditional Lyapunov analysis techniques for predictive control.

We note that efficiency can be detected through the summarizing states δ_{k+1}, δ_k. Consequently, the states x_k, z_k are detectable. This implies that the summarizing state δ_k is controllable. For clarity, we summarize the sequence of dependencies as follows.

- The states x_k, z_k and the data $m_{\mathcal{T}_k}$ up to time k, define $\eta_k(x_k, z_k)$ and δ_k.
- The control actions can be computed to give the $(u_k, w_k) = h(x_k, z_k, m_{\mathcal{T}_k}) := \tilde{h}(\delta_k)$.
- The states evolve as (16)

$$\begin{aligned}(x_{k+1}, z_{k+1}) &= \psi(x_k, z_k, h(x_k, z_k, m_{\mathcal{T}_k})) \\ &= \tilde{\psi}(x_k, z_k, m_{\mathcal{T}_k}).\end{aligned}$$

This defines $\eta_{k+1}(\tilde{\psi}(x_k, z_k, m_{\mathcal{T}_k}))$.
- The summarizing state evolves as

$$\delta_{k+1} = \Big(1 - \Big(\eta_{k+1}(\tilde{\psi}(x_k, z_k, m_{\mathcal{T}_k})) - \epsilon\Big)\Big) \cdot \delta_k$$
$$:= f(\delta_k, \tilde{h}(\delta_k)) := \tilde{f}(\delta_k).$$

Using this basic set of definitions, we now illustrate how to establish sufficient stability conditions for a given market design (market clearing procedure). In addition, we demonstrate that the current market design given by the incomplete solution of the game (17) and (18) is not stabilizing.

We extend the market clearing problem (18) by making use of the definition of the summarizing state as follows.

$$\min_{u_{\mathcal{T}_k}} \sum_{t \in \mathcal{T}_k^-} (\delta_{t+1} - \delta_t) \tag{20a}$$
$$\text{s.t. } u_{\mathcal{T}_k} \in \Omega^{ISO}(\hat{x}_k) \tag{20b}$$
$$\delta_{t+1} = (1 - (\eta_{t+1} - \epsilon)) \cdot \delta_t,\ t \in \mathcal{T}_k^- \tag{20c}$$
$$\eta_t \geq \epsilon,\ t \in \mathcal{T}_k \tag{20d}$$
$$\delta_k = \hat{\delta}_k, \tag{20e}$$

The detailed formulation of this problem is presented in the Appendix. The objective function of this market clearing problem will be used as a *summarizing market function*, which we define formally as

$$V_T(\delta_k) := -\sum_{t \in \mathcal{T}^-} (\delta_{t+1} - \delta_t) = (\delta_k - \delta_T). \tag{21}$$

Here, the subscript T indicates the length of the horizon. The solution of the game (17) and (20) provides the feedback law $(u_k, w_k) = \overline{h}(\delta_k)$. A crucial observation is that the summarizing market function can be used as a Lyapunov function that use to establish stability of the origin for the summarizing state δ_k.

Definition 2 A function $V_T(\delta_k)$ is a Lyapunov function for system $\delta_{k+1} = f(\delta_k, \overline{h}(\delta_k))$ if (1) it is positive definite: in a region Ω containing the origin if for $\delta_k \in \Omega$ we have $V_T(\delta_k) \geq 0$ for $\delta_k \geq 0$ for all k, and (2) it is non increasing: $\Delta V_T(\delta_k) := V_T(\delta_{k+1}) - V_T(\delta_k) \leq 0$, for all k.

We highlight that the proposed Lyapunov framework is a non-traditional extension of existing techniques. This has been motivated by the fact that the market system does not have a natural equilibrium which is a common characteristic among economic systems. The design of alternative Lyapunov analysis tools in non-traditional control settings is the subject of active research [14,9]. In this work, we have taken a step backwards and use the Lyapunov framework as a systematic construct to compare different market designs and to explain and quantify the impact of different parameters on market stability (constraints, forecast horizons, limited coordination, robustness, and so on). This is demonstrated in the following results.

5.1 Infinite Horizon

We now establish market stability using the market clearing cost as a Lyapunov function.

Theorem 3 *If the game given by* (17) *and* (20) *has a feasible solution, the summarizing cost function* (21) *with infinite horizon* $T=\infty$ *is a Lyapunov function and the market is stable.*

Proof: From feasibility of (20d) we have that $-(\delta_{t+1}-\delta_t)\geq 0,\ t=0,...,\infty$ so $V_\infty(\delta_k)=\sum_{t=k}^{\infty}-(\delta_{t+1}-\delta_t)\geq 0$. Consequently, positive definiteness follows. To prove that the function is non increasing, we consider the cost function of two consecutive problems generating two trajectories $\delta_t^k,\ t\in\{k..\infty\}$ and $\delta_t^{k+1},\ t\in\{k+1..\infty\}$, $\delta_k^k=\delta_k$ and $\delta_{k+1}^{k+1}=\delta_{k+1}$. We then have

$$\begin{aligned}\Delta V_\infty(\delta_k) &= V_\infty(\delta_{k+1})-V_\infty(\delta_k)\\ &= \sum_{t=k+1}^{\infty}(\delta_{t+1}^{k+1}-\delta_t^{k+1})-\sum_{t=k}^{\infty}(\delta_{t+1}^{k}-\delta_t^{k})\\ &= (\delta_{k+1}-\delta_k)\\ &= (1-(\eta_{k+1}-\epsilon))\cdot\delta_k-\delta_k\\ &= -(\eta_{k+1}-\epsilon)\cdot\delta_k\\ &\leq 0.\end{aligned}$$

The third equality follows from Bellman's principle of optimality [20]. The last inequality follows from feasibility. The proof is complete. □

With this, we have established that the decay of the proposed summarizing function is a sufficient condition for market stability in the sense of Definition 1. We note that if at any point we have that $\eta_{k+1}<\epsilon$, then $\delta_{k+1}>\delta_k$, and the decay condition will not hold.

A crucial observation in our analysis is the need of the incorporation of the stabilizing constraint (20d) in the clearing procedure. With this, the feasible set of the market clearing problem depends on the bidding states of the suppliers. A consequence is that the ISO and the suppliers might need to iterate several times to obtain a feasible solution to the game. Another consequence of this analysis is the fact that the existing market design where a single iterate is performed between the ISO and the suppliers *cannot be guaranteed to be stable* in the sense of Definition 1 since not every set of bidding parameters can be guaranteed to lead to a market clearing solution satisfying the stabilizing constraint. In other words, the stabilizing constraint acts as a *filter* that can be effectively used to avoid spurious bids or to determine appropriate clearing procedures (e.g., forecast horizons). Note that the current market design does not enable the ISO to correct the bidding quantities to stabilize the market. Hence, the market is more prone to be manipulated and destabilized by the suppliers if they do not have appropriate means to anticipate the ISO clearing prices (e.g., by price forecasting). For instance, if demands deviate significantly from expected conditions, bidding parameters can be inaccurate and can lead to strong price fluctuations. Finding a feasible solution to the game (17) and (20) with the stabilizing constraint avoids these problems. As can be seen, the proposed construct provides a mechanism to design and analyze market designs with stability guarantees.

5.2 Finite Horizon

An issue arising in actual market implementations is the fact that the market clearing problem is normally solved over a finite receding horizon $\mathcal{T}_k = \{k..k+T\}$. Hence, even if the infinite horizon game is feasible, the solution of the receding horizon game cannot be guaranteed to be feasible. Guaranteeing stability in this case requires the existence of a stable *terminal controller* able to stabilize the summarizing state beyond the current terminal time $k+T$ [20]. Constructing such a controller in a systematic manner remains in the proposed framework an open research question. It is possible, however, to establish sufficient stability conditions for finite horizon controllers. We consider the perturbation term,

$$\Xi_k^1 := |V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) - V_{T-1}(\delta_{k+1}, m_{\mathcal{T}_k})| \tag{22}$$

and we make explicit the dependence of the cost function on the data. We make the following assumption

Assumption 4 *The horizon T is sufficiently long such that there exists a finite $\alpha_T \geq 0$ satisfying $\Xi_k^1 \leq \alpha_T,\ \forall k$.*

Note that $\Xi_k^1 \to 0$ as $T \to \infty$ since the cost function $V_T(\cdot,\cdot)$ is positive definite.

Theorem 5 *Assume that the game defined* (17) *and* (20) *has a feasible solution* $\forall\, k$ *under the horizon T. If,*

$$\Xi_k^1 \leq (\delta_k - \delta_{k+1}), \forall\, k, \tag{23}$$

and Assumption 4 holds, then the market is stable.

Proof: From feasibility, the cost function is positive definite. To prove that it is nonincreasing under (23) we establish the following:

$$\begin{aligned}
&V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) - V_T(\delta_k, m_{\mathcal{T}_k}) \\
&\quad = -(\delta_k - \delta_{k+1}) + V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) - V_{T-1}(\delta_{k+1}, m_{\mathcal{T}_k}) \\
&\quad \leq 0.
\end{aligned}$$

From feasibility, we have that the lower bound $0 \leq (\delta_k - \delta_{k+1})$. An upper bound is given as follows. The difference between two terms in the right-hand side is bounded by the positive quantity α_T given in Assumption 4. In addition, since $V_T(\cdot,\cdot)$ is positive definite and nonincreasing the term in the left-hand side is negative. Rearranging terms in the above inequality, we have

$$(\delta_k - \delta_{k+1}) \leq V_T(\delta_k, m_{\mathcal{T}_k}) - V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) + \alpha_T.$$

The term $(\delta_k - \delta_{k+1})$ is bounded above by a positive quantity. This implies that $\delta_{k+1} \leq \delta_k$. Consequently, the sequence $\{\delta_k\}$ is bounded, and the conclusion follows. □

Current ISO operations have started to deploy look-ahead dispatch solution in trying to mitigate ramp effects and market volatility [29]. As can be seen, our analysis provides a framework to assess the effect of the look-ahead horizon on price stability. In particular, Theorem 5 provides a framework to compute estimates of Ξ_k^1 as a function of T to satisfy the decay of the Lyapunov function.

We note that the term $(\delta_k - \delta_{k+1})$ introduces some inherent robustness to the market design. This enables the market to sustain a certain level of errors introduced by finite horizons, suboptimal solutions, forecast errors, and so on. The effect of forecast errors and suboptimal solutions is analyzed in the following section.

5.3 Robust Stability

To analyze robustness properties, we consider the case in which the game data (e.g., demand, wind) cannot be forecast perfectly. To account for this case, the *true* value of the data at time k will be denoted as m_k and the true data trajectory over the horizon $\mathcal{T}_k$ as $m_{\mathcal{T}_k}$. The forecast error trajectory over $k...k+T$ generated at time k will be denoted as $\varepsilon_{\mathcal{T}_k} := \{\varepsilon_k(k), ..., \varepsilon_{k+T}(k)\}$ with $\varepsilon_k(k) = 0$ since the data at the current time k is assumed to be known. The symbol (k) is used to reflect the fact that the error trajectory is generated at time k using the most recent information. The forecast trajectory can be expressed as $\bar{m}_{\mathcal{T}_k} := m_{\mathcal{T}_k} + \varepsilon_{\mathcal{T}_k}$.

The forecast errors at time k, $\varepsilon_{\mathcal{T}_k}$, generate control actions that drive the summarizing state from δ_k to $\delta_{k+1} = f(\delta_k, \tilde{h}_k(\delta_k, \bar{m}_{\mathcal{T}_k}))$. This last state differs from the error-free state $\bar{\delta}_{k+1} = f(\delta_k, \tilde{h}_k(\delta_k, m_{\mathcal{T}_k}))$. The objective is to establish conditions under which the summarizing cost function with forecast errors still represents an improvement over the current cost. We follow the approach proposed in [21]. We define the following error terms:

$$\Xi_k^1 := |V_{T-1}(\bar{\delta}_{k+1}, m_{\mathcal{T}_k}) - V_T(\bar{\delta}_{k+1}, m_{\mathcal{T}_{k+1}})| \tag{24a}$$

$$\Xi_k^2 := |V_T(\bar{\delta}_{k+1}, m_{\mathcal{T}_{k+1}}) - V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}})|. \tag{24b}$$

Under numerical stability of the game, we have that

$$\begin{aligned} \Xi_k^2 &\leq L_V |\bar{\delta}_{k+1} - \delta_{k+1}| \\ &\leq L_V |f(\delta_k, \tilde{h}_k(\delta_k, \bar{m}_{\mathcal{T}_k})) - f(\delta_k, \tilde{h}_k(\delta_k, m_{\mathcal{T}_k}))| \\ &\leq L_V L_\delta L_h \|\varepsilon_{\mathcal{T}_k}\|. \end{aligned} \tag{25}$$

Theorem 6 *Assume that the game defined* (17) *and* (20) *has a feasible solution* $\forall\, k$ *under the horizon* T *and that there exists a finite* $\beta \geq 0$ *such that the error sequence remains bounded* $\varepsilon_{\mathcal{T}_k} \leq \beta$, $\forall k$. *If,*

$$\Xi_k^1 + \Xi_k^2 \leq (\delta_k - \bar{\delta}_{k+1}), \forall\, k, \tag{26}$$

then the market is stable.

Proof: From feasibility, the cost function $V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}})$ is positive definite. To prove that it is nonincreasing under (26), we establish the following.

$$\begin{aligned}
&V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) - V_T(\delta_k, m_{\mathcal{T}_k}) \\
&\quad = (\delta_k - \bar{\delta}_{k+1}) \\
&\qquad + V_{T-1}(\bar{\delta}_{k+1}, m_{\mathcal{T}_k}) - V_T(\bar{\delta}_{k+1}, m_{\mathcal{T}_{k+1}}) \\
&\qquad + V_T(\bar{\delta}_{k+1}, m_{\mathcal{T}_{k+1}}) - V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) \\
&\quad \leq -(\delta_k - \bar{\delta}_{k+1}) + \Xi_k^1 + \Xi_k^2.
\end{aligned}$$

The last inequality follows from the stability condition (26). The function is nonincreasing. We note the appearance of the extra term $\bar{\delta}_{k+1}$. To account for this, we compute the summation of the above inequality over $j = 0, ..., J$ to obtain

$$\begin{aligned}
&V_T(\delta_{k+1+J}, m_{\mathcal{T}_{k+1+J}}) - V_T(\delta_k, m_{\mathcal{T}_k}) \\
&\quad \leq -\sum_{j=0}^{J} \delta_{k+j} + \sum_{j=0}^{J} \bar{\delta}_{k+1+j} + \sum_{j=0}^{J} \Xi_{k+j}^1 + \sum_{j=0}^{J} \Xi_{k+j}^2 \\
&\quad \leq 0.
\end{aligned}$$

Since $\delta_k \geq 0$, we have that $0 \leq \sum_{j=0}^{J} \delta_{k+j}$. The second term in the right-hand side is bounded and positive since, from feasibility, we have $\bar{\delta}_{k+1+j} \leq \delta_{k+j}, \ \forall j, k$. The last two terms are positive and bounded as well by α_T and β, respectively. The function $V_T(\cdot, \cdot)$ is positive definite and nonincreasing, so the difference in the left-hand side is negative. Consequently, the sum $\sum_{j=0}^{J} \delta_{k+j}$ remains bounded. If we extend $J \to \infty$, the conclusion follows. □

Using the same construct, we can establish stability conditions for the case in which there is incomplete gaming at each step. This will introduce an additional error $\hat{\varepsilon}_k$ to the control action $\tilde{h}(\delta_k, \bar{m}_{\mathcal{T}_k})$ that will move the state from δ_k to δ_{k+1}. In this case, we have that

$$\begin{aligned}
\Xi_k^2 &\leq L_V |\bar{\delta}_{k+1} - \delta_{k+1}| \\
&\leq L_V |f(\delta_k, \tilde{h}_k(\delta_k, \bar{m}_{\mathcal{T}_k})) - f(\delta_k, \tilde{h}_k(\delta_k, m_{\mathcal{T}_k}) + \hat{\varepsilon}_k)| \\
&\leq L_V L_\delta \left(L_h \|\varepsilon_{\mathcal{T}_k}\| + \|\hat{\varepsilon}_k\| \right).
\end{aligned}$$

Since this bound is larger, the robust stability threshold (26) is narrower.

We note that the stability conditions (23) and (26) are only sufficient conditions for the market to remain stable. We also note that, because of forecast errors, the solution given by the game at k can be guaranteed only to satisfy the current demand but not the demands beyond $k+1$. Consequently, the cost function of the game given by $V_T(\delta_{k+1}, \bar{m}_{\mathcal{T}_{k+1}})$ cannot be used as a Lyapunov function. We therefore use the cost function of the game with no forecast errors $V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}})$ whose solution does satisfies the demands, and we deal with the forecast errors implicitly through the initial state δ_{k+1}. Finally, we note that the use of stochastic clearing formulations can potentially improve robustness and mitigate market volatility. In particular, the anticipatory properties of multi-stage stochastic formulations can be exploited to better manage ramping constraints.

For the finite horizon case with perfect forecast, we note that the term Ξ_k^1 is still present and depends on the data $m_{\mathcal{T}_k} := \{m_k, ..., m_{k+T}\}$. Consequently, *even if the forecast is perfect*, a strong change in the data m_{k+T} to m_{k+T+1} can break the stability condition (23) if the horizon is not long enough. This situation can arise, for instance, from a steep change in wind supply. This explains why the horizon should be sufficiently long so that the bound α_T is as small as possible and the market is more robust.

Consider now the limiting case in which the ramps do not constrain performance. We can show that the stability threshold is the same as that of the infinite horizon problem.

Theorem 7 *Assume that the ramp limits are given by* $\underline{r}^i, \overline{r}^i = (\overline{q}^i - \underline{q}^i)$. *Then, the market given by the game* (17) *and* (20) *is stable with* $\Xi_k^2 = \Xi_k^1 = 0,\ \forall k$.

Proof: In the absence of ramping constraints, the game problem is decoupled in time. Consequently, the optimal controls u_k, w_k are invariant to the forecast errors at the future times $k+1, ..., k+T$ and to the length of the horizon T. If we solve the game at k with $\bar{m}_{\mathcal{T}_k}$ we have that $\bar{\delta}_{k+1} = \delta_{k+1}$ since $\varepsilon_k(k) = 0,\ \forall k$. With this we have $\Xi_k^2 = 0,\ \forall k$. If we solve the game over a sequence of steps $i = 0, ..., T-1$ and collect the first terms in $V_T(\delta_{k+i}, \bar{m}_{\mathcal{T}_{k+i}})$ given by $(\delta_{k+i} - \delta_{k+1+i})$, we have that $\sum_{i=0}^{T}(\delta_{k+i} - \delta_{k+1+i}) = V_T(\delta_k, m_{\mathcal{T}_k}),\ \forall k$. We also have that we can extend the horizon and sequence as $J = T \to \infty$ so that $\Xi_k^1 = 0,\ \forall k$. Consequently, the sequence $\{\delta_k\}$ is bounded, and the market is stable. The proof is complete. □

This implies that the stability bounds of the market clearing problem are the same as those of the infinite horizon problem with perfect forecast information. In other words, as the generators become faster, market robustness increases, as expected. This result also implies that as the ramping capacity increases, the forecast horizon can be made shorter.

6 Numerical Case Study

In this section, we illustrate the effect of ramping constraints, foresight horizon, and limited coordination on market stability and price dynamics. We consider a market system with three suppliers and one demand. One of the suppliers has fast dynamics (high ramping capacity) but high cost such as natural gas generators, the second one has slow dynamics but also low cost such as a coal generator, and the third one is used as a slack generator with infinite ramp limits (equal to generation capacity) and a large cost. This last supplier acts as a slack to avoid infeasibility. The nominal parameters used are $\underline{q} = [0, 0, 0]$, $\overline{q} = [50, 70, 120]$, $\underline{r} = -[5, 10, 120]$, $\overline{r} = [5, 10, 120]$, $h = [4, 2, 5]$, and $g = [2, 1, 5]$. We used $\hat{q}_0 = [0, 40, 40]$ as initial conditions. We consider the demand profile presented in Figure 2, which is obtained from a periodic signal perturbed with Gaussian noise. We set the market stability threshold to $\epsilon = 0.65$.

To illustrate the main developments of the paper, we consider three market implementations. The first one uses a foresight horizon of six hours and performs a single

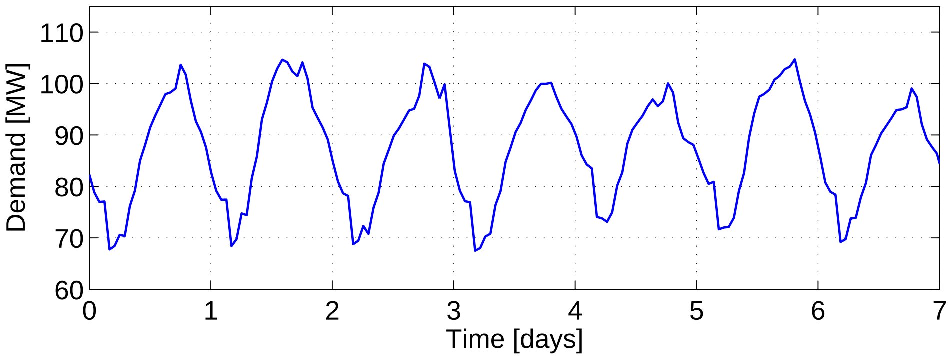

Fig. 2 Demand profile used for numerical case study.

Jacobi-like iteration at each clearing time (incomplete gaming). This implementation is labeled as ($T = 6Jac$) and represents current practice. The second implementation uses the same horizon length, but the game is converged to optimality ($T = 6Opt$) satisfying the stabilizing constraint. The third implementation uses an horizon of 24 hours, and the game is converged to optimality ($T = 24Opt$). To compute the reference social welfare used in the definition of efficiency, we also implemented an unconstrained market clearing procedure.

In Figure 3 we present the profiles of the summarizing state δ_t for the three market implementations, in Figure 4 we present efficiency profiles η_t, and in Figure 5 we present the resulting clearing price signals p_t. From Figure 3 it is clear that the summarizing state obtained from the suboptimal implementation $T = 6Jac$ is not strictly decreasing during days 1 and 3 and thus its market clearing cost cannot be used as a Lyapunov function. This indicates that the efficiency is crossing the threshold at certain times, as can be observed in Figure 4. This clearly illustrates that incomplete gaming can introduce market instability. The other two control implementations remain stable, but, as expected, a longer foresight horizon improves performance. This is observed from the faster decay of the summarizing state for $T = 24Opt$ when compared with $T = 6Opt$ and from the efficiency profiles. The efficiencies of $T = 24Opt$ remain farther away from the threshold. This illustrates that the length of the foresight horizon can have important effects on market stability. This is mainly because longer foresights can anticipate and manage ramping constraints more efficiently.

In Figure 5 we observe the spikes in the prices for $T = 6Jac$ during the first hours of the simulation and during the third day. In particular, note the strong price fluctuations when compared with the optimal unconstrained prices. These prices were obtained from the solution of the unconstrained market clearing problem. Note that in the absence of ramping constraints, the prices remain stable and nearly periodic. On the other hand, when the ramp constraints are active, strong price variations are observed. In particular, during the third day, the prices for $T = 6Jac$ reach levels of $150\$/MW$. The prices of $T = 24Opt$ stay well below $100\$/MW$ and much closer to the optimal unconstrained prices. These levels are a consequence of having a longer foresight horizon and converging the game to optimality to ensure that the efficiency is above the stability threshold. As a quantitative result, we computed the sum of squared errors $SSE = \sum_t |p_t - \bar{p}_t|^2$ over the entire simulation horizon of 7 days. Here, p_t are the constrained price signals, and $\bar{p}_t$ are the unconstrained price signals. For $T = 6Jac$ we obtained SSE=2.16×10^5 while for $T = 24Opt$ we have SSE=4.19×10^4. This is an improvement of nearly an order of magnitude. We have also observed that performing an extra Jacobi-like iteration for $T = 6Jac$ stabilizes the prices. In addition, we have observed that extending the horizon of $T = 24Opt$ does not improve its performance significantly.

In Figure 6 we present price profiles for $T = 6Jac$ and $T = 24Opt$ with relaxed ramp constraints. In this case, we increased the ramp limits from their nominal values to $\underline{r} = -[10, 20, 120]$, $\overline{r} = [10, 20, 120]$. As can be seen, the price signals for both implementations are close to those of the unconstrained clearing problem. The signals of $T = 24Opt$ get closer to the unconstrained reference faster because of a combined effect of complete gaming and forecast horizon. In particular, we observe that $T = 6Jac$ performs well in this case. The reason is that when the ramp limits are relaxed, subsequent gaming solutions become closer to each other. This case illustrates how ramping constraints can have strong effects on efficiency and stability and how alternative market designs can help mitigate those effects.

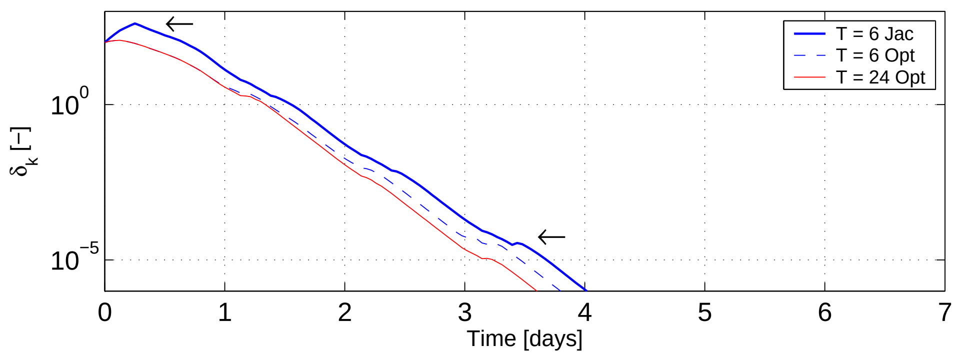

Fig. 3 Summarizing state for market implementations.

7 Conclusions and Future Work

We have established a framework to analyze and design stabilizing market designs. The framework incorporates physical constraints, efficiency concepts, and Lyapunov analysis tools. We explain how market stability issues can arise in current market designs as a result of limited coordination between the ISO and the suppliers and short foresight horizons. The framework is general and can be extended to consider other operational scenarios such as network constraints, forward and real-time markets, strategic behavior, stochastic formulations, and piece-wise supply functions. The key contribution of our work is the notion of incorporating market stability metrics and

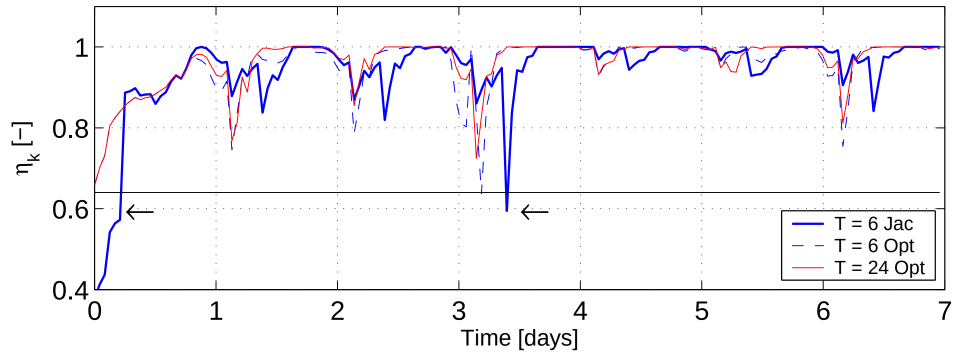

Fig. 4 Efficiencies for market implementations.

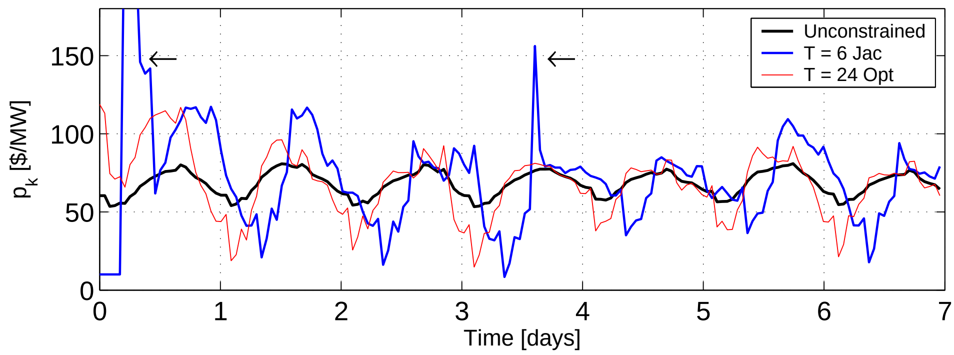

Fig. 5 Clearing prices for market implementations.

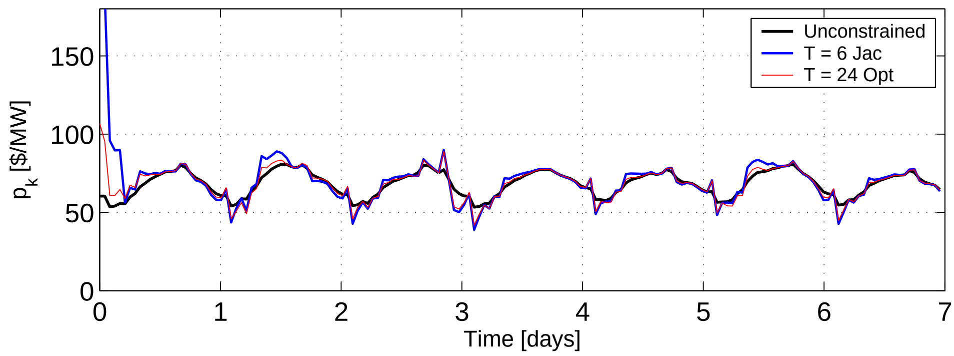

Fig. 6 Clearing prices for market implementations under relaxed ramp constraints.

constraints in clearing procedures. In particular, the proposed framework proposed to quantify the effect of different physical constraints and clearing practices on market stability.

We highlight that predictive control is a general framework that can capture stability issues under strategic behaviors and other types of physical constraints and decision-making objectives. For instance, strategic behavior under uncertainty can lead to more conservative or aggressive bids that affect pricing signals. Our stability framework can be extended to design clearing procedures that remain robust under different levels of risk aversion or anticipatory behavior. Another issue is the strategic manipulation of physical parameters such as ramp rates [22]. The analysis of this more complicated behaviors leads to theoretically and computationally more challenging formulations such as deterministic and stochastic equilibrium problems with equilibrium constraints (EPECs) [13].

The issue of limited coordination opens the door to several questions regarding appropriate distributed approaches to implement the bidding-clearing procedure in real-time. In particular, distributed iterations cannot be guaranteed to converge [10]. In any of these developments, we believe it is critical to establish a consistent framework as the one presented in this work to design and compare different market designs by characterizing their stabilizing and robustness properties. Finally, it seems necessary to establish formal connections between different market efficiency notions such as price stability as well as allocative and productive efficiencies [3] and to incorporate these metrics in clearing formulations. This can be done, for instance, by considering emission or fuel consumption constraints in the clearing formulation. In particular, we highlight that existing market design studies propose different efficiency metrics but no coherent framework exists to trade-off among them.

Acknowledgments

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A Problem Formulations and Optimality Conditions

A.1 Suppliers

For the supplier problem (3) we have the following. Given the prices $p_t,\ t \in \mathcal{T}$ solve

$$\min_{b_t^i} \sum_{t\in\mathcal{T}} \left(c_t^i(q_t^i) - p_t \cdot q_t^i\right) \tag{27a}$$

$$\text{s.t. } q_t^i = b_t^i \cdot p_t,\ t \in \mathcal{T} \tag{27b}$$

$$\underline{q}^i \le q_t^i \le \overline{q}^i,\ t \in \mathcal{T} \tag{27c}$$

$$b_t^i \ge 0,\ i \in \mathcal{S}, t \in \mathcal{T}. \tag{27d}$$

Since this problem is decoupled in time, we can derive its optimality conditions by looking at the Lagrange function at a time instant t:

$$\begin{aligned}\mathcal{L}_t^i(p_t) = c_t^i(q_t^i) - p_t \cdot q_t^i + \lambda^{q}{}_t^i \cdot \left(q_t^i - b_t^i \cdot p_t\right)\\ - \underline{\nu}^{q}{}_t^i \cdot (q_t^i - \underline{q}^i) - \overline{\nu}^{q}{}_t^i \cdot (\overline{q}^i - q_t^i) - \nu^{b}{}_t^i \cdot b_t^i,\ i \in \mathcal{S}, t \in \mathcal{T}.\end{aligned} \tag{28}$$

The optimality conditions are

$$\nabla_{q_t^i}\mathcal{L}_t^i(p_t) = h_t^i + g_t^i \cdot q_t^i - p_t + {\lambda^{q}}_t^i - {\underline{\nu}^{q}}_t^i + {\overline{\nu}^{q}}_t^i = 0,\ i \in \mathcal{S}, t \in \mathcal{T}. \tag{29a}$$

$$\nabla_{b_t^i}\mathcal{L}_t^i(p_t) = -{\lambda^{q}}_t^i \cdot p_t - {\nu^{b}}_t^i = 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{29b}$$

$$0 \leq {\underline{\nu}^{q}}_t^i \perp (q_t^i - \underline{q}^i) \geq 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{29c}$$

$$0 \leq {\overline{\nu}^{q}}_t^i \perp (\overline{q}^i - q_t^i) \geq 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{29d}$$

$$0 \leq {\nu^{b}}_t^i \perp b_t^i \geq 0,\ i \in \mathcal{S}, t \in \mathcal{T}. \tag{29e}$$

and,

$$q_t^i = b_t^i \cdot p_t = 0,\ i \in \mathcal{S}, t \in \mathcal{T}. \tag{30}$$

We note that we can also pose the supplier problem in terms of the quantities q_t^i and use the supply function $q_t^i = b_t^i \cdot p_t$ to recover the supply function parameters b_t^i exogenously. This leads to the following system of equations,

$$\nabla_{q_t^i}\mathcal{L}_t^i(p_t) = g_t^i \cdot q_t^i - p_t - {\underline{\nu}^{q}}_t^i + {\overline{\nu}^{q}}_t^i = -h_t^i,\ i \in \mathcal{S}, t \in \mathcal{T}. \tag{31a}$$

$$\nabla_{{\lambda^{q}}_t^i}\mathcal{L}_t^i(p_t) = q_t^i - b_t^i \cdot p_t = 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{31b}$$

$$0 \leq {\underline{\nu}^{q}}_t^i \perp (q_t^i - \underline{q}^i) \geq 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{31c}$$

$$0 \leq {\overline{\nu}^{q}}_t^i \perp (\overline{q}^i - q_t^i) \geq 0,\ i \in \mathcal{S}, t \in \mathcal{T}. \tag{31d}$$

We group the decision variables by defining the vectors

$$\begin{aligned} {q^s}^T &= [q_0^1, ..., q_0^S, ..., q_T^1, q_T^S]^T \\ b^T &= [b_0^1, ..., b_0^S, ..., b_T^1, b_T^S]^T, \end{aligned}$$

and $p = [p_0, ..., p_T]$. We group all the multipliers into a single vector ν^s. Finally, we linearize the system around a given solution q_*^s, ν_s^*, b_*, p_* and redefine the increments around the solution as $q \leftarrow \Delta q$,$\nu \leftarrow \Delta\nu$, $b \leftarrow \Delta b$, and $p \leftarrow \Delta p$. Using these modifications, we can pose the above system in condensed form as,

$$G \cdot q^s - A_p^s \cdot p + A_\nu^s \cdot \nu^s = r_q^s \tag{32a}$$

$$A_\nu^s \cdot q^s = r_\nu^s \tag{32b}$$

$$q^s + P \cdot b + B \cdot H \cdot p = r_b. \tag{32c}$$

Where G is a diagonal matrix with entries g_t^i, B is a diagonal matrix with entries b_{t*}^i, A_ν^s is the Jacobian matrix for the multipliers corresponding to active constraints, A_p^s is the Jacobian matrix for the prices (exogenous to the suppliers), H is a mapping matrix satisfying ${q^s}_* = B \cdot H \cdot p_*$, and P is a diagonal matrix with entries p_{t*} and satisfying $P = (I_\mathcal{S} \otimes I_\mathcal{T})\, p_*$. Finally, r_q^s, r_ν^s and r_b are right-hand side vectors.

A.2 Unconstrained ISO

For the unconstrained ISO market clearing problem (6) we have that the optimality conditions are decoupled in time as well. The Lagrange function at a time instant t is given by

$$\begin{aligned} \bar{\mathcal{L}}_t(b_t^i) = \sum_{i\in\mathcal{S}} \frac{1}{2\, b_t^i}(q_t^i)^2 - \bar{p}_t \left(\sum_{i\in\mathcal{S}} q_t^i - \sum_{j\in\mathcal{C}} d_j^t \right) \\ - \sum_{i\in\mathcal{S}} {\underline{\nu}^{q}}_t^i \cdot (q_t^i - \underline{q}^i) - \sum_{i\in\mathcal{S}} {\overline{\nu}^{q}}_t^i \cdot (\overline{q}^i - q_t^i),\ t \in \mathcal{T}. \end{aligned} \tag{33}$$

The optimality conditions are

$$\nabla_{q_t^i} \bar{\mathcal{L}}_t(b_t^i) = \frac{1}{b_t^i} q_t^i - p_t - \underline{\nu}^{q}{}_t^i + \overline{\nu}^{q}{}_t^i = 0,\ t \in \mathcal{T} \tag{34a}$$

$$0 \le \bar{p}_t \perp \left(\sum_{i\in\mathcal{S}} q_t^i - \sum_{j\in\mathcal{C}} d_j^t \right) \ge 0,\ t \in \mathcal{T} \tag{34b}$$

$$0 \le \underline{\nu}^{q}{}_t^i \perp (q_t^i - \underline{q}^i) \ge 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{34c}$$

$$0 \le \overline{\nu}^{q}{}_t^i \perp (\overline{q}^i - q_t^i) \ge 0,\ i \in \mathcal{S}, t \in \mathcal{T}. \tag{34d}$$

A.3 Constrained ISO

For the ISO problem (11) the Lagrange function is given by

$$\begin{aligned}\mathcal{L}(b_t^i) = \ & \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{S}} \frac{1}{2\,b_t^i}(q_t^i)^2 - \sum_{t\in\mathcal{T}} p_t \left(\sum_{i\in\mathcal{S}} q_t^i - \sum_{j\in\mathcal{C}} d_j^t \right) \\ & + \sum_{t\in\mathcal{T}^-} \sum_{i\in\mathcal{S}} \lambda_{t+1}^i (q_{t+1}^i - q_t^i - \Delta q_t^i) - \sum_{t\in\mathcal{T}^-} \sum_{i\in\mathcal{S}} \underline{\nu}^{\Delta}{}_t^i (\Delta q_t^i - \underline{r}^i) - \sum_{t\in\mathcal{T}^-} \sum_{i\in\mathcal{S}} \overline{\nu}^{\Delta}{}_t^i (\overline{r}^i - \Delta q_t^i) \\ & - \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{S}} \underline{\nu}^{q}{}_t^i (q_t^i - \underline{q}^i) - \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{S}} \overline{\nu}^{q}{}_t^i (\overline{q}^i - q_t^i) + \sum_{i\in\mathcal{S}} \lambda_k^i (q_k^i - \hat{q}_k^i).\end{aligned} \tag{35}$$

The optimality conditions are

$$\nabla_{q_T^i} \mathcal{L} = \frac{1}{b_T^i} q_T^i - p_T + \lambda_T^i - \underline{\nu}^{q}{}_t^i + \overline{\nu}^{q}{}_t^i = 0,\ i \in \mathcal{S} \tag{36a}$$

$$\nabla_{q_t^i} \mathcal{L} = \frac{1}{b_t^i} q_t^i - p_t + \lambda_t^i - \lambda_{t+1}^i - \underline{\nu}^{q}{}_t^i + \overline{\nu}^{q}{}_t^i = 0,\ i \in \mathcal{S}, t \in \mathcal{T}^- \tag{36b}$$

$$\nabla_{\Delta q_t^i} \mathcal{L} = -\lambda_{t+1}^i - \underline{\nu}^{\Delta}{}_t^i + \overline{\nu}^{\Delta}{}_t^i = 0,\ i \in \mathcal{S}, t \in \mathcal{T}^- \tag{36c}$$

$$\nabla_{\lambda_{t+1}^i} = q_{t+1}^i - q_t^i - \Delta q_t^i = 0,\ i \in \mathcal{S},\ \in \mathcal{T}^- \setminus \{k\} \tag{36d}$$

$$\nabla_{\lambda_0^i} = q_0^i - \hat{q}_k^i = 0,\ i \in \mathcal{S} \tag{36e}$$

$$0 \le p_t \perp \left(\sum_{i\in\mathcal{S}} q_t^i - \sum_{j\in\mathcal{C}} d_j^t \right) \ge 0,\ t \in \mathcal{T} \tag{36f}$$

$$0 \le \overline{\nu}^{\Delta}{}_t^i \perp \overline{r}^i - \Delta q_t^i \ge 0,\ i \in \mathcal{S}, t \in \mathcal{T}^- \tag{36g}$$

$$0 \le \underline{\nu}^{\Delta}{}_t^i \perp \Delta q_t^i - \underline{r}_i \ge 0,\ i \in \mathcal{S}, t \in \mathcal{T}^- \tag{36h}$$

$$0 \le \overline{\nu}^{q}{}_t^i \perp \overline{q}^i - q_t^i \ge 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{36i}$$

$$0 \le \underline{\nu}^{q}{}_t^i \perp q_t^i - \underline{q}^i \ge 0,\ i \in \mathcal{S}, t \in \mathcal{T}. \tag{36j}$$

We can pose the above system in block form and couple to the suppliers system by using the following modifications. We eliminate variables Δq_t^i, we define variable q^I and multiplier vectors ν^I and note the coupling with the suppliers problems through the variables p and b. Finally, by linearizing around a given solution q_*^I, ν^I, p_*, b_*, we obtain the following system,

$$B^{-2} \cdot b + B^{-1} \cdot q^I + A_p^I \cdot p + A_\nu^I \cdot \nu_I = r_q^I \tag{37a}$$

$$A_p^I \cdot q = r_p^I \tag{37b}$$

$$A_\nu^I \cdot q = r_\nu^I. \tag{37c}$$

Here, B is a diagonal matrix with entries b^i_{t*}, A^I_p is the Jacobian with respect to the prices, A^I_ν is the Jacobian with respect to the ISO multipliers, corresponding to the active constraints, and r^I_q, r^I_p and r^I_ν are right-hand side vectors.

A.4 Stabilizing ISO

The stabilizing ISO formulation (20) can be written as

$$\min_{q^i_t, \Delta q^i_t} \quad \delta_T \tag{38a}$$

$$\text{s.t. } \varphi_t = \sum_{i\in\mathcal{S}} \frac{1}{2\,b^i_t}(q^i_t)^2, \quad t\in\mathcal{T}_k \tag{38b}$$

$$\varphi_t \cdot \eta_t = \bar{\varphi}_t, \quad t\in\mathcal{T}_k \tag{38c}$$

$$\eta_t \geq \epsilon,\ t\in\mathcal{T}_k \tag{38d}$$

$$\delta_{t+1} = (1-(\eta_{t+1}-\epsilon))\cdot\delta_t,\ t\in\mathcal{T}^-_k \tag{38e}$$

$$\sum_{i\in\mathcal{S}} q^i_t \geq \sum_{j\in\mathcal{C}} d^t_j,\ t\in\mathcal{T}_k \tag{38f}$$

$$q^i_{t+1} = q^i_t + \Delta q^i_t,\ i\in\mathcal{S}, t\in\mathcal{T}^-_k \tag{38g}$$

$$\underline{r}^i \leq \Delta q^i_t \leq \overline{r}^i,\ i\in\mathcal{S}, t\in\mathcal{T}^-_k \tag{38h}$$

$$\underline{q}^i \leq q^i_t \leq \overline{q}^i,\ i\in\mathcal{S}, t\in\mathcal{T}_k \tag{38i}$$

$$q^i_k = \hat{q}^i_k, \delta_k = \hat{\delta}_k,\ i\in\mathcal{S}. \tag{38j}$$

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