EXPONENTIALLY CONVERGENT RECEDING HORIZON STRATEGY FOR CONSTRAINED OPTIMAL CONTROL[§]

WANTING XU[†] AND MIHAI ANITESCU[‡]

Abstract. Receding horizon control has been a widespread method in industrial control engineering as well as an extensively studied subject in control theory. In this work, we consider a lag L receding horizon strategy that applies the initial L optimal controls from each quadratic program to each receding horizon. We investigate a discrete-time and time-varying linear-quadratic optimal control problem that includes a nonzero reference trajectory and constraints on both state and control. We prove that, under boundedness and controllability conditions, the solution obtained by the receding horizon strategy converges to the solution of the full problem interval exponentially fast in the length of the receding horizon for some lag L. The exponential rate of convergence provides a systematic way of choosing the receding horizon length given a desired accuracy level. We illustrate our theoretical findings using a small, synthetic production cost model with real demand data.

Key words. constrained optimal control, receding horizon control, sensitivity analysis

AMS subject classifications. 49N10, 49N35, 49Q12

1. Introduction.

Receding horizon control (RHC), also known as model predictive control, has been a widely used feedback strategy in various industrial control applications [e.g., 1, 3, 6, 7, 21]. RHC can be applied to a broad class of optimal control problems, including those with nonlinear dynamics, time-delay systems, and constraints on state and control [14, 15]. The essence of RHC is to obtain the current control action by solving an optimal control problem defined on a finite horizon extending from the current time point k. The finite-horizon problem uses the current state of the system as its initial state, yields sequences of optimal controls and states, applies the optimal control at time point k to the system, and uses the optimal state at $k + 1$ as the initial state of the next receding horizon problem. This on-line feature of RHC makes the method adaptive to changing system parameters, since only a finite horizon extending into the future is required for the current control [14], and particularly attractive when off-line computation of the control policy is difficult [18].

Several results prove the stability of RHC for constrained linear and nonlinear systems. For example, reference [17] proves that RHC yields an asymptotically stable closed-loop system for continuous-time nonlinear systems. In addition, reference [11] establishes stability of RHC for discrete-time, time-varying, constrained nonlinear systems. Both references employ the value function as a Lyapunov function for the stability analysis. Stability results for linear systems can be found, for example, in [12, 13, 19, 20].

In this work, we consider a slight variation of the standard RHC described in the references above. In particular, on each receding horizon, instead of applying the optimal control at only the current time point k, we apply the optimal controls at the initial L time points for some lag $L > 0$, and the next receding horizon starts at time point $k + L$. The same receding horizon strategy is considered in other references for continuous time, in order to account for the sampling time; see, for example, [9, 10]. Building on our recent analytical developments in [24], we prove

[†]Department of Statistics, University of Chicago, Chicago, IL 60637 (wxu@galton.uchicago.edu).

[‡]Mathematics and Computer Science Division, Argonne National Laboratory, Lemont, IL 60439 (anitescu@mcs.anl.gov).

[§]Also, Preprint ANL/MCS-P9015-1017

that for an inequality-constrained time-varying linear system with a quadratic cost function and nonzero reference trajectory, the optimal states and controls obtained by this receding horizon strategy converge, for some lag L, to the solutions on the full problem interval exponentially fast in the length of the receding horizon. The appropriate lag L in the result is determined by the problem parameters and, in particular, by the controllability properties of the constrained system. Our analysis directly investigates the solutions of a related equality-constrained control problem and connects them to the original inequality-constrained problem through a sensitivity analysis. Specifically, we consider the following problem.

$$\min \quad \frac{1}{2}\sum_{k=n_1}^{n_2-1} u_k^T R_k u_k + (x_k - d_k)^T Q_k (x_k - d_k) \tag{1.1a}$$

$$+ (x_{n_2} - d_{n_2})^T Q_{n_2} (x_{n_2} - d_{n_2}) \tag{1.1b}$$

$$\text{s.t.} \quad x_{k+1} = A_k x_k + B_k u_k, \qquad n_1 \le k \le n_2 - 1, \qquad x_{n_1} = x_{n_1}^0, \tag{1.1c}$$

$$\widetilde{P}_{k+1} x_{k+1} + \widetilde{C}_k u_k \ge \widetilde{q}_k, \qquad n_1 \le k \le n_2 - 1 \tag{1.1d}$$

In (1.1), we refer to x_k, u_k, and d_k as the state, control, and reference trajectory, respectively. Problem (1.1) lacks nonlinear dynamics, which is studied in some stability analyses of RHC, for example, [9, 17]. However, we include the inequality path constraint (1.1d) of state and control as considered in [19, 20]. Furthermore, we allow a nonzero reference trajectory d_k and prove the exponential convergence of RHC solutions to the solutions of problem (1.1) instead of a fixed equilibrium point [9, 10].

In our proofs, we use two important results in optimal control theory. One is developed in [25], where the authors prove that for an unconstrained, switched-time, and discrete-time linear-quadratic optimal control problem, the optimal trajectory stabilizes exponentially under some mild conditions. They also give an estimate of the exponential rate, which we use here. The other one is established in [23], where the authors propose a Riccati-based approach for solving linear-quadratic optimal control problems subject to linear equality path constraints. They derive a solution procedure based on solving the KKT conditions via the Riccati recursion. We borrow similar manipulations and reductions of the KKT conditions here. Moreover, a few results regarding the Riccati recursion, closed-loop matrix, and sensitivity analysis are nearly the same as those in our previous work [24]. We present those proofs in the Appendix. We note that our previous work [24] had more complex algebra since it did not use the KKT-based ideas from [23], had bound constraints on control only, and did not investigate RHC convergence; these features are present in this work.

The rest of the article is organized as follows. In Section 2, we consider an equality constrained subproblem of (1.1) and investigate the dependences of solutions on the initial state and terminal reference. In Section 3, we define the lag L receding horizon strategy and prove the exponential convergence of RHC solutions based on results derived in Section 2. In Section 4, we demonstrate our theoretical findings using a synthetic production cost model with real demand data.

2. Path-constrained linear-quadratic problem. In this section, we mainly consider a subproblem of the constrained linear-quadratic optimal control problem (1.1). For (1.1), we have that $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $\widetilde{P}_k \in \mathbb{R}^{r\times n}$, $\widetilde{C}_k \in \mathbb{R}^{r\times m}$, and $Q_k \in \mathbb{R}^{n\times n}$, $R_k \in \mathbb{R}^{m\times m}$ are positive definite. We make the following uniform boundedness assumption.

ASSUMPTION 2.1. *For any n_1, n_2, and $n_1 \le k \le n_2$, we assume that* *(a)* $\|A_k\|_2 \le \widetilde{C}_A$, $\|B_k\|_2 \le \widetilde{C}_B$, $\|Q_k\|_2 \le \widetilde{C}_Q$, $\|R_k\|_2 \le C_R$, $\|\widetilde{P}_k\|_2 \le C_P$, $\|\widetilde{C}_k\|_2 \le C_C$, $\|\widetilde{q}_k\|_2 \le U$;

(b) $\lambda_{min}(Q_k) \ge \lambda_Q > 0$, $\lambda_{min}(R_k) \ge \lambda_R > 0$.

Note that we use symbols with a tilt for some upper bounds in Assumption 2.1 (a), since we reserve the corresponding straight symbols for the frequently used quantities defined later in Lemma 2.4. The subproblem of (1.1) we investigate is an equality-constrained problem obtained by considering some active subsets of the polyhedral path constraint (1.1d).

2.1. Equality-constrained subproblem. To define the equality-constrained subproblem, we let $I_k \subset \{1,\dots,r\}$ be some index set of the constraint (1.1d) attaining the bound. Let $P_k = \widetilde{P}_k(I_k,:)$ and $C_k = \widetilde{C}_k(I_k,:)$ be the corresponding submatrices, and denote $q_k = \widetilde{q}_k(I_k)$. Then the equality constraint corresponding to the index set $\mathcal{I} \triangleq \{I_k\}$ is $P_{k+1}x_{k+1} + C_k u_k = q_k$. The equality-constrained problem we consider is hence the following:

$$\text{(2.1a)} \qquad \min \quad \frac{1}{2}\sum_{k=n_1}^{n_2-1} u_k^T R_k u_k + (x_k - d_k)^T Q_k (x_k - d_k)$$

$$\text{(2.1b)} \qquad + (x_{n_2} - d_{n_2})^T Q_{n_2} (x_{n_2} - d_{n_2})$$

$$\text{(2.1c)} \qquad \text{s.t.} \quad x_{k+1} = A_k x_k + B_k u_k, \qquad n_1 \le k \le n_2 - 1, \qquad x_{n_1} = x_{n_1}^0,$$

$$\text{(2.1d)} \qquad E_k x_k + H_k u_k = q_k, \qquad n_1 \le k \le n_2 - 1,$$

where we denote that

$$\text{(2.2)} \qquad E_k = P_{k+1} A_k, \qquad H_k = P_{k+1} B_k + C_k.$$

If $\mathcal{I}$ is the active set of problem (1.1) at optimality, then problems (1.1) and (2.1) have the same solutions. Note that H_k in (2.2) is determined by the index set $\mathcal{I}$ encoding the equality constraints under consideration, and hence we define the following uniform boundedness property of H_k in terms of the index set.

DEFINITION 2.2. *Given an index set, let H_k be as in* (2.2)*. With some $\lambda_H > 0$, the index set is uniformly bounded below with respect to λ_H, denoted as UDB(λ_H), if for any $n_1 \le k \le n_2$, H_k has full row rank and*

$$\lambda_{min}(H_k H_k^T) \ge \lambda_H > 0.$$

Definition 2.2 restricts the total number of equality path constraints for an index set that is UDB(λ_H) and gives a uniform lower bound on the resulting matrix H_k. In the rest of this subsection, we restrict our attention to the index sets that are UDB(λ_H). First, we define some matrices frequently used in the subsection.

DEFINITION 2.3. *For some index set that is UDB(λ_H), we define the following matrices for $n_1 \le k \le n_2 - 1$:*

$$\begin{aligned}
&\hat{H}_k = (H_k R_k^{-1} H_k^T)^{-1}, \qquad \hat{Q}_k = Q_k + E_k^T \hat{H}_k E_k,\\
&\hat{A}_k = A_k - B_k R_k^{-1} H_k^T \hat{H}_k E_k, \qquad \hat{B}_k = B_k - B_k R_k^{-1} H_k^T \hat{H}_k H_k,\\
&X_k = R_k^{-1} - R_k^{-1} H_k^T \hat{H}_k H_k R_k^{-1}, \qquad \hat{R}_k = B_k X_k B_k^T,\\
&\hat{q}_k = B_k R_k^{-1} H_k^T \hat{H}_k q_k.
\end{aligned}$$

Note that $\hat{A}_k$ and $\hat{B}_k$ are modifications, inspired from [23], of A_k and B_k, respectively, by taking into account the equality constraints (2.1d) as determined by the index set. The rationale for Definition 2.3 will be made clear later in Lemma 2.8, which investigates the KKT conditions. To prepare for that, we first derive some properties for the matrices defined.

LEMMA 2.4. *Under Assumption 2.1, for any n_1, n_2, and $n_1 \le k \le n_2$, if the index set $\mathcal{I}$ is UDB(λ_H), then we have*

$$\|\hat{A}_k\|_2 \le C_A, \;\; \|\hat{B}_k\|_2 \le C_B, \;\; \|\hat{Q}_k\|_2 \le C_Q, \;\; \lambda_{min}(\hat{Q}_k) \ge \lambda_Q,$$
$$\|H_k\|_2 \le C_H, \;\; \|E_k\|_2 \le C_E, \;\; \|\hat{H}_k\|_2 \le C_{\hat{H}},$$

for some C_A, C_B, C_Q, C_H, C_E, $C_{\hat{H}} > 0$ independent of n_1, n_2, and the particular choice of $\mathcal{I}$. Here λ_Q is the same as that in Assumption 2.1.

Proof. From Assumption 2.1 and Definition 2.3, we have

$$\begin{aligned}
\|H_k\|_2 &\le \|P_{k+1}B_k\|_2 + \|C_k\|_2 \le C_P\widetilde{C}_B + C_C \triangleq C_H,\\
\|E_k\|_2 &\le \|P_{k+1}A_k\|_2 \le C_P\widetilde{C}_A \triangleq C_E,\\
\|\hat{H}_k\|_2 &\le (\lambda_H/C_R)^{-1} \triangleq C_{\hat{H}},\\
\|\hat{A}_k\|_2 &\le \widetilde{C}_A + \widetilde{C}_B C_H C_{\hat{H}} C_E/\lambda_R \triangleq C_A,\\
\|\hat{B}_k\|_2 &\le \widetilde{C}_B + \widetilde{C}_B C_H^2 C_{\hat{H}}/\lambda_R \triangleq C_B,\\
\|\hat{Q}_k\|_2 &\le \widetilde{C}_Q + C_E^2 C_{\hat{H}} \triangleq C_Q,
\end{aligned}$$

and $\lambda_{min}(\hat{Q}_k) \ge \lambda_{min}(Q) \ge \lambda_Q$. □

Note that throughout the article, we use the notations $A \succeq B$ and $A \succ B$ to mean $A-B$ is symmetric positive semidefinite and symmetric positive definite, respectively.

LEMMA 2.5. *For $n_1 \le k \le n_2 - 1$, if the index set $\mathcal{I}$ is UDB(λ_H), then we have $\hat{R}_k = \hat{B}_k R_k^{-1} \hat{B}_k$, and hence $\hat{R}_k \succeq \mathbf{0}$.*

Proof. Since $X_k = R_k^{-1} - R_k^{-1}H_k^T\hat{H}_k H_k R_k^{-1}$, we have $X_k^T = X_k$ and

$$\begin{aligned}
X_k^T R_k X_k &= R_k^{-1} - 2R_k^{-1}H_k^T\hat{H}_k H_k R_k^{-1} + R_k^{-1}H_k^T\hat{H}_k(H_k R_k^{-1}H_k^T)\hat{H}_k^T H_k R_k^{-1}\\
&\overset{\text{Def 2.3}}{=} R_k^{-1} - 2R_k^{-1}H_k^T\hat{H}_k H_k R_k^{-1} + R_k^{-1}H_k^T\cancel{\hat{H}_k\hat{H}_k^{-1}}\hat{H}_k^T H_k R_k^{-1}\\
&= R_k^{-1} - R_k^{-1}H_k^T\hat{H}_k H_k R_k^{-1}\\
&= X_k.
\end{aligned} \tag{2.3}$$

Note that $\hat{B}_k = B_k X_k R_k$ from Definition 2.3, so we have

$$\hat{R}_k = B_k X_k B_k^T \overset{(2.3)}{=} B_k X_k R_k X_k B_k^T = (B_k X_k R_k) R_k^{-1}(R_k X_k B_k^T) = \hat{B}_k R_k^{-1}\hat{B}_k^T.$$

Since $R_k \succ \mathbf{0}$, we have that $\hat{R}_k \succeq \mathbf{0}$. □

To prove the results in this subSection, we employ the approach in [23] by considering the KKT conditions of problem (2.1). We first define and derive properties for the following matrices, some of which are similar to those in [23].

DEFINITION 2.6. *For some index set that is UDB(λ_H), define the following backward recursions for $n_1 \le k \le n_2 - 1$:*

$$K_{n_2} = Q_{n_2}, \qquad T_{n_2} = -Q_{n_2}d_{n_2}, \tag{2.4a}$$
$$W_k = R_k + \hat{B}_k^T K_{k+1}\hat{B}_k, \tag{2.4b}$$
$$M_k = (I + \hat{R}_k K_{k+1})^{-1}, \tag{2.4c}$$

$$D_k = M_k \hat{A}_k, \tag{2.4d}$$

$$T_k = D_k^T T_{k+1} + \hat{A}_k^T K_{k+1} M_k \hat{q}_k - E_k^T \hat{H}_k q_k - Q_k d_k, \tag{2.4e}$$

$$K_k = \hat{Q}_k + \hat{A}_k^T K_{k+1} D_k. \tag{2.4f}$$

LEMMA 2.7. *For $n_1 \le k \le n_2 - 1$, if the index set $\mathcal{I}$ is UDB(λ_H), then M_k is well defined, $K_k \succ \mathbf{0}$, $W_k \succ \mathbf{0}$, $K_{k+1}M_k \succ \mathbf{0}$, and*

$$M_k^T = I - K_{k+1} M_k \hat{R}_k. \tag{2.5}$$

Proof. We prove the statement by backward induction based on (2.4a)–(2.4f). To start, we have $K_{n_2} = Q_{n_2} \succ \mathbf{0}$ as the induction basis. Suppose K_{k+1} is positive definite. Then we have $W_k \succ R_k \succ \mathbf{0}$, and

$$I + \hat{R}_k K_{k+1} = (K_{k+1}^{-1} + \hat{R}_k) K_{k+1}.$$

Since $\hat{R}_k \succeq \mathbf{0}$ as shown in Lemma 2.5, we have that $I + \hat{R}_k K_{k+1}$ is invertible and hence M_k is well defined. Also we have

$$K_{k+1} M_k \overset{(2.4c)}{=} K_{k+1}(I + \hat{R}_k K_{k+1})^{-1} = \left((I + \hat{R}_k K_{k+1}) K_{k+1}^{-1}\right)^{-1} = (K_{k+1}^{-1} + \hat{R}_k)^{-1} \succ \mathbf{0},$$

which implies that

$$K_k \overset{(2.4f),(2.4d)}{=} \hat{Q}_k + \hat{A}_k^T (K_{k+1} M_k) \hat{A}_k \succeq \hat{Q}_k \overset{\text{Def 2.3}}{\succeq} Q_k \succ \mathbf{0},$$

so that K_k is positive definite. By induction we have that M_k is well defined, $K_k \succ \mathbf{0}$, $W_k \succ \mathbf{0}$, and $K_{k+1} M_k \succ \mathbf{0}$ for all $n_1 \le k \le n_2 - 1$.

Note that since K_k is symmetric,

$$\begin{aligned} M_k^{-T}(I - K_{k+1} M_k \hat{R}_k) &\overset{(2.4c)}{=} (I + K_{k+1} \hat{R}_k)(I - K_{k+1} M_k \hat{R}_k) \\ &= I + K_{k+1}(I - M_k - \hat{R}_k K_{k+1} M_k) \hat{R}_k \\ &\overset{(2.4c)}{=} I + K_{k+1}(\cancel{I - M_k^{-1} M_k}) \hat{R}_k \\ &= I. \end{aligned}$$

Therefore (2.5) holds. ◻

Note that (2.5) is also stated (without proof) in [23]. Now we derive a recursion of the optimal states of problem (2.1) by investigating the KKT conditions.

LEMMA 2.8. *Let u_k^* and x_k^* be the optimal controls and states of problem* (2.1)*, and let λ_k^* and η_k^* be the Lagrange multipliers associated with the dynamical constraint* (2.1c) *and the equality constraint* (2.1d)*, respectively. For $n_1 \le k \le n_2 - 1$, if the index set $\mathcal{I}$ is UDB(λ_H), then we have*

$$u_k^* = R_k^{-1}(H_k^T \eta_k^* - B_k^T \lambda_k^*), \tag{2.6a}$$

$$\eta_k^* = \hat{H}_k(-E_k x_k^* + H_k R_k^{-1} B_k^T \lambda_k^* + q_k), \tag{2.6b}$$

$$\lambda_k^* = K_{k+1} x_{k+1}^* + T_{k+1}, \tag{2.6c}$$

$$x_{k+1}^* = D_k x_k^* - M_k \hat{R}_k T_{k+1} + M_k \hat{q}_k, \tag{2.6d}$$

where K_k and T_k are given by the backward recursions (2.4a)–(2.4f).

Proof. The KKT conditions of problem (2.1) are

$$R_k u_k^* + B_k^T \lambda_k^* - H_k^T \eta_k^* = 0, \qquad n_1 \le k \le n_2 - 1, \tag{2.7a}$$
$$Q_k(x_k^* - d_k) + A_k^T \lambda_k^* - \lambda_{k-1}^* - E_k^T \eta_k^* = 0, \qquad n_1 + 1 \le k \le n_2, \tag{2.7b}$$
$$Q_{n_2}(x_{n_2}^* - d_{n_2}) - \lambda_{n_2-1}^* = 0, \tag{2.7c}$$
$$x_{k+1}^* = A_k x_k^* + B_k u_k^*, \qquad n_1 \le k \le n_2 - 1, \tag{2.7d}$$
$$E_k x_k^* + H_k u_k^* = q_k, \qquad n_1 \le k \le n_2 - 1. \tag{2.7e}$$

Condition (2.7a) directly gives (2.6a). Substituting (2.6a) into (2.7e) gives

$$\begin{aligned} &E_k x_k^* + H_k R_k^{-1}(H_k^T \eta_k^* - B_k^T \lambda_k^*) = q_k, \\ &\Rightarrow E_k x_k^* + (H_k R_k^{-1} H_k^T)\eta_k^* - H_k R_k^{-1} B_k^T \lambda_k^* = q_k, \end{aligned}$$

and this gives (2.6b) from Definition 2.3. Substituting (2.6a) and (2.6b) into (2.7d) gives

$$\begin{aligned} x_{k+1}^* &= A_k x_k^* + B_k R_k^{-1}(H_k^T \eta_k^* - B_k^T \lambda_k^*) \\ &= A_k x_k^* + B_k R_k^{-1}\left(H_k^T \hat{H}_k(-E_k x_k^* + H_k R_k^{-1} B_k^T \lambda_k^* + q_k) - B_k^T \lambda_k^*\right) \\ &= (A_k - B_k R_k^{-1} H_k^T \hat{H}_k E_k) x_k^* - B_k(R_k^{-1} - R_k^{-1} H_k^T \hat{H}_k H_k R_k^{-1}) B_k^T \lambda_k^* \\ &\quad + B_k R_k^{-1} H_k^T \hat{H}_k q_k \\ &\overset{\text{Def 2.3}}{=} \hat{A}_k x_k^* - \hat{R}_k \lambda_k^* + \hat{q}_k. \end{aligned} \tag{2.8}$$

Substituting (2.6b) into (2.7b) gives

$$\begin{aligned} &Q_k(x_k^* - d_k) + A_k^T \lambda_k^* - \lambda_{k-1}^* - E_k^T \hat{H}_k(-E_k x_k^* + H_k R_k^{-1} B_k^T \lambda_k^* + q_k) = 0, \\ &\Rightarrow (Q_k + E_k^T \hat{H}_k E_k) x_k^* - Q_k d_k + (A_k - B_k R_k^{-1} H_k^T \hat{H}_k E_k)^T \lambda_k^* - \lambda_{k-1}^* - E_k^T \hat{H}_k q_k = 0. \end{aligned}$$

From Definition 2.3, we then have, for $n_1 \le k < n_2 - 1$,

$$\lambda_{k-1}^* = \hat{A}_k^T \lambda_k^* + \hat{Q}_k x_k^* - Q_k d_k - E_k^T \hat{H}_k q_k. \tag{2.9}$$

From (2.7c) we also have

$$\lambda_{n_2-1}^* = Q_{n_2}(x_{n_2}^* - d_{n_2}). \tag{2.10}$$

We prove (2.6c) and (2.6d) by backward induction. The statement (2.6c) holds for $k = n_2 - 1$ from (2.10) and (2.4a). Suppose (2.6c) holds for λ_k^*. Then substituting (2.6c) into (2.8) gives

$$x_{k+1}^* = \hat{A}_k x_k^* - \hat{R}_k K_{k+1} x_{k+1}^* - \hat{R}_k T_{k+1} + \hat{q}_k,$$

which leads to

$$(I + \hat{R}_k K_{k+1}) x_{k+1}^* = \hat{A}_k x_k^* - \hat{R}_k T_{k+1} + \hat{q}_k.$$

tTherefore from (2.4c) we have

$$x_{k+1}^* = M_k(\hat{A}_k x_k^* - \hat{R}_k T_{k+1} + \hat{q}_k), \tag{2.11}$$

and (2.6d) holds for x_{k+1}^*. Then for λ_{k-1}^*, using (2.9) and (2.6c), we have

$$\begin{aligned}
\lambda_{k-1}^* &= \hat{Q}_k x_k^* + \hat{A}_k^T (K_{k+1} x_{k+1}^* + T_{k+1}) - Q_k d_k - E_k^T \hat{H}_k q_k \\
&\overset{(2.11)}{=} \hat{Q}_k x_k^* + \hat{A}_k^T K_{k+1} M_k (\hat{A} x_k^* - \hat{R}_k T_{k+1} + \hat{q}_k) + \hat{A}_k T_{k+1} - Q_k d_k - E_k^T \hat{H}_k q_k \\
&= (\hat{Q}_k + \hat{A}_k^T K_{k+1} M_k \hat{A}_k) x_k^* + \hat{A}_k^T (I - K_{k+1} M_k \hat{R}_k) T_{k+1} \\
&\quad + \hat{A}_k^T K_{k+1} M_k \hat{q}_k - Q_k d_k - E_k^T \hat{H}_k q_k \\
&\overset{(2.4d),(2.5)}{=} (\hat{Q}_k + \hat{A}_k^T K_{k+1} D_k) x_k^* + \hat{A}_k^T M_k^T T_{k+1} + \hat{A}_k^T K_{k+1} M_k \hat{q}_k - Q_k d_k - E_k^T \hat{H}_k q_k \\
&\overset{(2.4f),(2.4d),(2.4e)}{=} K_k x_k^* + T_k,
\end{aligned}$$

and hence (2.6c) holds for λ_{k-1}^*. □

In the following, we investigate the notion of controllability for the system (2.1c)–(2.1d). We define the following controllability matrix for the index set $\mathcal{I}$ in terms of the sequence pair $\{\hat{A}_k, \hat{B}_k\}$ in Definition 2.3.

Definition 2.9. *For some $n_1 \le q \le n_2$, $t > 0$, and some index set $\mathcal{I}$ that is UDB(λ_H), define the controllability matrix associated with time steps $[q, q+t-1]$ as*

$$C_{q,t}(\mathcal{I}) = \left[\hat{B}_{q+t-1} \quad \hat{A}_{q+t-1}\hat{B}_{q+t-2} \quad \ldots, \left(\prod_{l=1}^{t-1} \hat{A}_{q+l}\right) \hat{B}_q\right].$$

To see the relationship between $C_{q,t}(\mathcal{I})$ and the controllability of the equality constrained system (2.1c)–(2.1d), we start by defining the notion of controllability for the system (2.1c)–(2.1d).

Definition 2.10. *Given an index set, define E_k and H_k as in (2.2). At time step q, the system*

$$\begin{aligned}
&x_{k+1} = A_k x_k + B_k u_k, \qquad q \le k \le n_2 - 1, \qquad x_q = x_q^0, \\
&E_k x_k + H_k u_k = q_k, \qquad q \le k \le n_2 - 1
\end{aligned} \tag{2.12}$$

is controllable in t steps if for any x_q^0 and $\overline{x}$, there exist admissible controls $\{\overline{u}_k\}_{k=q:q+t-1}$ and corresponding states $\{\overline{x}_k\}_{k=q+1:q+t}$ satisfying (2.12) and $\overline{x}_{q+t} = \overline{x}$.

Proposition 2.11. *If the index set $\mathcal{I}$ is UDB(λ_H) and the resulting constrained system (2.1c)–(2.1d) of problem (2.1) is controllable at time point q in t steps, then $C_{q,t}(\mathcal{I})$ has full row rank.*

Proof. The system (2.1c)–(2.1d) being controllable in t steps implies that there exist admissible controls $\{\overline{u}_k\}_{k=q:q+t-1}$ and corresponding states $\{\overline{x}_k\}_{k=q+1:q+t}$ so that $\overline{x}_{q+t} = \overline{x}$ for any $\overline{x}$. Then we have, for $q \le k \le q+t-1$,

$$\begin{aligned}
\overline{x}_{k+1} &= A_k \overline{x}_k + B_k \overline{u}_k \\
&\overset{(2.1d)}{=} A_k \overline{x}_k + B_k \overline{u}_k - B_k R_k^{-1} H_k^T \hat{H}_k (E_k \overline{x}_k + H_k \overline{u}_k - q_k) \\
&\overset{\text{Def 2.3}}{=} \hat{A}_k \overline{x}_k + \hat{B}_k \overline{u}_k + \hat{q}_k,
\end{aligned}$$

which means that the same sequences $\{\overline{u}_k\}$ and $\{\overline{x}_k\}$ also satisfy the linear dynamics

$$x_{k+1} = \hat{A}_k x_k + \hat{B}_k u_k + \hat{q}_k \tag{2.13}$$

and that $\overline{x}_{q+t} = \overline{x}$. In other words, (2.13) can be controlled in t steps to $\overline{x}$. Since $\overline{x}$ is arbitrary, it follows that $C_{q,t}(\mathcal{I})$ has full row rank. □

Proposition 2.11 connects the controllability of the equality-constrained system (2.1c)–(2.1d) to the full rank of a related controllability matrix $C_{q,t}(\mathcal{I})$. For our purpose, however, we need a uniform boundedness property of the controllability matrix, which is stronger than the standard assumption of merely full rankness.

DEFINITION 2.12. *For some index set $\mathcal{I}$ that is UDB(λ_H), let $\hat{A}_k$, $\hat{B}_k$ be as in Definition 2.3. With some $0 < t < n_2 - n_1$ and $\lambda_C > 0$, the index set is uniformly completely controllable with respect to λ_C, denoted as UCC(λ_C), if the sequence pair $\{\hat{A}_k, \hat{B}_k\}$ is uniformly completely controllable [11, Definition 3.1], i.e., for any $n_1 \leq q \leq n_2$,*

$$\lambda_{min}\left(C_{q,t}(\mathcal{I})C_{q,t}^T(\mathcal{I})\right) \geq \lambda_C > 0.$$

The main purpose of this subsection is to investigate the dependencies of the solutions of problem (2.1) on the initial value $x_{n_1}^0$ and terminal reference d_{n_2} for some index set that is UDB(λ_H) and UCC(λ_C). To start, we derive properties for the quantities defined in Definition 2.6. The proofs of the results regarding the Riccati matrix K_k and closed-loop matrix D_k are structurally the same as those in [24], and hence they are provided in the Appendix.

LEMMA 2.13. *For $n_1 \leq k \leq n_2 - 1$, if the index set $\mathcal{I}$ is UDB(λ_H), we have*

$$M_k = I - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}, \tag{2.14a}$$

$$K_k = \hat{Q}_k + \hat{A}_k^T K_{k+1} \hat{A}_k - \hat{A}_k^T K_{k+1} \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1} \hat{A}_k, \tag{2.14b}$$

where M_k, K_k, and W_k are from Definition 2.6.

Proof. Definition 2.6 and Lemma 2.5 imply that

$$M_k = \left(I + \hat{B}_k R_k^{-1} \hat{B}_k^T K_{k+1}\right)^{-1}.$$

Then we have

$$\begin{aligned}
&M_k^{-1}\left(I - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}\right) \\
&\quad= \left(I + \hat{B}_k R_k^{-1} \hat{B}_k^T K_{k+1}\right)\left(I - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}\right) \\
&\quad= I + \hat{B}_k R_k^{-1} \hat{B}_k^T K_{k+1} - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1} - \hat{B}_k R_k^{-1} \hat{B}_k^T K_{k+1} \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1} \\
&\quad= I + \hat{B}_k \left(R_k^{-1} - W_k^{-1} - R_k^{-1} \hat{B}_k^T K_{k+1} \hat{B}_k W_k^{-1}\right) \hat{B}_k^T K_{k+1} \\
&\quad= I + \hat{B}_k \left(R_k^{-1} W_k - I - R_k^{-1} \hat{B}_k^T K_{k+1} \hat{B}_k\right) W_k^{-1} \hat{B}_k^T K_{k+1} \\
&\quad= I + \hat{B}_k \left(R_k^{-1}\left(W_k - \hat{B}_k^T K_{k+1} \hat{B}_k\right) - I\right) W_k^{-1} \hat{B}_k^T K_{k+1} \\
&\quad\overset{(2.4b)}{=} I + \hat{B}_k \left(\cancel{R_k^{-1} R_k - I}\right) W_k^{-1} \hat{B}_k^T K_{k+1} \\
&\quad= I.
\end{aligned}$$

Hence, $M_k = I - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}$. Substituting (2.14a) into (2.4d) and (2.4f), we have

$$K_k = \hat{Q}_k + \hat{A}_k^T K_{k+1}\left(I - \hat{B}_k W_k^{-1} \hat{B}_k^T K_{k+1}\right)\hat{A}_k,$$

which proves (2.14b). ∎

PROPOSITION 2.14. *Under Assumption 2.1, if the index set $\mathcal{I}$ is UDB(λ_H) and UCC(λ_C), then for any $n_1 \leq q \leq n_2$, we have $\|K_q\|_2 \leq \beta$ for some $\beta > 0$ independent of n_1, n_2, and the particular choice of $\mathcal{I}$.*

Proof. See Appendix A.1; also see [24, Proposition 2.7]. ∎

PROPOSITION 2.15. *Under Assumption 2.1, for any $q \le j \le n_2 - 1$, if the index set $\mathcal{I}$ is UDB(λ_H) and UCC(λ_C), then we have*

$$\left\| \prod_{l=q}^{j} D_l \right\|_2 \le C_1 \rho^{j-q+1},$$

where $C_1 = \sqrt{\beta/\lambda_Q}$, $\rho = 1/\sqrt{1 + (\lambda_Q/\beta)}$, and C_1, ρ are independent of n_1, n_2, and the particular choice of $\mathcal{I}$.

Proof. See Appendix A.2; also see [24, Proposition 2.8]. ∎

In the following, we show the dependencies of the solutions to problem (2.1) on the initial state and terminal reference decay exponentially. To start, we prove a short lemma about the recursion defined in (2.4e).

LEMMA 2.16. *For $n_1 + 1 \le k \le n_2$, if the index set $\mathcal{I}$ is UDB(λ_H), then we have*

$$\|\nabla_{d_{n_2}} T_k\|_2 \le C_s \rho^{n_2 - k} \tag{2.15}$$

for some $C_s > 0$ independent of n_1, n_2, and the particular choice of $\mathcal{I}$.

Proof. Recursion (2.4e) gives

$$\nabla_{d_{n_2}} T_k = -\left(\prod_{l=k}^{n_2 - 1} D_l \right)^T Q_{n_2}.$$

The statement is proved by using Proposition 2.15 and taking $C_s = \widetilde{C}_Q C_1$ for $\widetilde{C}_Q$ defined in Assumption 2.1. ∎

PROPOSITION 2.17. *Let x_k^* and u_k^* be the optimal states and controls of problem (2.1). Under Assumption 2.1, if the index set $\mathcal{I}$ is UDB(λ_H) and UCC(λ_C), then*

$$\|\nabla_{x_{n_1}} x_k^*\|_2 \le Z_1 \rho^{k-n_1}, \ \|\nabla_{d_{n_2}} x_k^*\|_2 \le Z_2 \rho^{n_2-k}, \ n_1 + 1 \le k \le n_2,$$
$$\|\nabla_{x_{n_1}} u_k^*\|_2 \le Z_1 \rho^{k-n_1}, \ \|\nabla_{d_{n_2}} u_k^*\|_2 \le Z_2 \rho^{n_2-k}, \ n_1 \le k \le n_2 - 1,$$

for some $Z_1, Z_2 > 0$ independent of n_1, n_2, and the particular choice of $\mathcal{I}$.

Proof. FromAssumption 2.1 and Lemma 2.4 we have, for X_k and $\hat{R}_k$ defined in Definition 2.3,

$$\|X_k\|_2 \le 1/\lambda_R + C_H^2 C_{\hat{H}}/\lambda_R^2 \triangleq C_X, \qquad \|\hat{R}_k\|_2 \le \widetilde{C}_B^2 C_X \triangleq C_{\hat{R}}.$$

Lemmas 2.13 and 2.4 and Proposition 2.14 give

$$\|M_k\|_2 \le 1 + C_B^2 \beta/\lambda_R \triangleq C_M, \qquad \|D_k\|_2 \le \|M_k\|_2 \|\hat{A}_k\|_2 \le C_M C_A \triangleq C_D,$$

where the first inequality uses the relation $W_k \succeq R_k$, which is given by (2.4b) and Lemma 2.7. From Lemma 2.8 we have

$$x_{k+1}^* = D_k x_k^* - M_k \hat{R}_k T_{k+1} + M_k \hat{q}_k, \tag{2.16}$$

so

$$\|\nabla_{x_{n_1}} x_k^*\|_2 = \left\| \prod_{l=n_1}^{k-1} D_l \right\|_2 \le C_1 \rho^{k-n_1}, \tag{2.17}$$

which follows from Proposition 2.15. Also, from (2.16) we have

$$\nabla_{d_{n_2}} x^*_{k+1} = D_k \left(\nabla_{d_{n_2}} x^*_k\right) - M_k \hat{R}_k \left(\nabla_{d_{n_2}} T_{k+1}\right) \tag{2.18}$$

and $\nabla_{d_{n_2}} x_{n_1} = \mathbf{0}$. From Recursion (2.18) we have

$$\nabla_{d_{n_2}} x^*_k = -\sum_{i=n_1+1}^{k} \left(\prod_{l=i}^{k-1} D_l\right) M_{i-1} \hat{R}_{i-1} \left(\nabla_{d_{n_2}} T_i\right),$$

from which, using Proposition 2.15 and Lemma 2.16, we have, for $C \triangleq C_1 C_M C_{\hat{R}} C_s$,

$$\begin{aligned} \|\nabla_{d_{n_2}} x^*_k\|_2 &\le \sum_{i=n_1+1}^{k} C_1 \rho^{k-i} C_M C_{\hat{R}} C_s \rho^{n_2-i} \\ &= C\rho^{n_2-k} \sum_{i=n_1+1}^{k} \rho^{2(k-i)} \\ &\le \frac{C}{1-\rho^2} \rho^{n_2-k}. \end{aligned} \tag{2.19}$$

Equation (2.6a) gives

$$\begin{aligned} u^*_k &= R_k^{-1}(H_k^T \eta^*_k - B_k^T \lambda^*_k) \\ &\overset{(2.6b)}{=} R_k^{-1} H_k^T \hat{H}_k(-E_k x^*_k + H_k R_k^{-1} B_k^T \lambda^*_k + q_k) - R_k^{-1} B_k^T \lambda^*_k \\ &\overset{\text{Def } 2.3}{=} -R_k^{-1} H_k^T \hat{H}_k E_k x^*_k - X_k B_k^T \lambda^*_k + R_k^{-1} H_k^T \hat{H}_k q_k \\ &\overset{(2.6c)}{=} -R_k^{-1} H_k^T \hat{H}_k E_k x^*_k + R_k^{-1} H_k^T \hat{H}_k q_k - X_k B_k^T (K_{k+1} x^*_{k+1} + T_{k+1}) \\ &\overset{(2.6d)}{=} -R_k^{-1} H_k^T \hat{H}_k E_k x^*_k + R_k^{-1} H_k^T \hat{H}_k q_k \\ &- X_k B_k^T K_{k+1}(D_k x^*_k - M_k \hat{R}_k T_{k+1} + M_k \hat{q}_k) - X_k B_k^T T_{k+1} \\ &= -(R_k^{-1} H_k^T \hat{H}_k E_k + X_k B_k^T K_{k+1} D_k) x^*_k + R_k^{-1} H_k^T \hat{H}_k q_k \\ &- X_k B_k^T (I - K_{k+1} M_k \hat{R}_k) T_{k+1} - X_k B_k^T K_{k+1} M_k \hat{q}_k. \end{aligned}$$

In this expression, for the term multiplying x^*_k and T_{k+1}, we have the following from Assumption 2.1 and Lemma 2.4, respectively:

$$\begin{aligned} &\|R_k^{-1} H_k^T \hat{H}_k E_k + X_k B_k^T K_{k+1} D_k\|_2 \le C_H C_{\hat{H}} C_E / \lambda_R + C_X \widetilde{C}_B \beta C_D \triangleq Y_1, \\ &\|X_k B_k^T (I - K_{k+1} M_k \hat{R}_k)\|_2 \le C_X \widetilde{C}_B (1 + \beta C_M C_{\hat{R}}) \triangleq Y_2. \end{aligned}$$

Note that from (2.4e), T_k does not depend on the initial value x_{n_1}. Therefore we have

$$\|\nabla_{x_{n_1}} u^*_k\|_2 \le Y_1 \|\nabla_{x_{n_1}} x^*_k\|_2 \overset{(2.17)}{\le} Y_1 C_1 \rho^{k-n_1} \tag{2.20}$$

and

$$\begin{aligned} \|\nabla_{d_{n_2}} u^*_k\|_2 &\le Y_1 \|\nabla_{d_{n_2}} x^*_k\|_2 + Y_2 \|\nabla_{d_{n_2}} T_{k+1}\|_2 \\ &\overset{(2.19),\text{Lemma } 2.16}{\le} \frac{Y_1 C}{1-\rho^2} \rho^{n_2-k} + \frac{Y_2 C_s}{\rho} \rho^{n_2-k}. \end{aligned} \tag{2.21}$$

Considering (2.17), (2.19), (2.20), and (2.21) and letting

$$Z_1 = \max(C_1, Y_1C_1), \qquad Z_2 = \max(\frac{C}{1-\rho^2}, \frac{Y_1C}{1-\rho^2} + \frac{Y_2C_s}{\rho}),$$

prove the statement. ∎

Proposition 2.17 is the main result of this subsection. It shows that the effect of the initial state (or terminal reference) on the solutions of problem (2.1) decays exponentially fast in the time distance between the solution and the initial (or terminal) time point. Moreover, under the uniform boundedness Assumption 2.1, the decay rate is independent of the problem interval $[n_1, n_2]$, and the particular choice of the index set given it is UDB(λ_H) and UCC(λ_C). This property is essential for proving that a receding horizon strategy approximates the solution on the full horizon in Section 3. We now conclude this subsection with a boundedness result of the solutions and adjoint variables of problem (2.1).

ASSUMPTION 2.18. *For any n_1, n_2 and $n_1 \le k \le n_2$, we have $\|d_k\|_2 \le m_0$ and $\|x^0_{n_1}\|_2 \le u_0$.*

LEMMA 2.19. *Let x^*_k and λ^*_k be the optimal states and adjoint variables of problem (2.1), respectively. Under Assumptions 2.1 and 2.18, if the index set $\mathcal{I}$ is UDB(λ_H) and UCC(λ_C), then we have*

$$\|x^*_k\|_2 \le C_g,\ n_1+1 \le k \le n_2; \ \ \|\lambda^*_k\|_2 \le C_\lambda,\ n_1 \le k \le n_2 - 1$$

for some C_g, $C_\lambda > 0$ independent of n_1, n_2, and the particular choice of $\mathcal{I}$.

Proof. In (2.4e), denote $\overline{T}_{n_2} = T_{n_2}$ and for $k < n_2$,

$$\overline{T}_k = \hat{A}^T_k K_{k+1} M_k \hat{q}_k - E^T_k \hat{H}_k q_k - Q_k d_k,$$

so that $T_k = D^T_k T_{k+1} + \overline{T}_k$ and $T_{n_2} = -Q_{n_2} d_{n_2}$. As a result, we have

$$T_k = \sum_{i=k}^{n_2} \left(\prod_{l=k}^{i-1} D_l \right)^T \overline{T}_i. \tag{2.22}$$

From Assumption 2.1 and Definition 2.3 we have

$$\|q_k\|_2 \le U, \qquad \|\hat{q}_k\|_2 \le \widetilde{C}_B C_H C_{\hat{H}} U/\lambda_R \triangleq C_{\hat{q}}.$$

From Lemmas 2.13 and 2.4 and Proposition 2.14 we have $\|M_k\|_2 \le 1 + C_B^2\beta/\lambda_R \triangleq C_M$. Then using Assumption 2.1, Lemma 2.4, and Proposition 2.14, we have

$$\|\overline{T}_k\|_2 \le C_A \beta C_M C_{\hat{q}} + C_E C_{\hat{H}} U + \widetilde{C}_Q m_0 \triangleq C_{\overline{T}}.$$

Combining the above with Proposition 2.15, we have

$$\|T_k\|_2 \le \sum_{i=k}^{n_2} C_{\overline{T}} C_1 \rho^{i-k} \le C_{\overline{T}} C_1/(1-\rho) \triangleq C_T. \tag{2.23}$$

Denote $G_k \triangleq -M_k \hat{R}_k T_{k+1} + M_k \hat{q}_k$. Then, from Lemma 2.8, we have $x^*_{k+1} = D_k x^*_k + G_k$. Thus,

$$x^*_k = \sum_{i=n_1}^{k-1} \left(\prod_{l=i+1}^{k-1} D_l \right) G_i + \left(\prod_{l=n_1}^{k-1} D_l \right) x_{n_1}.$$

Note that $\|G_k\|_2 \le C_M C_{\hat{R}} C_T + C_M C_{\hat{q}} \triangleq C_G,$. Thus, from Proposition 2.15 we have

$$\text{(2.24)}\quad \|x_k^*\| \le \sum_{i=n_1}^{k-1} \left(C_G C_1 \rho^{k-i-1}\right) + C_1 u_0 \rho^{k-n_1} \le C_1 C_G/(1-\rho) + C_1 u_0 \triangleq C_g.$$

Next, we prove that the bound on λ_k^*. Lemma 2.8 gives

$$\lambda_k^* = K_{k+1} x_{k+1}^* + T_{k+1}.$$

Using Proposition 2.14, (2.23) and (2.24), we have

$$\|\lambda_k^*\|_2 \le \beta \|x_{k+1}^*\|_2 + C_T \le \beta C_g + C_T \triangleq C_\lambda.$$

This completes the proof. ∎

2.2. Path-constrained inequality problem. In this subsection, we return to the inequality-constrained problem (1.1). Using the results established for (2.1), we investigate the solutions and adjoint variables of (1.1). We make the following controllability assumption of the active set of problem (1.1) at optimality.

ASSUMPTION 2.20. *Let $\mathcal{A}$ be the active set of problem* (1.1) *at optimality. Then*
- *(a) $\mathcal{A}$ is UDB(λ_H) as defined in Definition 2.2;*
- *(b) the equality-constrained system* (2.1c)–(2.1d) *corresponding to $\mathcal{A}$ is controllable in t steps as defined in Definition 2.10 for any $n_1 \le q \le n_2$;*
- *(c) under (b), $C_{q,t}(\mathcal{A})$ has full row rank by Proposition 2.11, and we further assume that $\mathcal{A}$ is UCC(λ_C) as defined in Definition 2.12.*

COROLLARY 2.21. *Let x_k^* and λ_k^* be respectively the optimal states and adjoint variables of problem* (1.1). *Under Assumptions 2.1, 2.18, and 2.20, we have*

$$\|x_k^*\|_2 \le C_g,\ n_1+1 \le k \le n_2;\ \ \|\lambda_k^*\|_2 \le C_\lambda,\ n_1 \le k \le n_2 - 1$$

for C_g, $C_\lambda > 0$ defined in Lemma 2.19.

Proof. Note that when the index set for problem (2.1) is the active set $\mathcal{A}$ of problem (1.1) at optimality, problems (1.1) and (2.1) have the same solutions and adjoint variables. Since $\mathcal{A}$ is UDB(λ_H) and UCC(λ_C) by Assumption 2.20, applying Lemma 2.19 gives the result. ∎

3. Lag L receding horizon strategy. In this section, we prove an exponentially decaying approximation error for a lag L receding horizon strategy. Let $N > L$ be the length of each but the last receding horizon, and let $n_0 = \lfloor (n_2 - n_1 - N + 1)/L \rfloor + 1$ be the number of receding horizons. Then for $i = 1, \dots, n_0$, define the ith receding horizon $\mathcal{R}_i = [n_1'(i), n_2'(i)]$ as

$$\text{(3.1)}\qquad n_1'(i) = n_1 + L(i-1), \qquad n_2'(i) = \begin{cases} n_1'(i) + N - 1, & 1 \le i \le n_0 - 1, \\ n_2, & i = n_0. \end{cases}$$

For simplicity, we denote $m \triangleq n_1'(n_0)$ to be the starting index of the last receding horizon. Note that with (3.1), we have $n_1'(i+1) = n_1'(i) + L$ and that the length $N_1 = n_2 - m + 1$ of the last receding horizon satisfies $N \le N_1 < N + L$. On a receding horizon $\mathcal{R}_i$, we define the following parametrized problem whose parameters are the initial state and terminal reference.

Definition 3.1. *For $i = 1, \ldots, n_0$, define the parametrized problem P_θ^i with $\theta = (\theta^{(h)}, \theta^{(d)})$ as follows:*

$$\min \quad \frac{1}{2} \sum_{k=n_1'(i)}^{n_2'(i)-1} w_k^T R_k w_k + (h_k - d_k)^T Q_k (h_k - d_k) \tag{3.2a}$$

$$+ (h_{n_2'(i)} - d_{n_2'(i)})^T Q_{n_2'(i)} (h_{n_2'(i)} - d_{n_2'(i)}) \tag{3.2b}$$

$$\text{s.t.} \quad h_{k+1} = A_k h_k + B_k w_k, \qquad n_1'(i) \le k \le n_2'(i) - 1 \tag{3.2c}$$

$$\widetilde{P}_{k+1} h_{k+1} + \widetilde{C}_k w_k \ge \widetilde{q}_k, \qquad n_1'(i) \le k \le n_2'(i) - 1 \tag{3.2d}$$

$$h_{n_1'(i)} = \theta^{(h)}, \ \ d_{n_2'(i)} = \theta^{(d)}, \tag{3.2e}$$

where $d_{n_1'(i):n_2'(i)-1}$ is the reference trajectory of problem (1.1).

The parametrized problem P_θ^i is essentially a subproblem of (1.1) restricted on the receding horizon $\mathcal{R}_i$ with terminal reference parametrized by $\theta^{(d)}$ and reinitialized with $\theta^{(h)}$. Denote $x_k^*(P_\theta^i)$, $u_k^*(P_\theta^i)$ as the optimal state and control, respectively, of problem P_θ^i at some time point $k \in \mathcal{R}_i$. Then the RHC policy (e.g., [9, 14]) is the sequence $\{\widetilde{u}_k\}_{k=n_1}^{n_2}$ defined as

$$\begin{aligned} &\widetilde{u}_{n_1'(i)+j-1} = u_{n_1'(i)+j-1}^*(P_{\theta_0(i)}^i), \ \ 1 \le j \le L, \ \ 1 \le i \le n_0 - 1, \\ &\widetilde{u}_k = u_k^*(P_{\theta_0(n_0)}^{n_0}), \ \ m \le k \le n_2 - 1, \end{aligned} \tag{3.3}$$

where $\theta_0(i) = (\widetilde{x}_{n_1'(i)}, d_{n_2'(i)})$, and the state sequence $\{\widetilde{x}_k\}_{k=n_1}^{n_2}$ is defined as

$$\begin{aligned} &\widetilde{x}_{n_1'(1)} = x_{n_1}^0, \\ &\widetilde{x}_{n_1'(i)+j} = x_{n_1'(i)+j}^*(P_{\theta_0(i)}^i), \ \ 1 \le j \le L, \ \ 1 \le i \le n_0 - 1, \\ &\widetilde{x}_k = x_k^*(P_{\theta_0(n_0)}^{n_0}), \ \ m+1 \le k \le n_2. \end{aligned} \tag{3.4}$$

In other words, the RHC policies $\widetilde{u}_{n_1'(i)+j-1}$ for $1 \le j \le L$ and $1 \le i \le n_0 - 1$ are obtained by solving problem $P_{\theta_0(i)}^i$ on $\mathcal{R}_i$ initialized with $\widetilde{x}_{n_1'(i)} = \widetilde{x}_{n_1'(i-1)+L}$, which in turn is obtained by solving $P_{\theta_0(i-1)}^{i-1}$ on $\mathcal{R}_{i-1}$. On the last receding horizon, $\widetilde{u}_k$ for $m \le k \le n_2 - 1$ are defined as the optimal controls of problem $P_{\theta_0(n_0)}^{n_0}$ on $\mathcal{R}_{n_0}$. To bound the error of this RHC strategy, we need to relate the solutions of problems $P_{\theta_0(i)}^i$ to those of problem (1.1). To start, we consider a different choice of the parameter $\theta_1(i)$. The following result establishes a connection between the solutions of (1.1) and those of $P_{\theta_1(i)}^i$.

Proposition 3.2. *Let $(u_{n_1:n_2-1}^*, x_{n_1+1:n_2}^*)$ and λ_k^* be the solutions and optimal adjoint variables of problem* (1.1) *with some initial value $x_{n_1}^0$, and let η_k^* be the optimal Lagrange multipliers associated with the path constraints* (1.1d). *For each $i = 1, \ldots, n_0$, define*

$$\begin{aligned} \hat{h}_{n_1'(i)} &= \begin{cases} x_{n_1}^0, & i = 1 \\ x_{n_1'(i)}^*, & i = 2, \ldots, n_0, \end{cases} \\ \hat{d}_{n_2'(i)} &= \begin{cases} -Q_{n_2'(i)}^{-1} \left(\lambda_{n_2'(i)-1}^* + \widetilde{P}_{n_2'(i)}^T \eta_{n_2'(i)-1}^* \right) + x_{n_2'(i)}^*, & i = 1, \ldots, n_0 - 1 \\ d_{n_2}, & i = n_0. \end{cases} \end{aligned} \tag{3.5}$$

Then $(u_{n_1'(i):n_2'(i)-1}^*, x_{n_1'(i)+1:n_2'(i)}^*)$ satisfies the KKT conditions and the second-order sufficient conditions of problem $P_{\theta_1(i)}^i$ with $\theta_1(i) = (\hat{h}_{n_1'(i)}, \hat{d}_{n_2'(i)})$.

Proof. The KKT conditions for problem (1.1) are

$$R_k u_k^* + B_k^T \lambda_k^* - \widetilde{C}_k^T \eta_k^* = 0,\ k \in [n_1, n_2 - 1] \tag{3.6a}$$
$$Q_k(x_k^* - d_k) + A_k^T \lambda_k^* - \lambda_{k-1}^* - \widetilde{P}_k^T \eta_{k-1}^* = 0,\ k \in [n_1 + 1, n_2 - 1] \tag{3.6b}$$
$$Q_{n_2}(x_{n_2}^* - d_{n_2}) - \lambda_{n_2-1}^* - \widetilde{P}_{n_2}^T \eta_{n_2-1}^* = 0, \tag{3.6c}$$
$$x_{k+1}^* = A_k x_k^* + B_k u_k^*,\ k \in [n_1, n_2 - 1], \tag{3.6d}$$
$$x_{n_1} = x_{n_1}^0, \tag{3.6e}$$
$$\widetilde{P}_{k+1} x_{k+1}^* + \widetilde{C}_k u_k^* \geq \widetilde{q}_k,\ k \in [n_1, n_2 - 1] \tag{3.6f}$$
$$\eta_k^* \geq 0,\ k \in [n_1, n_2 - 1] \tag{3.6g}$$
$$\eta_k^*(j) \left(\widetilde{P}_{k+1}(j,:) x_{k+1}^* + \widetilde{C}_k(j,:) u_k^* - \widetilde{q}_k(j) \right) = 0,\ k \in [n_1, n_2 - 1]. \tag{3.6h}$$

Then for problem $P^i_{\theta_1(i)}$, the KKT conditions are satisfied by $(u^*_{n_1'(i):n_2'(i)-1}, x^*_{n_1'(i)+1:n_2'(i)})$ with the same Lagrange multipliers λ_k^* and η_k^* as follows. Note that for problem $P^i_{\theta_1(i)}$, the initial state and terminal reference are $\hat{h}_{n_1'(i)}$ and $\hat{d}_{n_2'(i)}$ defined in (3.5):

$$R_k u_k^* + B_k^T \lambda_k^* - \widetilde{C}_k^T \eta_k^* = 0,\ k \in [n_1'(i), n_2'(i) - 1] \tag{3.7a}$$
$$Q_k(x_k^* - d_k) + A_k^T \lambda_k^* - \lambda_{k-1}^* - \widetilde{P}_k^T \eta_{k-1}^* = 0,\ k \in [n_1'(i) + 1, n_2'(i) - 1] \tag{3.7b}$$
$$Q_{n_2'(i)}(x_{n_2'(i)}^* - \hat{d}_{n_2'(i)}) - \lambda_{n_2'(i)-1}^* - \widetilde{P}_{n_2'(i)}^T \eta_{n_2'(i)-1}^* = 0, \tag{3.7c}$$
$$x_{k+1}^* = A_k x_k^* + B_k u_k^*,\ k \in [n_1'(i), n_2'(i) - 1], \tag{3.7d}$$
$$x_{n_1'(i)} = \hat{h}_{n_1'(i)}, \tag{3.7e}$$
$$\widetilde{P}_{k+1} x_{k+1}^* + \widetilde{C}_k u_k^* \geq \widetilde{q}_k,\ k \in [n_1'(i), n_2'(i) - 1] \tag{3.7f}$$
$$\eta_k^* \geq 0,\ k \in [n_1'(i), n_2'(i) - 1] \tag{3.7g}$$
$$\eta_k^*(j) \left(\widetilde{P}_{k+1}(j,:) x_{k+1}^* + \widetilde{C}_k(j,:) u_k^* - \widetilde{q}_k(j) \right) = 0,\ k \in [n_1'(i), n_2'(i) - 1], \tag{3.7h}$$

where (3.7a)–(3.7b) and (3.7f)–(3.7h) directly follow from (3.6a)–(3.6b) and (3.6f)–(3.6h), respectively. Equation (3.7c) follows from the definition of $\hat{d}_{n_2'(i)}$ in (3.5). Equation (3.7d) follows from (3.6d), (3.7e), and (3.5). The second-order condition is satisfied by virtue of the strong convexity of the problem. □

LEMMA 3.3. *Under Assumptions 2.1, 2.18, and 2.20, for $\hat{h}_{n_1'(i)}$ and $\hat{d}_{n_2'(i)}$ defined in* (3.5)*, we have, for $i = 1, \ldots, n_0$,*

$$\|\hat{h}_{n_1'(i)}\|_2,\ \|\hat{d}_{n_2'(i)}\|_2 \leq C_\theta$$

for some $C_\theta > 0$ independent of i, n_1, and n_2.

Proof. For $i = 1, \ldots, n_0 - 1$, KKT conditions of problem (1.1) give

$$\lambda^*_{n_2'(i)-1} + \widetilde{P}^T_{n_2'(i)} \eta^*_{n_2'(i)-1} = Q_{n_2'(i)}(x^*_{n_2'(i)} - d_{n_2'(i)}) + A^T_{n_2'(i)} \lambda^*_{n_2'(i)}.$$

Then Assumptions 2.1 and 2.18 and Corollary 2.21 give

$$\|\lambda^*_{n_2'(i)-1} + \widetilde{P}^T_{n_2'(i)} \eta^*_{n_2'(i)-1}\|_2 \leq \widetilde{C}_Q(C_g + m_0) + \widetilde{C}_A C_\lambda.$$

Combining this with (3.5), we have

$$\|\hat{d}_{n_2'(i)}\|_2 \leq \left(\widetilde{C}_Q(C_g + m_0) + \widetilde{C}_A C_\lambda \right) / \lambda_Q + C_g \triangleq C_{\hat{d}},$$

for $i = 1, \ldots, n_0 - 1$. From (3.5), we have $\|\hat{d}_{n_2'(i)}\| \leq \min(C_{\hat{d}}, m_0)$ for $i = 1, \ldots, n_0$. Similarly, from Assumption 2.18, Corollary 2.21, and (3.5) we have $\|\hat{h}_{n_1'(i)}\|_2 \leq \min(u_0, C_g)$ for $i = 1, \ldots, n_0$. Taking $C_\theta = \min(C_{\hat{d}}, m_0, u_0, C_g)$ concludes the proof. ∎

Proposition 3.2 shows that the solution of $P^i_{\theta_1(i)}$ is identical to the solution, restricted to $\mathcal{R}_i$, of problem (1.1). However, problem $P^i_{\theta_1(i)}$ is only notional and cannot be defined without first solving problem (1.1). Hence we need to investigate the relationship between solutions of problems $P^i_{\theta_1(i)}$ and $P^i_{\theta_0(i)}$, the latter of which gives the RHC solutions. Since problem $P^i_{\theta_1(i)}$ can be viewed as resulting from a perturbation of the parameters of problem $P^i_{\theta_0(i)}$, we employ the following parametric sensitivity results derived from [5].

DEFINITION 3.4. *For $\theta \in \mathbb{R}^q$, define the one-sided directional derivative of $y(\theta)$ along a direction $p \in \mathbb{R}^q$ at θ_0 as*

$$D_p y(\theta_0) = \lim_{t \downarrow 0} \frac{y(\theta_0 + tp) - y(\theta_0)}{t},$$

given that the limit exists.

LEMMA 3.5. *Consider the following parametrized quadratic programming problem,*

$$\begin{aligned} &\min \quad f(y,\theta) \triangleq y^T G y/2 + y^T c(\theta) + \theta^T F \theta + y^T c_1 + \theta^T c_2 + C \\ &\text{s.t.} \quad Ay - r \leq 0 \\ &\qquad\quad By - d(\theta) = 0, \end{aligned} \tag{3.8}$$

where G, F are positive definite, $\theta \in \mathbb{R}^q$, and $A^T = [a_1, \ldots, a_m] \in \mathbb{R}^{n \times m}$. Denote the solution of problem (3.8) as $y(\theta)$. When $\theta = \theta_0$, let $y_0 = y(\theta_0)$ and the Lagrange multiplier corresponding to y_0 be $\bar{\lambda}$. Denote $I(y_0, \theta_0) = \{i : a_i^T y_0 = r_i, i = 1, \ldots, m\}$ be the set of active inequality constraints, $I_+(y_0, \theta_0, \bar{\lambda}) = \{i \in I(y_0, \theta_0) : \bar{\lambda}_i > 0\}$ and $I_0(y_0, \theta_0, \bar{\lambda}) = \{i \in I(y_0, \theta_0) : \bar{\lambda}_i = 0\}$. If the linear independence constraint qualification holds at $y(\theta_0)$, then for any direction $p \in \mathbb{R}^q$, we have

$$D_p y(\theta_0) = \left(\left. \frac{d y^*_{I'(\theta_0)}(\theta)}{d\theta} \right|_{\theta = \theta_0} \right) p,$$

where $y^*_{I'(\theta_0)}(\theta)$ is the solution of the problem

$$\begin{aligned} &\min \quad f(y,\theta) = y^T G y/2 + y^T c(\theta) + \theta^T F \theta + y^T c_1 + \theta^T c_2 + C \\ &\text{s.t.} \quad A_{I'(\theta_0)} y - r' = 0 \\ &\qquad\quad By - d(\theta) = 0 \end{aligned} \tag{3.9}$$

and where $I'(\theta_0) = I_+(y_0, \theta_0, \bar{\lambda}) \cup I_1$ for some $I_1 \subset I_0(y_0, \theta_0, \bar{\lambda})$ and $A_{I'(\theta_0)} = [a_i^T]_{i \in I'(\theta_0)}$, $r' = [r_i]_{i \in I'(\theta_0)}$.

Proof. See Appendix A.3; see also [24, Lemma 3.5]. ∎

Note that problem P^i_θ has the same structure as that defined by (3.8), and Lemma 3.5 connects the dependence on parameters of the solutions for the inequality-constrained problem (3.8) with that of a related equality-constrained problem (3.9), whose equality constraints are subsetted from the active constraints of (3.8) at optimality. The equality-constrained problem has smooth and regular KKT conditions which facilitate the derivation for the dependence of solutions on parameters as shown in Section 2.1. Now we are ready to investigate the effect on solutions of perturbing the parameters of P_θ^i. Since the proof for each receding horizon is the same, for notational simplicity we suppress the dependence of $n_1'(i)$, $n_2'(i)$ and P_θ^i on i whenever the index of the receding horizon under consideration is clear. To connect the solutions of P_{θ_1} and P_{θ_0}, we consider a continuously indexed family of problems P_{θ_s} for $\theta_s = \theta_0 + s(\theta_1 - \theta_0)$ and $s \in [0,1]$. Let $P_{k+1}(s)h_{k+1} + C_k(s)w_k = q_k(s)$ be the active constraints of problem P_{θ_s} at optimality. We let

$$E_k(s) = P_{k+1}(s)A_k, \quad H_k(s) = P_{k+1}(s)B_k + C_k(s). \tag{3.10}$$

Thus, the active constraints of P_{θ_s} are

$$h_{k+1} = A_k h_k + B_k w_k \qquad n_1' \le k \le n_2' - 1, \qquad h_{n_1'} = \theta_s^{(h)}, \tag{3.11a}$$

$$E_k(s)h_k + H_k(s)w_k = q_k(s), \qquad n_1' \le k \le n_2' - 1, \tag{3.11b}$$

where $\theta_s = (\theta_s^{(h)}, \theta_s^{(d)})$. As in Assumption 2.20 (a), we make the following uniform boundedness assumption about the active constraints of P_{θ_s}.

ASSUMPTION 3.6. *For $i = 1, \ldots, n_0$ and $s \in [0,1]$, let $\theta_0(i) = (\widetilde{x}_{n_1'(i)}, d_{n_2'(i)})$, $\theta_1(i) = (\hat{h}_{n_1'(i)}, \hat{d}_{n_2'(i)})$ as defined in (3.5), and $\theta_s(i) = \theta_0(i) + s(\theta_1(i) - \theta_0(i))$. Then the active sets of problems $P^i_{\theta_s(i)}$ at optimality are UDB(λ_H) as in Definition 2.2.*

In particular, Assumption 3.6 implies that $H_k(s)$ defined in (3.10) has full row rank, with which we can now apply Lemma 3.5 to problem P_{θ_s}.

LEMMA 3.7. *Denote $\theta_0 = (\widetilde{x}_{n_1'}, d_{n_2'})$ and $\theta_1 = (\hat{h}_{n_1'}, \hat{d}_{n_2'})$ as defined in (3.5). For $\theta = (\theta^{(h)}, \theta^{(d)})$, let $x(\theta)$ be the solution of problem P_θ. Under Assumption 3.6, for $s \in [0,1]$ and $\theta_s = \theta_0 + s(\theta_1 - \theta_0)$, we have*

$$D_{\theta_1 - \theta_0} x(\theta_s) = \left(\frac{dy_s(\theta)}{d\theta}\Big|_{\theta = \theta_0 + s(\theta_1 - \theta_0)} \right)(\theta_1 - \theta_0),$$

and $y_s(\theta)$ is the solution of the following equality-constrained problem:

$$\min \quad \frac{1}{2}\sum_{k=n_1'}^{n_2'-1} w_k^T R_k w_k + (h_k - d_k)^T Q_k (h_k - d_k) \tag{3.12a}$$

$$+ (h_{n_2'} - \theta_s^{(d)})^T Q_{n_2'} (h_{n_2'} - \theta_s^{(d)}) \tag{3.12b}$$

$$\text{s.t.} \quad h_{k+1} = A_k h_k + B_k w_k, \qquad n_1' \le k \le n_2' - 1, \qquad h_{n_1'} = \theta_s^{(h)}, \tag{3.12c}$$

$$E_k'(s)h_k + H_k'(s)w_k = q_k'(s), \qquad n_1' \le k \le n_2' - 1, \tag{3.12d}$$

where rows of $E_k'(s)$ and $H_k'(s)$ are respectively subsets of rows of $E_k(s)$ and $H_k(s)$ defined by the active constraints of P_{θ_s} at optimality as in (3.10). In other words, $E_k'(s)h_k + H_k'(s)w_k = q_k'(s)$ is a subset of the equality constraints (3.11b).

Proof. Problem P_{θ_s} is an instance of problem (3.8) with the following parameters:

$$G = \operatorname{diag}\left(\begin{bmatrix} R_{n_1'} & \ldots & R_{n_2'-1} & Q_{n_1'+1} & \ldots & Q_{n_2'} \end{bmatrix}\right), \quad c(\theta) = \begin{bmatrix} \mathbf{0}_{(n_2'-n_1')m + (n_2'-n_1'-1)n} \\ -Q_{n_2'}\theta_s^{(d)} \end{bmatrix},$$

$$A = \left[\begin{array}{ccc|ccc} -\widetilde{C}_{n_1'} & & & -\widetilde{P}_{n_1'+1} & & \\ & \ddots & & & \ddots & \\ & & -\widetilde{C}_{n_2'-1} & & & -\widetilde{P}_{n_2'} \end{array}\right], \quad r = \begin{bmatrix} -\widetilde{q}_{n_1'} \\ \vdots \\ -\widetilde{q}_{n_2'-1} \end{bmatrix}, \quad F = \begin{bmatrix} Q_{n_1'}/2 & \\ & Q_{n_2'}/2 \end{bmatrix},$$

$$B = \left[\begin{array}{ccc|cccc} -B_{n_1'} & & & I & & & \\ & \ddots & & -A_{n_1'+1} & I & & \\ & & & & & \ddots & \\ & & -B_{n_2'-1} & & & -A_{n_2'-1} & I \end{array}\right], \quad d(\theta) = \begin{bmatrix} A_{n_1'}\theta_s^{(h)} \\ \mathbf{0}_{(n_2'-n_1'-1)n} \end{bmatrix}.$$

Here $Ax \le r$ and $Bx = d(\theta)$ correspond respectively to the inequality constraints (3.2d) and the dynamical constraints (3.2c). Note that A and B have the same number of columns. G and F are positive definite from Assumption 2.1. The quantities c_1, c_2, and C of problem (3.8) do not enter in the proof, so their definitions are not shown.

Let $\overline{A}(s)$ be the matrix whose rows are subsets of rows of A corresponding to the active constraints $P_{k+1}(s)h_{k+1} + C_k(s)w_k = q_k(s)$ at optimality for problem P_{θ_s}. Then we define

$$\overline{A}(s) \triangleq \left[\begin{array}{ccc|ccc} -C_{n_1'}(s) & & & -P_{n_1'+1}(s) & & \\ & \ddots & & & \ddots & \\ & & -C_{n_2'-1}(s) & & & -P_{n_2'}(s) \end{array}\right].$$

In the following we show that rows of $\overline{A}(s)$ and B are linearly independent. Denote

$$\alpha^T = \left[\alpha_{n_1'+1}^T, \ldots, \alpha_{n_2'}^T\right], \qquad \beta^T = \left[\beta_{n_1'}^T, \ldots, \beta_{n_2'-1}^T\right].$$

Then

$$\alpha^T \overline{A}(s) + \beta^T B = \left[\begin{array}{ccc|ccc} x_{n_1'} & \ldots & x_{n_2'-1} & y_{n_1'+1} \cdots & y_{n_2'} \end{array}\right],$$

where

$$x_k = -\alpha_{k+1}^T C_k(s) - \beta_k^T B_k, \qquad y_k = \begin{cases} \beta_{k-1}^T - \beta_k^T A_k - \alpha_k^T P_k(s), & k < n_2' \\ \beta_{n_2'-1}^T - \alpha_{n_2'}^T P_{n_2'}(s), & k = n_2'. \end{cases}$$

Now we let $\alpha^T \overline{A}(s) + \beta^T B = \mathbf{0}$. We show that $\alpha, \beta = \mathbf{0}$ by backward induction. We have

$$\begin{aligned} \begin{cases} x_{n_2'-1} = \mathbf{0} \\ y_{n_2'} = \mathbf{0} \end{cases} &\Rightarrow \begin{cases} \alpha_{n_2'}^T C_{n_2'-1}(s) + \beta_{n_2'-1}^T B_{n_2'-1} = \mathbf{0} \\ \beta_{n_2'-1}^T = \alpha_{n_2'}^T P_{n_2'}(s) \end{cases} \\ &\Rightarrow \alpha_{n_2'}^T C_{n_2'-1}(s) + \alpha_{n_2'}^T P_{n_2'}(s) B_{n_2'-1} = \mathbf{0} \\ &\overset{(3.10),\text{Asmp } 3.6}{\Rightarrow} \alpha_{n_2'} = \mathbf{0} \Rightarrow \beta_{n_2'-1} = \mathbf{0}. \end{aligned}$$

Suppose $\alpha_k, \beta_{k-1} = \mathbf{0}$ for some $n_1' + 1 < k \le n_2'$. Then we have

$$\begin{aligned} \begin{cases} x_{k-2} = \mathbf{0} \\ y_{k-1} = \mathbf{0} \end{cases} &\Rightarrow \begin{cases} \alpha_{k-1}^T C_{k-2}(s) + \beta_{k-2}^T B_{k-2} = \mathbf{0} \\ \beta_{k-2}^T = \cancel{\beta_{k-1}^T} A_{k-1} + \alpha_{k-1}^T P_{k-1}(s) \end{cases} \\ &\Rightarrow \alpha_{k-1}^T C_{k-2}(s) + \alpha_{k-1}^T P_{k-1}(s) B_{k-2} = \mathbf{0} \\ &\overset{(3.10),\text{Asmp } 3.6}{\Rightarrow} \alpha_{k-1} = \mathbf{0} \Rightarrow \beta_{k-2} = \mathbf{0}. \end{aligned}$$

So LICQ holds for problem P_{θ_s} at optimality. Directly applying Lemma 3.5 concludes the proof. ∎

Problem (3.12) is an equality-constrained problem for which the results derived in Section 2, especially the exponential decay property of the dependence of solutions on the initial state and terminal reference, can be applied under certain assumptions. In the following, we investigate the controllability conditions for problem (3.12). Denote the active set of problem $P^i_{\theta_s(i)}$ and the index set for the corresponding equality constraints of problem (3.12) as $\mathcal{A}_s(i)$ and $\mathcal{I}_s(i)$, respectively.

LEMMA 3.8. *Under Assumption 3.6, for* $s \in [0,1]$ *and* $i = 1, \ldots, n_0$*, we have that* $\mathcal{I}_s(i)$ *is UDB*(λ_H) *as defined in Definition 2.2.*

Proof. Since $\mathcal{I}_s(i) \subset \mathcal{A}_s(i)$, rows of $H'_k(s)$ are subsets of rows of $H_k(s)$, then the conclusion follows from Assumption 3.6. ∎

LEMMA 3.9. *If the equality-constrained system* (3.11) *corresponding to the active sets of* P_{θ_s} *is controllable at* q *in* t *steps, then the controllability matrix* $C_{q,t}(\mathcal{I}_s)$ *defined by the subsetted system* (3.12c)–(3.12d) *has full row rank.*

Proof. If system (3.11) can be controlled to an arbitrary state $\overline{x}$ in t steps with some admissible controls and corresponding states, then the subsetted system (3.12c)–(3.12d) can also be controlled to $\overline{x}$ in t steps with the same controls and states, because the feasible set defined by (3.12c)–(3.12d) contains that defined by (3.11). As a result, from Proposition 2.11 $C_{q,t}(\mathcal{I}_s)$ has full row rank. ∎

ASSUMPTION 3.10. *For* $i = 1, \ldots, n_0$ *and* $s \in [0,1]$*, let* $\theta_0(i) = (\widetilde{x}_{n'_1(i)}, d_{n'_2(i)})$, $\theta_1(i) = (\hat{h}_{n'_1(i)}, \hat{d}_{n'_2(i)})$ *as defined in* (3.5)*, and* $\theta_s(i) = \theta_0(i) + s(\theta_1(i) - \theta_0(i))$*. We have the following for any* $n'_1(i) \le q \le n'_2(i)$, $1 \le i \le n_0$*:*

(a) the system (3.11) *defined by* $\mathcal{A}_s(i)$ *is controllable at* q *for* t *steps;*

(b) under (a), $C_{q,t}(\mathcal{A}_s(i))$ *and* $C_{q,t}(\mathcal{I}_s(i))$ *have full row rank by Proposition 2.11 and Lemma 3.9, respectively; and we further assume that both* $\mathcal{A}_s(i)$ *and* $\mathcal{I}_s(i)$ *are UCC*(λ_C)*.*

Assumption 3.10(a) assumes controllability of problem P_{θ_s} only at optimality. Proposition 2.11 and Lemma 3.9 imply that $C_{q,t}(\mathcal{A}_s)$ and $C_{q,t}(\mathcal{I}_s)$ are bounded below, and Assumption 3.10 (b) in addition assumes that the lower bounds are uniform for all time points and receding horizons. Now we are ready to bound the distance between solutions of P_{θ_0} and P_{θ_1}, which by Proposition 3.2, is also the distance between solutions of RHC and problem (1.1).

THEOREM 3.11. *For* $1 \le i \le n_0$*, let* $(u^*_{n'_1(i):n'_2(i)-1}, x^*_{n'_1(i)+1:n'_2(i)})$ *be the solution of problem* $P^i_{\theta_1(i)}$*, which from Proposition 3.2 is exactly the solution of problem* (1.1) *restricted to* $\mathcal{R}_i$*. Let* $\widetilde{u}_k$ *and* $\widetilde{x}_k$ *be the receding horizon control and state defined in* (3.3) *and* (3.4)*, respectively. Under Assumptions 2.1, 2.18, 2.20, and 3.6, we have, for some lag* L *so that* $Z_1\rho^L < 1$ *and receding horizon length* $N > L$*,*

$$\|x^*_k - \widetilde{x}_k\|_2, \|u^*_k - \widetilde{u}_k\|_2 \le C_d Z_2 \left(1 + \frac{Z_1}{1 - Z_1\rho^L}\right) \rho^{N-L-1}$$

for $n_1 \le k \le n_2 - 1$*. Here* $C_d > 0$ *is independent of* N*,* n_1*, and* n_2*;* Z_1*,* Z_2*, and* ρ *are as in Proposition 2.17.*

Proof. Assumption 2.1 and Lemma 3.3 give, for $i = 1, \ldots, n_0$,

$$\|\hat{d}_{n'_2(i)} - d_{n'_2(i)}\|_2 \le C_\theta + m_0 \triangleq C_d. \tag{3.13}$$

Let $\theta_0(i) = (\widetilde{x}_{n'_1(i)}, d_{n'_2(i)})$, $\theta_1(i) = (\hat{h}_{n'_1(i)}, \hat{d}_{n'_2(i)})$ as defined in (3.5), and $\theta_s(i) = \theta_0(i) + s(\theta_1(i) - \theta_0(i))$ for $s \in [0,1]$. Note that for $1 \le j \le L$, $x^*_{n'_1(i)+j}$ is the optimal state of problem (1.1), which by Proposition 3.2 is also that of problem $P^i_{\theta_1(i)}$; while

$\widetilde{x}_{n'_1(i)+j}$ is the optimal state of problem $P^i_{\theta_0(i)}$ by (3.4). Denote $\widetilde{s}^*_k(\theta_s(i))$ and $\widetilde{p}^*_k(\theta_s(i))$ for $n'_1(i) \le k \le n'_2(i)$ as the optimal control and state of problem $P^i_{\theta_s(i)}$, and $s^*_k(\theta_s(i))$ and $p^*_k(\theta_s(i))$ as those of the corresponding subsetted equality-constrained problem (3.12). Then for $1 \le i \le n_0 - 1$, we have from Proposition 3.2 and Lemma 3.7

$$\begin{aligned}
&x^*_{n'_1(i)+j} - \widetilde{x}_{n'_1(i)+j} \\
&= \int_0^1 D_{\theta_1(i)-\theta_0(i)} \widetilde{p}^*_{n'_1(i)+j}(\theta_s(i))\, ds \\
&= \int_0^1 \begin{bmatrix} \nabla_{h_{n'_1(i)}} p^*_{n'_1(i)+j}(\theta_s(i)) & \nabla_{d_{n'_2(i)}} p^*_{n'_1(i)+j}(\theta_s(i)) \end{bmatrix} \begin{bmatrix} \hat{h}_{n'_1(i)} - \widetilde{x}_{n'_1(i)} \\ \hat{d}_{n'_2(i)} - d_{n'_2(i)} \end{bmatrix} ds.
\end{aligned}$$

Lemma 3.8 states that the index set of the corresponding problem (3.12) is UDB(λ_H), and Assumption 3.10(b) further states it is UCC(λ_C). Note that the exponential bounds obtained in Proposition 2.17 are independent of the problem interval and the particular choice of the equality constraint index set, which is UDB(λ_H) and UCC(λ_C). Therefore, applying Proposition 2.17, we have, for $1 \le i \le n_0 - 1$ and $1 \le j \le L$,

$$\begin{aligned}
\|x^*_{n'_1(i)+j} - \widetilde{x}_{n'_1(i)+j}\|_2 &\overset{(3.13)}{\le} C_d Z_2 \rho^{n'_2(i)-n'_1(i)-j} + Z_1 \rho^j \|\hat{h}_{n'_1(i)} - \widetilde{x}_{n'_1(i)}\|_2 \\
&\overset{(3.5)}{=} C_d Z_2 \rho^{n'_2(i)-n'_1(i)-j} + Z_1 \rho^j \|x^*_{n'_1(i)} - \widetilde{x}_{n'_1(i)}\|_2 \\
&= C_d Z_2 \rho^{N-j-1} + Z_1 \rho^j \|x^*_{n'_1(i)} - \widetilde{x}_{n'_1(i)}\|_2.
\end{aligned} \tag{3.14}$$

When $j = L$, note that $n'_1(i) + L = n'_1(i+1)$, so (3.14) becomes

$$\|x^*_{n'_1(i+1)} - \widetilde{x}_{n'_1(i+1)}\|_2 \le C_d Z_2 \rho^{N-L-1} + Z_1 \rho^L \|x^*_{n'_1(i)} - \widetilde{x}_{n'_1(i)}\|_2.$$

From this recursion, at the starting index of $\mathcal{R}_i$ for $1 \le i \le n_0$ we have

$$\begin{aligned}
\|x^*_{n'_1(i)} - \widetilde{x}_{n'_1(i)}\|_2 &\le \frac{1 - (Z_1\rho^L)^{i-1}}{1 - Z_1\rho^L} C_d Z_2 \rho^{N-L-1} \\
&\le C_d Z_2 \rho^{N-L-1} / (1 - Z_1 \rho^L).
\end{aligned} \tag{3.15}$$

Note that L is chosen so that $Z_1 \rho^L < 1$. Substituting (3.15) into (3.14), we have, for $1 \le i \le n_0 - 1$ and $1 \le j \le L$,

$$\begin{aligned}
\|x^*_{n'_1(i)+j} - \widetilde{x}_{n'_1(i)+j}\|_2 &\le C_d Z_2 \rho^{N-j-1} + Z_1 \rho^j \|x^*_{n'_1(i)} - \widetilde{x}_{n'_1(i)}\|_2 \\
&\le C_d Z_2 \rho^{N-L-1} + Z_1 \|x^*_{n'_1(i)} - \widetilde{x}_{n'_1(i)}\|_2 \\
&\overset{(3.15)}{\le} C_d Z_2 \left(1 + \frac{Z_1}{1 - Z_1 \rho^L}\right) \rho^{N-L-1}.
\end{aligned} \tag{3.16}$$

Now we prove the approximation error bound for the RHC policies. For $1 \le i \le n_0 - 1$ and $1 \le j \le L$, $u^*_{n'_1(i)+j-1}$ is the optimal control of problem (1.1), which by Proposition 3.2 is also that of problem $P^i_{\theta_1(i)}$, whereas $\widetilde{u}_{n'_1(i)+j-1}$ is the optimal control of problem $P^i_{\theta_0(i)}$ by (3.3). Therefore, by Proposition 3.2 and Lemma 3.7 we have

$$u^*_{n'_1(i)+j-1} - \widetilde{u}_{n'_1(i)+j-1}$$

$$= \int_0^1 D_{\theta_1(i)-\theta_0(i)} \widetilde{s}^*_{n_1'(i)+j-1}(\theta_s(i))\, ds$$
$$= \int_0^1 \begin{bmatrix} \nabla_{h_{n_1'(i)}} s^*_{n_1'(i)+j-1}(\theta_s(i)) & \nabla_{d_{n_2'(i)}} s^*_{n_1'(i)+j-1}(\theta_s(i)) \end{bmatrix} \begin{bmatrix} \hat{h}_{n_1'(i)} - \widetilde{x}_{n_1'(i)} \\ \hat{d}_{n_2'(i)} - d_{n_2'(i)} \end{bmatrix} ds.$$

Proposition 2.17 and (3.13) give

$$\begin{aligned} \|u^*_{n_1'(i)+j-1} - \widetilde{u}_{n_1'(i)+j-1}\|_2 &\le C_d Z_2 \rho^{n_2'(i)-n_1'(i)-j+1} + Z_1 \rho^{j-1} \|\hat{h}_{n_1'(i)} - \widetilde{x}_{n_1'(i)}\|_2 \\ &\overset{(3.5)}{=} C_d Z_2 \rho^{N-j} + Z_1 \rho^{j-1} \|x^*_{n_1'(i)} - \widetilde{x}_{n_1'(i)}\|_2 \\ &\le C_d Z_2 \rho^{N-L-1} + Z_1 \|x^*_{n_1'(i)} - \widetilde{x}_{n_1'(i)}\|_2 \\ &\overset{(3.15)}{\le} C_d Z_2 \left(1 + \frac{Z_1}{1 - Z_1 \rho^L}\right) \rho^{N-L-1}. \end{aligned} \tag{3.17}$$

On the last receding horizon $\mathcal{R}_{n_0} = [n_1'(n_0), n_2]$, we have, for $m+1 \le k \le n_2$,

$$\begin{aligned} x^*_k - \widetilde{x}_k &= \int_0^1 D_{\theta_1(n_0)-\theta_0(n_0)} \widetilde{p}^*_k(\theta_s(n_0))\, ds \\ &= \int_0^1 \begin{bmatrix} \nabla_{h_{n_1'(n_0)}} p^*_k(\theta_s(n_0)) & \nabla_{d_{n_2}} p^*_k(\theta_s(n_0)) \end{bmatrix} \begin{bmatrix} \hat{h}_{n_1'(n_0)} - \widetilde{x}_{n_1'(n_0)} \\ \mathbf{0} \end{bmatrix} ds, \end{aligned}$$

and for $m \le k \le n_2 - 1$,

$$\begin{aligned} u^*_k - \widetilde{u}_k &= \int_0^1 D_{\theta_1(n_0)-\theta_0(n_0)} \widetilde{s}^*_k(\theta_s(n_0))\, ds \\ &= \int_0^1 \begin{bmatrix} \nabla_{h_{n_1'(n_0)}} s^*_k(\theta_s(n_0)) & \nabla_{d_{n_2}} s^*_k(\theta_s(n_0)) \end{bmatrix} \begin{bmatrix} \hat{h}_{n_1'(n_0)} - \widetilde{x}_{n_1'(n_0)} \\ \mathbf{0} \end{bmatrix} ds, \end{aligned}$$

from which we have

$$\begin{aligned} \|x^*_k - \widetilde{x}_k\|_2, \|u^*_k - \widetilde{u}_k\|_2 &\overset{\text{Prop 2.17}}{\le} Z_1 \rho^{k-n_1'(n_0)} \|\hat{h}_{n_1'(n_0)} - \widetilde{x}_{n_1'(n_0)}\|_2 \\ &\overset{(3.5)}{=} Z_1 \rho^{k-n_1'(n_0)} \|x^*_{n_1'(n_0)} - \widetilde{x}_{n_1'(n_0)}\|_2 \\ &\overset{(3.15)}{\le} C_d Z_2 Z_1 \rho^{N-L-1} / (1 - Z_1 \rho^L). \end{aligned} \tag{3.18}$$

Combining (3.15), (3.16), (3.17) and (3.18) concludes this proof. ∎

Note that since the quantities Z_1 and ρ are independent of the problem interval $[n_1, n_2]$, so is the choice of L. Therefore, Theorem 3.11 proves the approximation error of the RHC solution with an appropriate lag L decays exponentially in the length N of the receding horizon regardless of the full problem interval under the uniform boundedness and controllability conditions. The exponential decay rate provides systematic means to choose the length of the receding horizon given a desired accuracy level. We also note that it is a bit surprising that the choice $L = 1$ may sometimes not satisfy our assumptions; that case occurs when $Z_1 \rho > 1$. This seems to indicate that it is sometimes better to wait for a few lags before applying a newly computed control, which may be counterintuitive if the goal is stability. While we cannot yet state whether this condition is also necessary for some cases, we point out that certain types of instability in rolling horizon control due to frequent policy updates have been identified in inventory management and carry the name of “nervousness” [22].

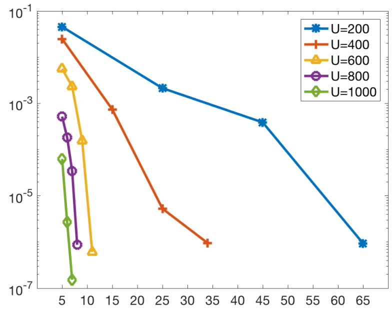

Fig. 4.1: Relative error in states $\|x^* - \widetilde{x}\|_2/\|x^*\|_2$ at each receding horizon length (hour) N for $U = 200, \dots, 1000$ and $G = 12,000$.

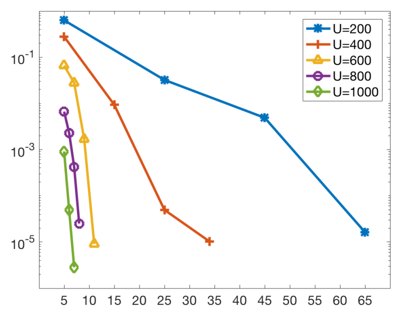

Fig. 4.2: Relative error in controls $\|u^* - \widetilde{u}\|_2/\|u^*\|_2$ at each receding horizon length (hour) N for $U = 200, \dots, 1000$ and $G = 12,000$.

4. Numerical results. In this section, we apply the receding horizon strategy to the following production cost model and verify some of the theoretical results.

$$\min \sum_{k=1}^{N} c_1(x_k - d_k)^2 + c_2 x_k^2 + u_k^2 \tag{4.1a}$$

$$\text{s.t.} \quad x_{k+1} = x_k + u_k \tag{4.1b}$$

$$0 \le x_k \le G \tag{4.1c}$$

$$-U \le u_k \le U \tag{4.1d}$$

In this model, d_k is the hourly electricity demand to be satisfied, for which we employ the estimated hourly demand data in the northern Illinois region for year 2016 provided by PJM Interconnection [8]. The demand can be satisfied by two generators: one with a high quadratic cost $c_1 = 10$ and the other one with a low quadratic cost $c_2 = 5$. The low-cost generator has a limited capacity to change its output, modeled by the box constraints (4.1d) on the controls u_k and the dynamical constraints (4.1b); and it also has a limited generation level, modeled by the upper bound (4.1c) on the generation x_k. The generator with a high cost can change its output rapidly and hence serve the remaining loads $d_k - x_k$. One example of such a situation is the combination of a fast but expensive gas plant and a cheap but slow coal plant. We initialize problem (4.1) by setting the initial state x_1^0 to be the average demand of year 2015 on the same hour as the initial time point. We note that problem (4.1) has the form of problem (1.1).

We implement the receding horizon strategy described in Section 3 with lag $L = 1$. Specifically, we solve a short version of problem (4.1) on a receding horizon $\mathcal{R}_i = [n_1'(i), n_2'(i)]$ with length N and initial value $\widetilde{x}_{n_1'(i)}$, obtain the optimal control $\widetilde{u}_{n_1'(i)}$ and state $\widetilde{x}_{n_1'(i)+1}$, then reinitialize at time point $n_1'(i) + 1$ with $\widetilde{x}_{n_1'(i)+1}$ to solve the problem on the next receding horizon $\mathcal{R}_{i+1} = [n_1'(i) + 1, n_2'(i) + 1]$. Problem (4.1) is solved on the full horizon and each receding horizon using the Ipopt software [4]. The model was defined by using the Julia/JuMP interface [16].

We investigate the solution accuracy of the receding horizon strategy with different choices of the generation upper bound G and the bound U on control to verify

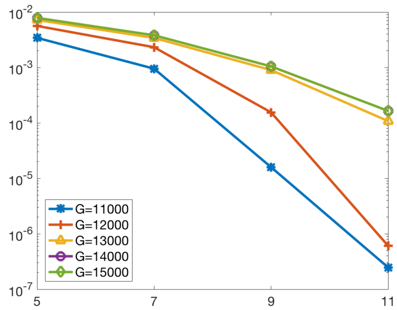

Fig. 4.3: Relative error in states $\|x^* - \widetilde{x}\|_2/\|x^*\|_2$ at each receding horizon length (hour) N for $G = 11000, \ldots, 15000$ and $U = 600$.

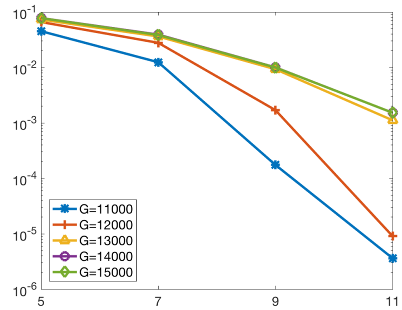

Fig. 4.4: Relative error in controls $\|u^* - \widetilde{u}\|_2/\|u^*\|_2$ at each receding horizon length (hour) N for $G = 11000, \ldots, 15000$ and $U = 600$.

our theoretical findings. Denote $x^* = \{x_k^*\}$ and $u^* = \{u_k^*\}$ as the optimal state and control of problem (4.1) on the full horizon, and $\widetilde{x} = \{\widetilde{x}_k\}$ and $\widetilde{u} = \{\widetilde{u}_k\}$ as those obtained by the receding horizon strategy. Figures 4.1 and 4.2 show the relative approximation errors of the optimal states and controls, respectively, for a fixed G. We observe exponential decay of the approximation error in the length of the receding horizon for all cases tested. Moreover, the rate of decay is faster for a larger bound U on the control. The decay rate ρ in Theorem 3.11 depends on the quantity β defined in Proposition 2.14, which in turn depends on the uniform lower bound λ_C of the controllability matrix defined in Definitions 2.9 and 2.12. A larger bound on the control improves the controllability of the problem and should lead to a faster rate of convergence, as is indeed observed here. Figures 4.3 and 4.4 plot the relative errors for different upper bound G on the generation. Similarly, we observe exponential rate of decay for the approximation error in the length of receding horizon. Furthermore, in this case the decay rate is larger for a smaller choice of G. Recall that G is the generation upper limit of the slow plant and our model (4.1) assumes the remaining load will be satisfied fully by a high-cost fast plant. A smaller G indicates that more demand is met by the fast plant, and therefore the system is more controllable, resulting in a faster rate of convergence (for example, if $G = 0$, then we get the optimal solution to be $x_k = u_k = 0$, $\forall k$ and the convergence occurs in one step for any horizon). In summary, our numerical experiments verify the exponential decay of the approximation errors for RHC as proved in Theorem 3.11.

5. Conclusions.

RHC has made a significant impact on industrial control engineering and received extensive study of its theoretical characteristics. We investigate the convergence of its solution with respect to the length of the receding horizon for a linear-quadratic path-constrained optimal control problem.

The version of RHC considered in this work applies the model predictive control every L steps. Our theoretical result, Theorem 3.11, shows that, under some boundedness and controllability conditions, the RHC solution converges to the full horizon solution exponentially fast in the length of receding horizon for a certain choice of L. The exponential rate of convergence allows a principled way of choosing the length of the receding horizon and the control frequency, both important parameters for applications, to achieve a desired accuracy. Our problem admits a nonzero reference trajectory, which to the best of our knowledge is not assumed in the existing stability analysis of RHC. The inclusion of a reference trajectory makes the analysis different from previous approaches since now the convergence is with respect to the solution of the full horizon problem instead of a fixed equilibrium point. Therefore our proofs do not rely on the value function, as most RHC stability analyses do, but instead expose the solution properties of an equality-constrained subproblem and then use sensitivity analysis to connect it to the solution of the original problem. We verify numerically the exponential rate of convergence for a small, synthetic production cost model under various parameter settings. In this example, a lag $L = 1$ is sufficient to observe the exponential decay for the approximation error of the RHC solutions.

The class of optimal control problems investigated here is only one instance of the problems to which RHC can be applied. In particular, although we consider state and control constraints that are common in RHC literature, we do not include other intricate but practical features such as nonlinear dynamics or time delay. Moreover, our theory certifies only that an L, which is computable in terms of the problem data, exists; but $L = 1$ may not always satisfy our conditions. In future work, we will investigate extending the results to other complicating features and determining whether there exist cases where smaller L decreases the performance, as our analysis and the"nervousness" [22] concept in inventory management seem to suggest.

Acknowledgments. We thank Prof. V. Zavala for pointing us to references about stability issues in rolling horizon control. We thank the anonymous referee of [24] who suggested that we look into RHC. This material was based upon work supported by the U.S. Department of Energy, Office of Science, under Contract DE-AC02-06CH11347. M. A. acknowledges partial NSF funding through awards FP061151-01-PR and CNS-1545046.

References.

[1] John Bellingham, Arthur Richards, and Jonathan P How, *Receding horizon control of autonomous aerial vehicles*, in American Control Conference, 2002. Proceedings of the 2002, vol. 5, IEEE, 2002, pp. 3741–3746.

[2] Dimitri P Bertsekas, *Dynamic programming and optimal control*, vol. 1, Athena Scientific, Belmont, MA, 1995.

[3] Raktim Bhattacharya, Gary J Balas, Alpay Kaya, and Andy Packard, *Nonlinear receding horizon control of F-16 aircraft*, in American Control Conference, 2001. Proceedings of the 2001, vol. 1, IEEE, 2001, pp. 518–522.

[4] Lorenz T Biegler and Victor M Zavala, *Large-scale nonlinear programming using Ipopt: An integrating framework for enterprise-wide dynamic optimization*, Computers & Chemical Engineering, 33 (2009), pp. 575–582.

[5] J Frédéric Bonnans and Alexander Shapiro, *Perturbation analysis of optimization problems*, Springer Science & Business Media, 2013.

[6] William B Dunbar and Derek S Caveney, *Distributed receding horizon control of vehicle platoons: Stability and string stability*, IEEE Transactions on Automatic Control, 57 (2012), pp. 620–633.

[7] Ryan Franz, Mark Milam, and John Hauser, *Applied receding horizon control of the Caltech Ducted Fan*, in American Control Conference, 2002. Proceedings of the 2002, vol. 5, IEEE, 2002, pp. 3735–3740.

[8] PJM Interconnection, *Estimated hourly load data*. `http://www.pjm.com/markets-and-operations/energy/real-time/loadhryr.aspx`. Accessed: 2017-07-23.

[9] Ali Jadbabaie and John Hauser, *On the stability of receding horizon control with a general terminal cost*, IEEE Transactions on Automatic Control, 50 (2005), pp. 674–678.

[10] Ali Jadbabaie, Jie Yu, and John Hauser, *Unconstrained receding-horizon control of nonlinear systems*, IEEE Transactions on Automatic Control, 46 (2001), pp. 776–783.

[11] SS a Keerthi and Elmer G Gilbert, *Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations*, Journal of optimization theory and applications, 57 (1988), pp. 265–293.

[12] W Kwon and A Pearson, *A modified quadratic cost problem and feedback stabilization of a linear system*, IEEE Transactions on Automatic Control, 22 (1977), pp. 838–842.

[13] W Hi Kwon, AM Bruckstein, and T Kailath, *Stabilizing state-feedback design via the moving horizon method*, International Journal of Control, 37 (1983), pp. 631–643.

[14] Wook Hyun Kwon and Soo Hee Han, *Receding horizon control: model predictive control for state models*, Springer Science & Business Media, 2006.

[15] Wook Hyun Kwon, YS Lee, and Soo Hee Han, *General receding horizon control for linear time-delay systems*, Automatica, 40 (2004), pp. 1603–1611.

[16] Miles Lubin and Iain Dunning, *Computing in operations research using Julia*, INFORMS Journal on Computing, 27 (2015), pp. 238–248.

[17] David Q Mayne and Hannah Michalska, *Receding horizon control of nonlinear systems*, IEEE Transactions on automatic control, 35 (1990), pp. 814–824.

[18] David Q Mayne, James B Rawlings, Christopher V Rao, and Pierre OM Scokaert, *Constrained model predictive control: Stability and optimality*, Automatica, 36 (2000), pp. 789–814.

[19] James A Primbs and Vesna Nevistić, *Feasibility and stability of constrained finite receding horizon control*, Automatica, 36 (2000), pp. 965–971.

[20] James B Rawlings and Kenneth R Muske, *The stability of constrained receding horizon control*, IEEE transactions on automatic control, 38 (1993), pp. 1512–1516.

[21] Jacques Richalet, A Rault, JL Testud, and J Papon, *Model predictive heuristic control: Applications to industrial processes*, Automatica, 14 (1978), pp. 413–428.

[22] Suresh Sethi and Gerhard Sorger, *A theory of rolling horizon decision making*, Annals of Operations Research, 29 (1991), pp. 387–415.

[23] Athanasios Sideris and Luis A Rodriguez, *A Riccati approach to equality constrained Linear Quadratic Optimal control*, in Proc. American Control Conference, 2010, pp. 5167–5172.

[24] Wanting Xu and Mihai Anitescu, *Exponentially accurate temporal decomposition for long-horizon linear-quadratic dynamic optimization*, Preprint ANL/MCS-P6022-0616, Argonne National Laboratory, Lemont, IL, 60439, June 2016. Available for download at https://doi.org/10.6084/m9.figshare.5504032.v1.

[25] Wei Zhang, Jianghai Hu, and Alessandro Abate, *On the value functions of the discrete-time switched LQR problem*, IEEE Transactions on Automatic Control, 54 (2009), pp. 2669–2674.

Appendix A. Proofs of results in Sections 2 and 3.

A.1. Proof of Proposition 2.14. For any $x_q \in \mathbb{R}^n$, consider the standard linear-quadratic problem:

$$\min \sum_{k=q}^{n_2-1} u_k^T R_k u_k + x_k^T \hat{Q}_k x_k + x_{n_2}^T Q_{n_2} x_{n_2} \tag{A.1a}$$

$$\text{s.t. } x_{k+1} = \hat{A}_k x_k + \hat{B}_k u_k, \qquad q \le k \le n_2 - 1. \tag{A.1b}$$

For $k \ge q$, successively applying (A.1b) gives, for $j \ge 0$,

(A.2)

$$x_{q+j} - \left(\prod_{l=0}^{j-1} \hat{A}_{q+l}\right) x_q = \left[\hat{B}_{q+j-1} \quad \hat{A}_{q+j-1}\hat{B}_{q+j-2} \quad \dots \left(\prod_{l=1}^{j-1} \hat{A}_{q+l}\right) \hat{B}_q\right] \begin{bmatrix} u_{q+j-1} \\ \vdots \\ u_q \end{bmatrix},$$

and for $j = t$, (A.2) reduces to

$$x_{q+t} - \left(\prod_{l=0}^{t-1} \hat{A}_{q+l}\right) x_q = C_{q,t} \begin{bmatrix} u_{q+t-1} \\ \vdots \\ u_q \end{bmatrix}.$$

The index set being UCC(λ_C) implies that $C_{q,t}$ is uniformly completely controllable and in particular that $C_{q,t}$ has full row rank. Therefore, there exists $\hat{u} = (\hat{u}_q^T, \dots, \hat{u}_{q+t-1}^T)^T$ so that

$$-\left(\prod_{l=0}^{t-1} \hat{A}_l\right) x_q = C_{q,t} \begin{bmatrix} \hat{u}_{q+t-1} \\ \vdots \\ \hat{u}_q \end{bmatrix}. \tag{A.3}$$

Several $\hat{u}$ satisfy this relationship; we consider the one defined by

$$\hat{u} = -C_{q,t}^T (C_{q,t} C_{q,t}^T)^{-1} \left(\prod_{l=0}^{t-1} \hat{A}_{q+l}\right) x_q.$$

Denote the corresponding states generated with $\hat{u}_{q:q+t-1}$ as $\hat{x}_{q:q+t}$. Then $\hat{x}_{q+t} = \mathbf{0}$ by (A.3).

Lemma 2.4 implies that

$$\begin{aligned} &\max_{1 \le j \le t} \left\| \left[\hat{B}_{q+j-1} \quad \hat{A}_{q+j-1}\hat{B}_{q+j-2} \quad \dots \left(\prod_{l=1}^{j-1} \hat{A}_{q+l}\right) \hat{B}_q\right] \right\|_2 \\ &\le \max_{1 \le j \le t} \left(C_B + C_A C_B + \dots + C_A^{j-1} C_B \right) \\ &\le \frac{C_B \left(1 - C_A^t\right)}{1 - C_A} \triangleq M. \end{aligned}$$

Then from Definition 2.12 and Lemma 2.4, we have

$$\|\hat{u}\| \le \frac{M}{\lambda_C} C_A^t \|x_q\|. \tag{A.4}$$

From (A.2), we have, for $1 \le j \le t-1$,

$$\|\hat{x}_{q+j}\| \le C_A^j \|x_q\| + M\|\hat{u}\| \le \left(C_A^j + \frac{M^2}{\lambda_C} C_A^t \right) \|x_q\|. \tag{A.5}$$

Now we let $\hat{u}_k = \mathbf{0}$ for $k \ge q+t$. Then it follows that $\hat{x}_k = \mathbf{0}$ for $k \ge q+t$. Also note that since (A.1) is a standard linear-quadratic regulator problem, the optimal value is given by $x_q^T K_q x_q$ [2]. As a result, we have the following.

$$\begin{aligned}
x_q^T K_q x_q \quad &= \quad \min_{u_k} \sum_{k=q}^{n_2-1} x_k^T \hat{Q}_k x_k + u_k^T R_k u_k + x_{n_2}^T Q_{n_2} x_{n_2} \\
&\le \quad \sum_{k=q}^{n_2-1} \hat{x}_k^T \hat{Q}_k \hat{x}_k + \hat{u}_k^T R_k \hat{u}_k + \hat{x}_{n_2}^T Q_{n_2} \hat{x}_{n_2} \\
&\le \quad \sum_{k=q}^{q+t-1} \hat{x}_k^T \hat{Q}_k \hat{x}_k + \hat{u}_k^T R_k \hat{u}_k \\
&\le \quad C_Q \sum_{k=q}^{q+t-1} \|\hat{x}_k\|^2 + C_R \sum_{k=q}^{q+t-1} \|\hat{u}_k\|^2 \\
&\overset{(A.4),(A.5)}{\le} \quad C_Q \left(1 + \sum_{i=1}^{t-1} \left(C_A^i + \frac{M^2}{\lambda_C} C_A^t \right)^2 \right) \|x_q\|^2 + C_R \frac{M^2 C_A^{2t}}{\lambda_C^2} \|x_q\|^2
\end{aligned}$$

Letting

$$\beta = C_Q \left(1 + \sum_{i=1}^{t-1} \left(C_A^i + \frac{M^2}{\lambda_C} C_A^t \right)^2 \right) + C_R \frac{M^2 C_A^{2t}}{\lambda_C^2}$$

completes the proof. Note that β depends only on the quantities in Assumption 2.1, Definitions 2.2 and 2.12, and Lemma 2.4, which are independent of n_1, n_2, and the particular choice of $\mathcal{I}$ given it is UDB(λ_H) and UCC(λ_C).

A.2. Proof of Proposition 2.15. Define $L_k = -W_k^{-1} \hat{B}_k^T K_{k+1} \hat{A}_k$. Then from Lemma 2.13 and (2.4d) we have $D_k = \hat{A}_k + \hat{B}_k L_k$. In [2] the recursion (2.14b) is shown to be equivalent to

$$K_k = {D_k}^T K_{k+1} D_k + \hat{Q}_k + {L_k}^T R_k L_k. \tag{A.6}$$

For $q \le j \le n_2 - 1$, define $x_{j+1} = D_j x_j$. Then (A.6) and Proposition 2.14 imply that

$$\begin{aligned}
x_j^T K_j x_j &\ge x_{j+1}^T K_{j+1} x_{j+1} + x_j^T \hat{Q}_j x_j \\
&\overset{\text{Prop 2.14}}{\ge} x_{j+1}^T K_{j+1} x_{j+1} + \frac{\lambda_Q}{\beta} x_j^T K_j x_j \\
&\overset{(A.6)}{\ge} \left(1 + \frac{\lambda_Q}{\beta} \right) x_{j+1}^T K_{j+1} x_{j+1}.
\end{aligned} \tag{A.7}$$

Here we used the bounds from Lemma 2.4 and the fact that $x_j^T K_j x_j \ge x_{j+1}^T K_{j+1} x_{j+1}$, as implied by (A.6) and the positive definiteness of $\hat{Q}_k$ and R_k. Also we have

$$x_j^T K_j x_j \overset{(A.6),\text{Lemma 2.7}}{\ge} x_j^T \hat{Q}_j x_j \ge \lambda_Q \|x_j\|^2. \tag{A.8}$$

As a result, for $n_2 - 1 \geq j \geq q$, we have the following:

$$\left\|\prod_{l=q}^{j} D_l x_q\right\|^2 = \|x_{j+1}\|^2 \overset{\text{(A.8)}}{\leq} \frac{1}{\lambda_Q} x_{j+1}^T K_{j+1} x_{j+1}$$
$$\overset{\text{(A.7)}}{\leq} \frac{1}{\lambda_Q(1+\lambda_Q/\beta)} x_j^T K_j x_j$$
$$\overset{\text{(A.7)}}{\leq} \frac{1}{\lambda_Q}\left(\frac{1}{1+\lambda_Q/\beta}\right)^{j-q+1} x_q^T K_q x_q$$
$$\overset{\text{Prop 2.14}}{\leq} \frac{\beta}{\lambda_Q}\left(\frac{1}{1+\lambda_Q/\beta}\right)^{j-q+1} \|x_q\|^2,$$

where the third inequality is obtained by repeatedly applying (A.7).

A.3. Proof of Lemma 3.5. Let

$$\begin{aligned} L(y,\theta) = {} & y^T G y/2 + y^T c(\theta) + \lambda^T(Ay - r) + \phi^T(By - d(\theta)) \\ & + \theta^T F \theta + y^T c_1 + \theta^T c_2 + C \end{aligned} \tag{A.9}$$

be the Lagrangian of problem (3.8). Then we have

$$\nabla^2_{(y,\theta)} L = \begin{bmatrix} G & \nabla_\theta c \\ \nabla_\theta^T c & * \end{bmatrix}.$$

Since G and F are positive definite and LICQ holds at y_0, then from [5, Theorem 5.53] and [5, Remark 5.55] we have

$$\begin{aligned} D_p y(\theta_0) &= \text{argmin}_{h\in S} \begin{bmatrix} h^T & p^T \end{bmatrix} \left(\nabla^2_{(y,\theta)} L(y_0,\theta_0)\right) \begin{bmatrix} h \\ p \end{bmatrix} \\ &= \text{argmin}_{h\in S}\ h^T G h/2 + p^T\left(\nabla_\theta^T c(\theta_0)\right) h, \end{aligned} \tag{A.10}$$

where S is the solution of the following linearized problem,

$$\begin{aligned} \min_h \quad & (Gy_0 + c(\theta_0) + c_1)^T h + \left(\nabla_\theta^T c(\theta_0) y_0 + 2F\theta_0 + c_2\right)^T p \\ \text{s.t.} \quad & Bh - (\nabla_\theta d(\theta_0))\, p = 0 \\ & A_{I(y_0,\theta_0)} h \leq 0, \end{aligned} \tag{A.11}$$

and S is given by

$$S = \left\{ h : \begin{bmatrix} B & -\nabla_\theta d(\theta_0) \end{bmatrix} \begin{bmatrix} h \\ p \end{bmatrix} = 0,\ \begin{bmatrix} A_{I_+(y_0,\theta_0,\bar\lambda)} & 0 \end{bmatrix} \begin{bmatrix} h \\ p \end{bmatrix} = 0,\ \begin{bmatrix} A_{I_0(y_0,\theta_0,\bar\lambda)} & 0 \end{bmatrix} \begin{bmatrix} h \\ p \end{bmatrix} \leq 0 \right\}.$$

Thus the directional derivative $D_p y(\theta_0)$ of $y(\theta)$ along direction p at θ_0 is the solution of the problem

$$\begin{aligned} \min_h \quad & h^T G h/2 + p^T\left(\nabla_\theta^T c(\theta_0)\right) h \\ \text{s.t.} \quad & Bh - (\nabla_\theta d(\theta_0))\, p = 0 \\ & A_{I_+(y_0,\theta_0,\bar\lambda)} h = 0 \\ & A_{I_0(y_0,\theta_0,\bar\lambda)} h \leq 0. \end{aligned} \tag{A.12}$$

Let I_1 be the set of active inequality constraints of problem (A.12). Then $I_1 \subset I_0(y_0, \theta_0, \bar{\lambda})$, and let $I'(\theta_0) = I_1 \cup I_+(y_0, \theta_0, \bar{\lambda})$. The KKT condition of problem (A.12) is hence

$$\widetilde{G} \triangleq \begin{bmatrix} G & A_{I'(\theta_0)}^T & B^T \\ A_{I'(\theta_0)} & 0 & 0 \\ B & 0 & 0 \end{bmatrix}, \qquad \widetilde{G} \begin{bmatrix} h^* \\ \phi_1^* \\ \phi_2^* \end{bmatrix} = \begin{bmatrix} -\nabla_\theta c(\theta_0) p \\ 0 \\ \nabla_\theta d(\theta_0) p \end{bmatrix}$$

for some Lagrange multipliers ϕ_1^* and ϕ_2^*. Since LICQ holds at y_0, rows of $A_{I'(\theta_0)}$ and B are linearly independent. Together with the fact that G is positive definite, we have that $\widetilde{G}$ is invertible. Denote the first row of $\widetilde{G}^{-1}$ to be $\begin{bmatrix} p_{11} & p_{12} & p_{13} \end{bmatrix}$. Then, we have

$$D_p y(\theta_0) = h^* = \left(-p_{11} \nabla_\theta c(\theta_0) + p_{13} \nabla_\theta d(\theta_0)\right) p.$$

On the other hand, for problem (3.9) with $I'(\theta_0)$ constructed above, the KKT condition is

$$\widetilde{G} \begin{bmatrix} y^*_{I'(\theta_0)}(\theta) \\ \psi_1^* \\ \psi_2^* \end{bmatrix} = \begin{bmatrix} -c(\theta) \\ r' \\ d(\theta) \end{bmatrix},$$

for some Lagrange multipliers ψ_1^* and ψ_2^*. Since $\widetilde{G}$ is invertible, we have $y^*_{I'(\theta_0)}(\theta) = -p_{11} c(\theta) + p_{12} r' + p_{13} d(\theta)$. It follows that

$$\left. \frac{d y^*_{I'(\theta_0)}(\theta)}{d\theta} \right|_{\theta = \theta_0} = -p_{11} \nabla_\theta c(\theta_0) + p_{13} \nabla_\theta d(\theta_0).$$

As a result, we have

$$D_p y(\theta_0) = \left(\left. \frac{d y^*_{I'(\theta_0)}(\theta)}{d\theta} \right|_{\theta = \theta_0} \right) p,$$

which proves the claim.