

Constrained-based simulation of Multi-Rigid-Body Dynamics with Contact and Friction

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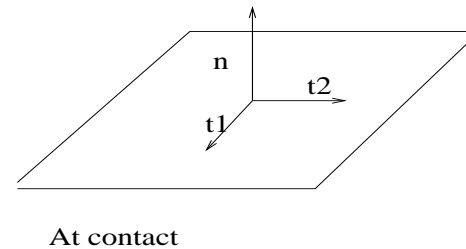
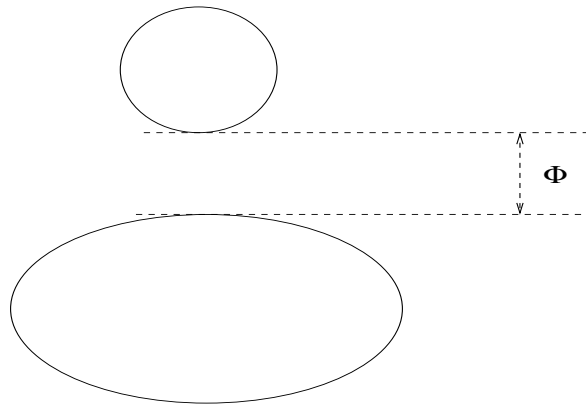
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ICIAM 2003.

Objective

Simulate the behaviour of a system of rigid bodies subject to contact, friction and joint constraints with

- (generalized) positions q .
- (generalized) velocities v .
- mass $M(q)$ (positive definite),
- subject to external force $k(t, q, v)$.



Contact and (Coulomb) Friction Model

- Noninterpenetration Constraints $\Phi(q) \geq 0$ complementary to $c_n \geq 0$ (normal impulse/force).
- Tangent space generators: $\hat{D}(q) = [\hat{t}_1(q), \hat{t}_2(q)]$, tangent force multipliers: $\beta \in R^2$. Tangent force/impulse becomes $\hat{D}(q)\beta$.
- Conic constraints: $\|\beta\| \leq \mu c_n$, where μ is the friction coefficient.
- Max Dissipation Constraints: $\beta = \operatorname{argmin}_{\|\hat{\beta}\| \leq \mu c_n} v^T \hat{D}(q)\hat{\beta}$.

Acceleration Formulation

$$M(q) \frac{d^2 q}{dt^2} - \sum_{i=1}^m \nu^{(i)} c_\nu^{(i)} - \sum_{j=1}^p \left(n^{(j)}(q) c_n^{(j)} + D^{(j)}(q) \beta^{(j)} \right) = k(t, q, \frac{dq}{dt})$$

$$\Theta^{(i)}(q) = 0, \quad i = 1 \dots m$$

$$\Phi^{(j)}(q) \geq 0, \quad \text{compl. to} \quad c_n^{(j)} \geq 0, \quad j = 1 \dots p$$

$$\beta = \operatorname{argmin}_{\widehat{\beta}^{(j)}} v^T D(q)^{(j)} \widehat{\beta}^{(j)} \quad \text{subject to} \quad \left\| \widehat{\beta}^{(j)} \right\| \leq \mu^{(j)} c_n^{(j)}, \quad j = 1 \dots p$$

Here $\nu^{(i)} = \nabla \Theta^{(i)}$, $n^{(j)} = \nabla \Phi^{(j)}$.

It is known that these problems do not have a classical solution even in 2 dimensions, where the discretized cone coincides with the total cone.

Time-stepping scheme

Use Euler method, half-explicit in velocities, linearizing the geometrical constraints. **Fundamental variables: velocities and impulses ($h \times \text{force}$).**

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^m \nu^{(i)} c_\nu^{(i)} - \sum_{j \in \mathcal{A}} (n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)}) = hk$$

$$\nu^{(i)T} v^{l+1} = 0, \quad i = 1..m$$

$$\rho^{(j)} = n^{(j)T} v^{l+1} \geq 0, \quad \text{compl. to } c_n^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\sigma^{(j)} = \lambda^{(j)} e^{(j)} + D^{(j)T} v^{l+1} \geq 0, \quad \text{compl. to } \beta^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\zeta^{(j)} = \mu^{(j)} c_n^{(j)} - e^{(j)T} \beta^{(j)} \geq 0, \quad \text{compl. to } \lambda^{(j)} \geq 0, \quad j \in \mathcal{A}.$$

Here $\nu^{(i)} = \nabla \Theta^{(i)}$, $n^{(j)} = \nabla \Phi^{(j)}$. h is the time step. The set \mathcal{A} consists of the **active** constraints. (Anitescu and Potra, 1997) based on (Stewart and Trinkle, 1996),

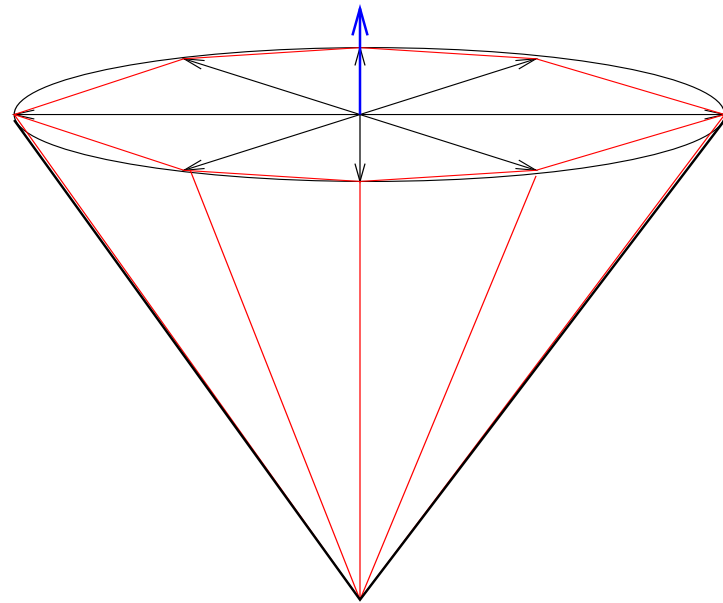
The time-stepping scheme has a solution although the classical formulation doesn't!

Discretized Friction Model

- d_i (GC) is the column corresponding to $t(\alpha_i)$, $\alpha_i \in [0, \pi]$, $i = 1, 2, \dots, l$, $D(q) = [d_1, d_2, \dots, d_l]$.
- To each tangential direction we attach a force $\beta_i \geq 0$, $i = 1, 2, \dots, l$. We denote by $\beta = (\beta_1, \beta_2, \dots, \beta_l)$.
- The frictional constraints become

$$\beta = \operatorname{argmin}_{\hat{\beta} \geq 0} v^T D(q) \hat{\beta} \quad \text{subject to} \quad \left\| \hat{\beta} \right\|_1 \leq \mu c_n.$$

Polygonal cone approximation to the Coulomb cone (3D).



Matrix Form of the Integration Step

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & 0 \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} -Mv^{(l)} - hk \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \quad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

The need for constraint stabilization

The positions are updated by $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$.

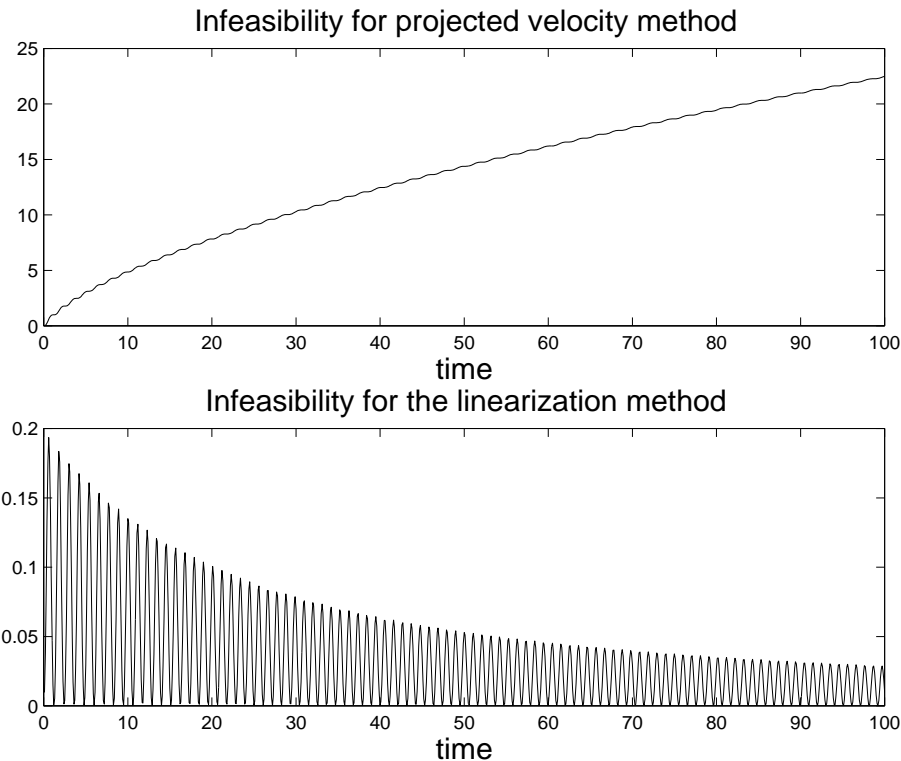
Due to the index reduction, the (geometrical) joint and non interpenetration constraints, which define the feasible set

$$\mathcal{F} = \left\{ q \mid \Theta^{(i)}(q) = 0, 1 \leq i \leq m, \Phi^{(j)}(q) \geq 0, 1 \leq j \leq n_{total} \right\}$$

are replaced by constraints at the velocity level.

This may create **constraint drift**, in which the constraint infeasibility keeps growing. **In interactive simulation this is particularly annoying, since geometrical inconsistency is easy to notice.**

Example of constraint drift



Comparison of constraint error the original method and modified method (to be presented later) for a pendulum example. **Ratio of infeasibilities gets to about 10^3 !**

Preventing constraint drift

- Change the approach to a nonlinear and potentially nonconvex optimization problem.
- Perform a nonlinear projection after each LCP (and eventually preserving the good energy properties (Anitescu and Potra 2002)). If there is no friction the projection may be more costly than the LCP.
- Perform **one step** of an SQP applied to the nonlinear projection problem (Cline and Pai, 2003). Needs an additional Quadratic Program/step. Extension of (Ascher, Chin, Reich 1994) from DAE.
- Modify the right hand side of the LCP with an appropriate function of the infeasibility (parameter-free, (Jean, 1999,w/o analysis) and this work, (Anitescu and Hart 02)) and (Miller and Christiansen 02) and (Anitescu, Miller and Hart 03). This approach uses no additional subproblems or projections.

Stabilized time-stepping scheme

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^m \nu^{(i)} c_\nu^{(i)} - \sum_{j \in \mathcal{A}} (n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)}) = hk$$

$$\nu^{(i)T} v^{l+1} + \frac{\Theta^{(i)}(q^{(1)})}{h_1} = 0, \quad i = 1, \dots, m$$

$$n^{(j)T} v^{l+1} + \frac{\Phi^{(j)}(q^{(1)})}{h_1} - \left(\mu^{(j)} \lambda^{(j)} \right) \geq 0, \quad \text{compl. to } c_n^{(j)} \geq 0, j \in \mathcal{A}$$

$$\lambda^{(j)} e^{(j)} + D^{(j)T} v^{l+1} \geq 0, \quad \text{compl. to } \beta^{(j)} \geq 0, j \in \mathcal{A}$$

$$\mu^{(j)} c_n^{(j)} - e^{(j)T} \beta^{(j)} \geq 0, \quad \text{compl. to } \lambda^{(j)} \geq 0, j \in \mathcal{A}.$$

Recall, $\nu^{(i)} = \nabla \Theta^{(i)}$, $n^{(j)} = \nabla \Phi^{(j)}$, $q^{(l+1)} = q^{(l)} + hv^{(l+1)}$.

The term $(-\mu^{(j)} \lambda^{(j)})$ corresponds to the convex relaxation algorithm proposed by (Anitescu and Hart, 2002). Stabilization works w or w/o it.

Why does it work?

Consider the one contact case where we enforce the constraint by a penalty (smoothing method).

$$\Phi^{(1)}(q) \geq 0 \text{ enforced by penalty force } \theta^{(1)}(q) = \gamma^{(1)} \left(\Phi_{-}^{(j)}(q) \right)^2$$

where $\gamma^{(1)}$ is a very large parameter. Dynamics becomes

$$\begin{aligned} \frac{dq}{dt} &= v. \\ M \frac{dv}{dt} &= k(t, q, v) + \theta^{(1)}(q) \nabla_q \Phi^{(1)}(q). \end{aligned}$$

Apply backward Euler, where $\Phi^{(1)}(q^{(l+1)})$ is replaced by its linearization

$$\Phi^{(1)}(q^{(l+1)}) \approx \Phi^{(1)}(q^{(l)}) + h_l \nabla_q \Phi^{(1)}(q^{(l)})^T v^{(l+1)}$$

Take the limit as time step h_l is fixed and $\gamma^{(1)} \rightarrow \infty$ and

Why does it work (2)?

... we obtain:

$$\begin{aligned}q^{(l+1)} &= q^{(l)} + h_l v^{(l+1)}. \\M \frac{v^{(l+1)} - v^{(l)}}{h_l} &= k(t^{(l)}, q^{(l)}, v^{(l)}) + \sum_{j=1}^m c^{(j),(l+1)} \nabla_q \Phi^{(j)}(q^{(l+1)}) \\0 \leq c^{(j),(l+1)} &\perp \Phi^{(j)}(q^{(l)}) + h_l \nabla \Phi(q^{(l)})^T v^{(l+1)}\end{aligned}$$

which is precisely our scheme.

Assumption: pointed friction cone assumption

Friction cone: set of all possible constraint reaction impulses:

$$FC(q) = \left\{ t = \tilde{\nu}c_\nu + \tilde{n}c_n + \tilde{D}\tilde{\beta} \mid c_n \geq 0, \tilde{\beta} \geq 0, \right. \\ \left. \left\| \beta^{(j)} \right\|_1 \leq \mu^{(j)} c_n^{(j)}, \forall j \in \mathcal{A} \right\}$$

- **Definition (Pang and Stewart 1999)** $FC(q)$ is **pointed** if

$$0 = \tilde{\nu}c_\nu + \tilde{n}c_n + \tilde{D}\tilde{\beta} \in FC(q), c_n \geq 0, \tilde{\beta} \geq 0 \Rightarrow (c_\nu, c_n, \tilde{\beta}) = 0$$

- This assumption is essential in ensuring convergence to a differential inclusions as $h_l \rightarrow 0$ (Stewart 2000) and solvability of time-stepping scheme under more general conditions (Pang and Stewart 1999).
- This assumption implies that the configuration is disassemblable at q (Anitescu, Cremer and Potra 1995).

Main Result: assumptions and setup

- **Definition:** The active set $\mathcal{A} = \{j | 1 \leq j \leq n, \Phi^{(j)}(q^{(l)}) \leq \hat{\epsilon}\}$.
- The friction cone is uniformly pointed for all configurations.
- The geometrical data of the problem, are twice continuously differentiable **in a neighborhood of the feasible set F.**
- The external force increases at most linearly in position and velocity.
- The ratio between successive time steps is uniformly lower bounded
$$\frac{h_l}{h_{l-1}} \geq \frac{1}{c_h}$$
- The mass matrix is constant (Newton-Euler body coordinates).
- There is no initial infeasibility and **the coefficient of restitution is 0.**

Main result, continued

Then for any fixed time interval T , there exists an $H > 0$, and a $V > 0$ and $C > 0$ such that if $h_l < H, \forall l$, we have that

1. $v^{(l)} \leq V, \forall l$.
2. If $j \notin \mathcal{A}$ then $\Phi^{(j)}(q^{(l+1)}) \geq 0$.
3. The infeasibility, defined as

$$I(q) = \max_{1 \leq i \leq m, 1 \leq j \leq n_{total}} \left\{ 0, \left| \Theta^{(i)}(q) \right|, \Phi^{(j)}(q) \right\}$$

satisfies $I(q^{(l+1)}) \leq C \|h_l v^{(l+1)}\|^2, \forall l$.

- a) The last conclusion demonstrates **constraint stabilization**.
- b) $H = \infty$ if geometrical data has uniformly bounded second derivatives.

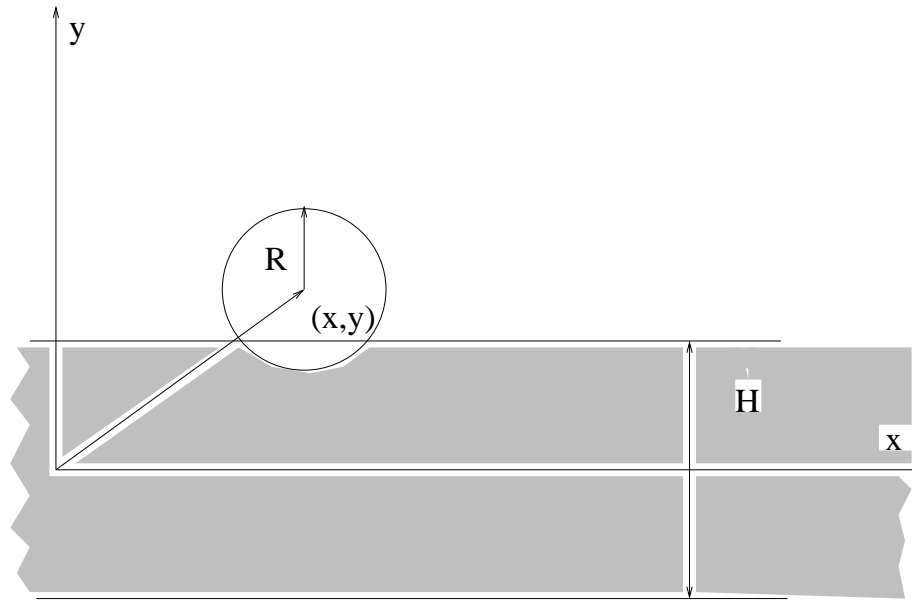
Main technical tool

$$\begin{aligned}
 \min_{\boldsymbol{v}} \quad & \frac{1}{2} \boldsymbol{v}^T M^{(l)} \boldsymbol{v} + \widehat{\boldsymbol{k}}^{(l)T} \boldsymbol{v} \\
 \text{subject to} \quad & n^{(j)T} \boldsymbol{v} + \mu^{(j)} d_i^{(j)T} \boldsymbol{v} \geq -\Gamma^{(j)} - \frac{\boldsymbol{\Phi}^{(j)}(\mathbf{q}^{(1)})}{\mathbf{h}_1} \\
 & i = 1, 2, \dots, m_C^{(j)}, \quad j \in \mathcal{A} \\
 & \boldsymbol{\nu}^{(i)T} \boldsymbol{v} = -\frac{\boldsymbol{\Theta}^{(i)}(\mathbf{q}^{(1)})}{\mathbf{h}_1}, \quad i = 1, 2, \dots, m.
 \end{aligned}$$

- The solution of the LCP for both the original and convex relaxation scheme is a solution of this QP for some $\Gamma^{(j)} \geq 0$, $j \in \mathcal{A}$ and $\widehat{\boldsymbol{k}}^{(l)}$.
- The QP satisfies MFCQ if and only if the friction cone is pointed.
- There exists a $c > 0$, such that, for all l ,

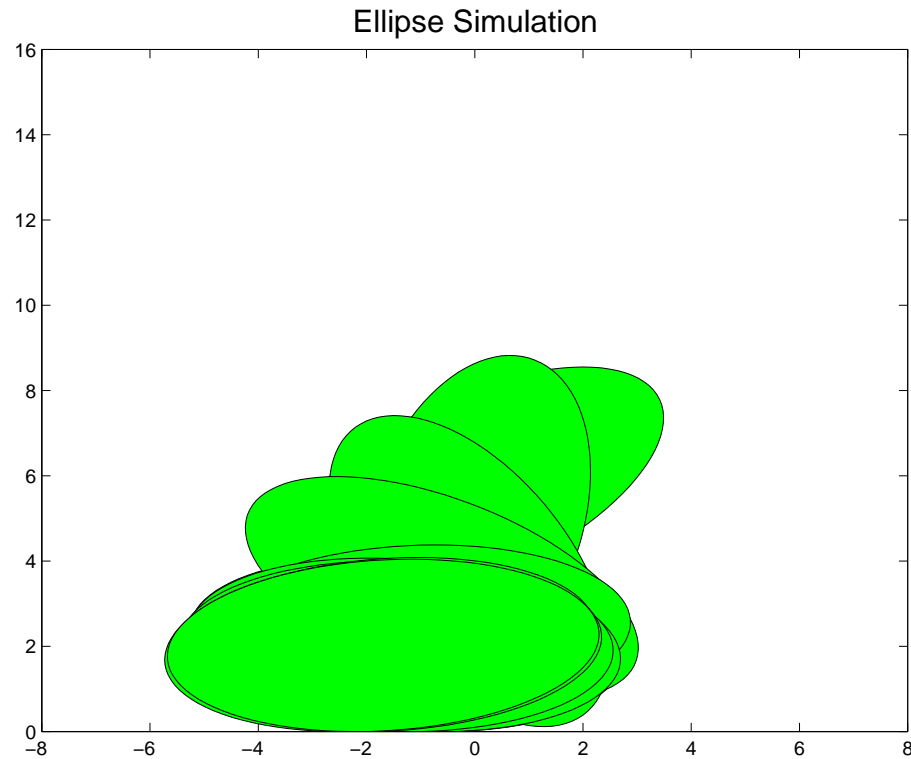
$$\begin{aligned}
 \boldsymbol{v}^{(l+1)T} M^{(l)} \boldsymbol{v}^{(l+1)} & \leq \boldsymbol{v}^{(l)T} M^{(l)} \boldsymbol{v}^{(l)} + h_l^2 \boldsymbol{k}^{(l)T} M^{(l)-1} \boldsymbol{k}^{(l)} \\
 & + 2h_l \boldsymbol{v}^{(l)T} \boldsymbol{k}^{(l)} + c \frac{I(\boldsymbol{q}^{(l)})^2}{h_{l-1}^2} \\
 I(\boldsymbol{q}^{(l+1)}) & \leq ch_l^2 v_{l+1}^2, \text{ if } [\boldsymbol{q}^{(l)}, \boldsymbol{q}^{(l+1)}] \in \mathcal{N}(F)
 \end{aligned}$$

Example of non differentiability of the signed distance



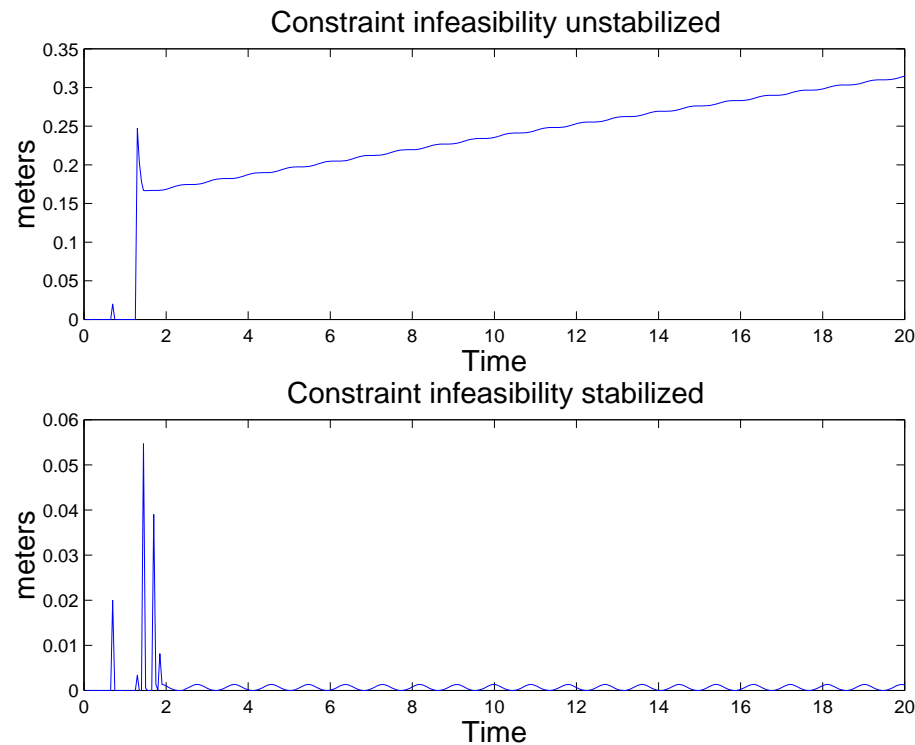
- Signed distance: $d_{12}(q) = |y| - R - \frac{H}{2}$ is not differentiable everywhere!
- It is, however, differentiable over the set $d_{12}(q) \geq -\epsilon$ for any $\epsilon < R + \frac{H}{2}$. . We are OK if the infeasibility is not too large.

Elliptic body simulation



We present ten frames of the simulation of an elliptic body that is dropped on the table. There is an initial angular velocity of 3, the body has axes 4 and 8 and is dropped from a height of 8.

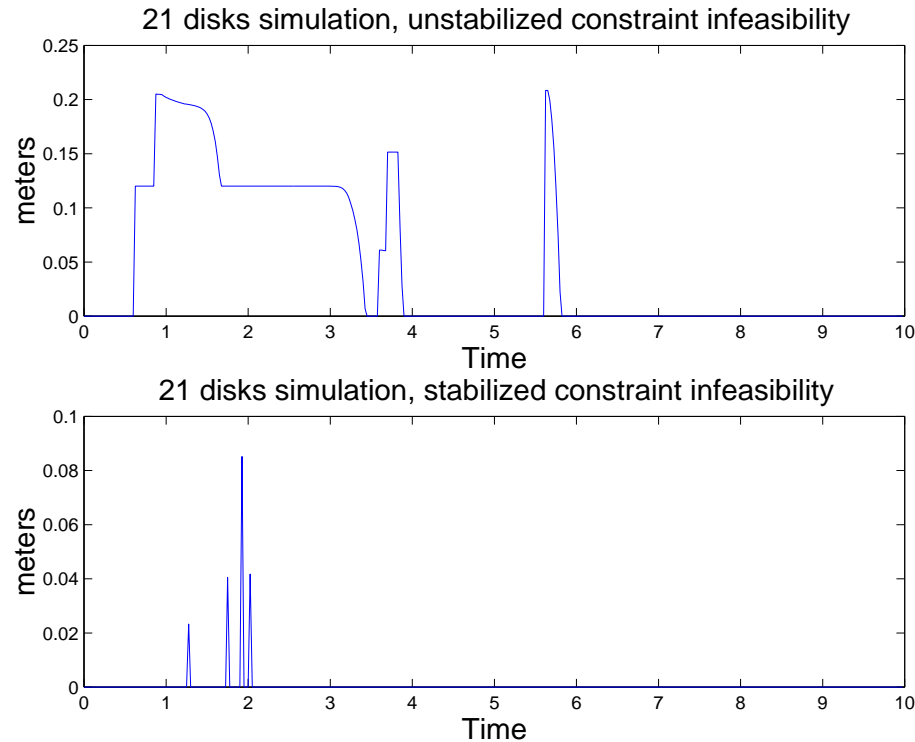
Infeasibility behavior unstabilized versus stabilized method



We see that drift becomes catastrophic for the unstabilized method, whereas remains in a narrow range for the stabilized method.

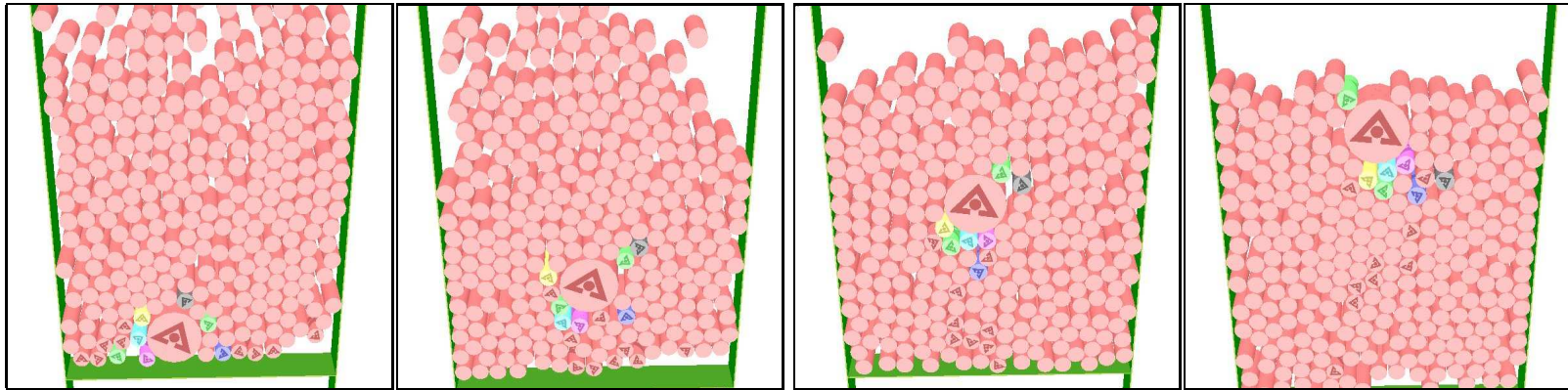
Constraint stabilization is accomplished!

Infeasibility comparison for 21 body simulation



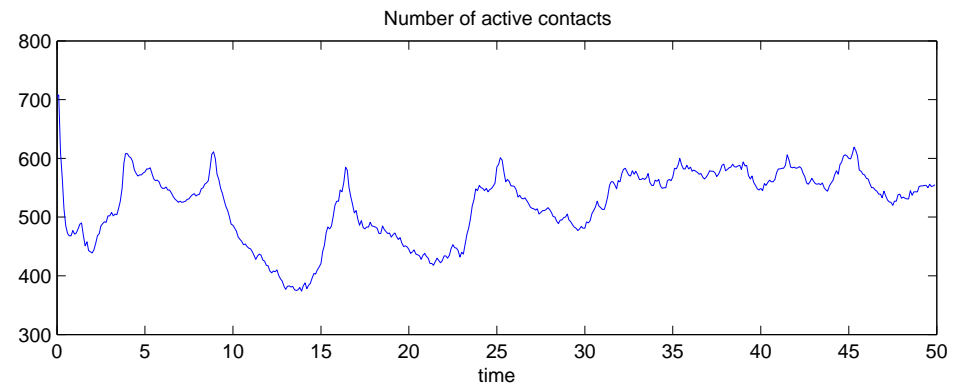
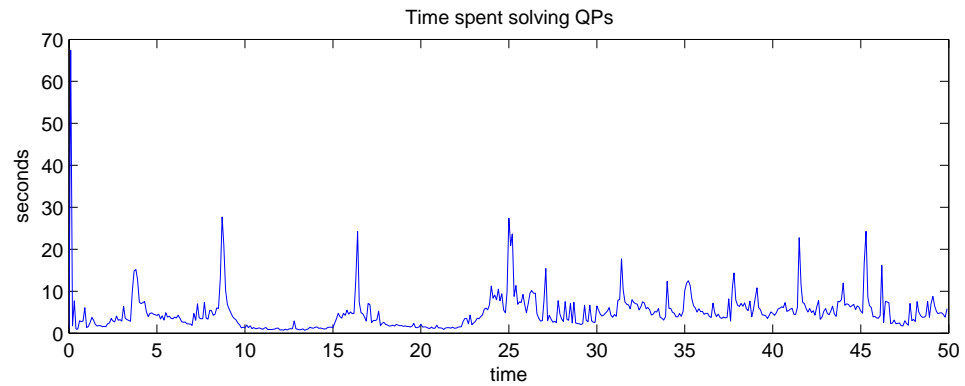
21 disks of radius 3 on a plank starting from the cannoball arrangement at rest. Stabilized method still has a much lower infeasibility. The time-step is 50ms, and the LCP is solved in at most 30 ms at every step.

Brazil nut effect simulation



- Time step of 100ms, for 50s. 270 bodies.
- Convex Relaxation Method. One QP/step. No collision backtrack.
- Friction is 0.5, restitution coefficient is 0.5.
- Large ball emerges after about 40 shakes. Results in the same order of magnitude as MD simulations (but with 4 orders of magnitude larger time step).

Brazil nut effect simulations performance



Conclusions and future work

- We define a method that achieves constraint stabilization while solving only linear complementarity problem per step.
- Our method does not need to stop and detect collisions explicitly and can advance with a constant time step and predictable amount of effort per step.
- The method has been extended to a parametric version that is used in a robotic grasp simulator (Miller and Christiansen, 2002) and (Anitescu, Miller and Hart, 2003).
- **Future work:** Nonsmooth particles and stabilization proof for nonzero coefficient of restitution. **Fast QP Solver.** Convergence for relaxation scheme as $h \rightarrow 0$?