Solving nonconvex problems of nonsmooth dynamics by convex relaxation

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Nonsmooth rigid multibody dynamics (NRMD) methods attempt to predict the position and velocity evolution of a group of rigid particles subject to certain constraints and forces.

- non-interpenetration.
- collision.
- joint constraints
- adhesion
- Dry friction – Coulomb model.
- global forces: electrostatic, gravitational.

These we cover in our approach.
Areas that use NRMD

- granular and rock dynamics.
- masonry stability analysis.
- simulation of concrete obstacle response to explosion.
- tumbling mill design (mineral processing industry).
- interactive virtual reality.
- robot simulation and design.
Model Requirements and Notations

- MBD system: generalized positions \( q \) and velocities \( v \). Dynamic parameters: mass \( M(q) \) (positive definite), external force \( k(t, q, v) \).
- Non interpenetration constraints: \( \Phi^{(j)}(q) \geq 0, 1 \leq j \leq n_{total} \) and compressive contact forces at a contact.
- Joint (bilateral) constraints: \( \Theta^{(i)}(q) = 0, 1 \leq i \leq m \).
- Frictional Constraints: Coulomb friction, for friction coefficients \( \mu^{(j)} \).
- Dynamical Constraints: Newton laws, conservation of impulse at collision.
Normal velocity: $v_n$
Normal impulse: $c_n$

Contact Model

- Contact configuration described by the (generalized) distance function $d = \Phi(q)$, which is defined for some values of the interpenetration. Feasible set: $\Phi(q) \geq 0$.

- Contact forces are compressive, $c_n \geq 0$.

- Contact forces act only when the contact constraint is exactly satisfied, or

  $\Phi(q)$ is complementary to $c_n$ or $\Phi(q)c_n = 0$, or $\Phi(q) \perp c_n$. 
**Coulomb Friction Model**

- Tangent space generators: \( \hat{D}(q) = [\hat{d}_1(q), \hat{d}_2(q)] \), tangent force multipliers: \( \beta \in \mathbb{R}^2 \), tangent force \( D(q)\beta \).

- **Conic constraints**: \( ||\beta|| \leq \mu c_n \), where \( \mu \) is the friction coefficient.

- **Max Dissipation Constraints**: \( \beta = \text{argmin}_{||\beta|| \leq \mu c_n} v^T \hat{D}(q)\beta. \)

- \( v_T \), the tangential velocity, satisfies \( |v_T| = \lambda = -v^T \hat{D}(q) \frac{\beta}{||\beta||}. \) \( \lambda \) is the Lagrange multiplier of the conic constraint.

- **Discretized Constraints**: The set \( \hat{D}(q)\beta \) where \( ||\beta|| \leq \mu c_n \) is approximated by a polygonal convex subset: \( D(q)\tilde{\beta}, \tilde{\beta} \geq 0, \)

\[
\left\| \tilde{\beta} \right\|_1 \leq \mu c_n. \text{ Here } D(q) = [d_1(q), d_2(q), \ldots, d_m(q)].
\]

For simplicity, we denote \( \tilde{\beta} \) the vector of force multipliers by \( \beta \).
Defining the friction cone

For one contact:

\[
FC^{(j)}(q) = \left\{ c_n^{(j)} n^{(j)} + \beta_1^{(j)} t_1^{(j)} + \beta_2^{(j)} t_2^{(j)} \right\}
\]

\[
c_n^{(j)} \geq 0, \quad \sqrt{(\beta_1^{(j)})^2 + (\beta_2^{(j)})^2} \leq \mu^{(j)} c_n^{(j)}
\]

The total friction cone:

\[
FC(q) = \left\{ \sum_{j=1,2,\ldots,p} c_n^{(j)} n^{(j)} + \beta_1^{(j)} t_1^{(j)} + \beta_2^{(j)} t_2^{(j)} \right\}
\]

\[
\sqrt{(\beta_1^{(j)})^2 + (\beta_2^{(j)})^2} \leq \mu^{(j)} c_n^{(j)},
\]

\[
c_n^{(j)} \geq 0 \perp \Phi^{(j)}(q) = 0, \quad j = 1, 2, \ldots, p \}
\]

We have

\[
FC(q) = \sum_{j=1,2,\ldots,p, \Phi^{(j)}(q)=0} FC^{(j)}(q).
\]
**Acceleration Formulation**

\[
M(q) \frac{d^2 q}{dt^2} - \sum_{i=1}^{m} \nu^{(i)} c^{(i)}_{\nu} - \sum_{j=1}^{p} \left( n^{(j)}(q)c^{(j)}_{n} + D^{(j)}(q)\beta^{(j)} \right) = k(t, q, \frac{dq}{dt})
\]

\[
\Theta^{(i)}(q) = 0, \quad i = 1 \ldots m
\]

\[
\Phi^{(j)}(q) \geq 0, \quad \text{compl. to} \quad c^{(j)}_{n} \geq 0, \quad j = 1 \ldots p
\]

\[
\beta = \arg\min_{\tilde{\beta}(j)} v^{T} D(q)^{(j)} \tilde{\beta}(j)
\]

subject to \[
\left\| \tilde{\beta}(j) \right\|_{1} \leq \mu^{(j)} c^{(j)}_{n}, \quad j = 1 \ldots p
\]

Here \[
\nu^{(i)} = \nabla \Theta^{(i)}, \quad n^{(j)} = \nabla \Phi^{(j)}.
\]

It is known that these problems do not have a classical solution even in 2 dimensions, where the discretized cone coincides with the total cone. Painlevé’s paradox.
A Painleve paradox example

\[ I = \frac{m}{16} \]
\[ \theta = 72^\circ \]
\[ \omega = 0 \]
\[ 16(\cos^2 \theta - \mu \cos \theta \sin \theta) = -2 \]
\[ \mu = 0.75 \]

\[ p = r - \frac{l}{2} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \]

**Constraint:** \[ \hat{n}p \geq 0 \] (defined everywhere).

\[ \hat{n}\ddot{p} = -g + f_N \left( \frac{1}{m} + \frac{l}{2I} (\cos^2(\theta) - \mu \sin(\theta) \cos(\theta)) \right) \]

\[ \hat{n}\ddot{p}_a = -g - \frac{f_N}{m} \]

Painleve Paradox: No classical solutions!
Continuous formulation in terms of friction cone

\[ M \frac{dv}{dt} = f_C(q, v) + k(q, v) + \rho \]
\[ \frac{dq}{dt} = v. \]
\[ \rho = \sum_{j=1}^{p} \rho^{(j)}(t). \]
\[ \rho^{(j)}(t) \in FC^{(j)}(q(t)) \]
\[ \Phi^{(j)}(q) \geq 0, \]
\[ \|\rho^{(j)}\| \Phi^{(j)}(q) = 0, \quad j = 1, 2, \ldots, p. \]

However, we cannot expect even that the velocity is continuous!. So we must consider a weaker form of differential relationship
Measure Differential Inclusions

We must now assign a meaning to

\[ M \frac{dv}{dt} - f_c(q, v) - k(t, q, v) \in FC(q). \]

**Definition** If \( \nu \) is a measure and \( K(\cdot) \) is a convex-set valued mapping, we say that \( \nu \) satisfies the differential inclusions

\[ \frac{dv}{dt} \in K(t) \]

if, for all continuous \( \phi \geq 0 \) with compact support, not identically 0, we have that

\[ \frac{\int \phi(t) \nu(dt)}{\int \phi(t)dt} \in \bigcup_{\tau: \phi(\tau) \neq 0} K(\tau). \]
Weaker formulation for NRMD

Find $q(\cdot), v(\cdot)$ such that

1. $v(0)$ is a function of bounded variation (but may be discontinuous).

2. $q(\cdot)$ is a continuous, locally Lipschitz function that satisfies

$$q(t) = q(0) + \int_0^t v(\tau) d\tau$$

3. The measure $dv(t)$, which exists due to $v$ being a bounded variation function, must satisfy, (where $f_c(q, v)$ is the Coriolis and Centripetal Force)

$$\frac{d(Mv)}{dt} - k(t, v) - f_c(q, v) \in FC(q(t))$$

4. $\Phi^{(j)}(q) \geq 0, \forall j = 1, 2, \ldots, p.$
Linearization method

For time-stepping scheme, the geometrical constraints are enforced by linearization.

\[
\nabla \Phi(q^{(l)})^T v^{(l+1)} \geq 0 \implies \Phi^{(j)}(q^{(l)}) + \gamma h_l \nabla \Phi(q^{(l)})^T v^{(l+1)} \geq 0.
\]

\[
\nabla \Theta(q^{(l)})^T v^{(l+1)} = 0 \implies \Theta^{(j)}(q^{(l)}) + \gamma h_l \nabla \Theta(q^{(l)})^T v^{(l+1)} = 0.
\]

Here \( \gamma \in (0, 1] \). \( \gamma = 1 \) corresponds to exact linearization.
**Time-stepping scheme**

Euler method, half-explicit in velocities, linearization for constraints. Maximum dissipation principle enforced through optimality conditions.

\[
M(v^{l+1} - v^{l}) - \sum_{i=1}^{m} \nu^{(i)}c^{(i)} - \sum_{j \in A} (n^{(j)}c^{(j)} + D^{(j)}\beta^{(j)}) = hk
\]

\[
\nu^{(i)T}v^{l+1} = -\gamma \frac{\Theta^{(i)}}{h},
\]

\[
\rho^{(j)} = n^{(j)^T}v^{l+1} \geq -\gamma \frac{\Phi^{(j)}(q)}{h}, \quad \text{compl. to} \quad c^{(j)} \geq 0, \quad j \in A
\]

\[
\sigma^{(j)} = \lambda^{(j)}e^{(j)} + D^{(j)^T}v^{l+1} \geq 0, \quad \text{compl. to} \quad \beta^{(j)} \geq 0, \quad j \in A
\]

\[
\zeta^{(j)} = \mu^{(j)}c^{(j)} - e^{(j)^T}\beta^{(j)} \geq 0, \quad \text{compl. to} \quad \lambda^{(j)} \geq 0, \quad j \in A.
\]

Here \(\nu^{(i)} = \nabla\Theta^{(i)}\), \(n^{(j)} = \nabla\Phi^{(j)}\). \(h\) is the time step. The set \(A\) consists of the active constraints. Stewart and Trinkle, 1996, MA and Potra, 1997: Scheme has a solution although the classical formulation doesn’t!
Matrix Form of the Integration Step

\[
\begin{bmatrix}
M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\
\tilde{\nu}^T & 0 & 0 & 0 & 0 \\
\tilde{n}^T & 0 & 0 & 0 & 0 \\
\tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\
0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0
\end{bmatrix}
\begin{bmatrix}
v^{(l+1)} \\
\tilde{c}_\nu \\
\tilde{c}_n \\
\tilde{\beta} \\
\tilde{\lambda}
\end{bmatrix}
+ \begin{bmatrix}
-Mv^{(l)} - hk \\
\gamma \\
\Delta \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\tilde{\rho} \\
\tilde{\sigma} \\
\tilde{\zeta}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\tilde{c}_n \\
\tilde{\beta} \\
\tilde{\lambda}
\end{bmatrix}^T
\begin{bmatrix}
\tilde{\rho} \\
\tilde{\sigma} \\
\tilde{\zeta}
\end{bmatrix} = 0,
\begin{bmatrix}
\tilde{c}_n \\
\tilde{\beta} \\
\tilde{\lambda}
\end{bmatrix} \geq 0,
\begin{bmatrix}
\tilde{\rho} \\
\tilde{\sigma} \\
\tilde{\zeta}
\end{bmatrix} \geq 0.
\]
Regularity Conditions: Friction cone assumptions

Define $\varepsilon$ cone

$$
\varepsilon \widehat{FC}(q) = \sum_{\Phi^{(j)}(q) \leq \varepsilon} FC^{(j)}(q).
$$

Pointed friction cone assumption: $\exists K_\varepsilon, K^*_\varepsilon$, and $t(q, \varepsilon) \in \varepsilon \widehat{FC}(q)$ and $v(q, \varepsilon) \in \varepsilon \widehat{FC}^*(q)$, such that, $\forall q \in R^n$, and $\forall \varepsilon \in [0, \bar{\varepsilon}]$, we have that

- $t(q, \varepsilon)^T w \geq K_\varepsilon \|t(q, \varepsilon)\| \|w\|$, $\forall w \in \varepsilon \widehat{FC}(q)$.
- $n^{(j)^T} v(q, \varepsilon) \geq \mu \sqrt{t_1^{(j)^T} v(q, \varepsilon) + t_2^{(j)^T} v(q, \varepsilon) + K^*_\varepsilon \|v(q, \varepsilon)\|}$, $j = 1, 2, \ldots, p.$
(Stewart) Assume

H1 The functions $n^{(j)}(q)$, $t^{(j)}_{1}(q)$, $t^{(j)}_{2}(q)$ are smooth and globally Lipschitz, and they are bounded in the 2-norm.

H2 The mass matrix $M$ is positive definite.

H3 The external force increases at most linearly with the velocity and position.

H4 The uniform pointed friction cone assumption holds.

Then there exists a subsequence $h_{k} \to 0$ where

- $q^{h_{k}}(\cdot) \to q(\cdot)$ uniformly.
- $v^{h_{k}}(\cdot) \to v(\cdot)$ pointwise a.e.
- $d\nu^{h_{k}}(\cdot) \to d\nu(\cdot)$ weak * as Borel measures. in $[0,T]$, and every such subsequence converges to a solution $(q(\cdot), v(\cdot))$ of MDI.
Solving the LCP

Is it possible to obtain an algorithm that has modest conceptual complexity?

- **Lemke’s method** after reduction to proper LCP works, but for larger scale problems alternatives to it are desirable. Works well for tens of bodies, most of the time.

- **Interior Point methods** work for the frictionless problem (since matrices are PSD), but their applicability to the problem with friction depends on the convexity of the solution set.

- Is the solution set of the complementarity problem convex?
Nonconvex solution set

Force Balance:

\[ \sum_{j=1}^{6} c^{(j)} n^{(j)} - hmg \begin{pmatrix} n_m \\ 0_3 \end{pmatrix} = 0. \]

\[ \mu c^{(j)} \geq 0 \quad \perp \quad \lambda^{(j)} \geq 0, \quad j = 1, 2, \ldots, 6. \]
Nonconvex solution set

The following solutions

1. \( c_n^{(1)} = c_n^{(3)} = c_n^{(5)} = \frac{hmg}{3}, c_n^{(2)} = c_n^{(4)} = c_n^{(6)} = 0, \quad \lambda^{(1)} = \lambda^{(3)} = \lambda^{(5)} = 0, \lambda^{(2)} = \lambda^{(4)} = \lambda^{(6)} = 1, \)

2. \( c_n^{(1)} = c_n^{(3)} = c_n^{(5)} = 0, c_n^{(2)} = c_n^{(4)} = c_n^{(6)} = \frac{hmg}{3}, \quad \lambda^{(1)} = \lambda^{(3)} = \lambda^{(5)} = 1, \lambda^{(2)} = \lambda^{(4)} = \lambda^{(6)} = 0. \)

The average of these solutions satisfies \( c_n^{(j)} = \frac{hmg}{6}, \lambda^{(j)} = \frac{1}{2}, \) for \( j = 1, 2, \ldots, 6, \) which violate

\[ \mu c_n^{(j)} \geq 0 \quad \perp \quad \lambda^{(j)} \geq 0, \quad j = 1, 2, \ldots, 6, \]

The average of these solutions, that both induce \( v = 0, \) violates,

\[ \beta_1^{(2)} \geq 0 \quad \perp \quad \lambda^{(2)} \geq 0. \]

For any \( \mu > 0 \) the LCP matrix is no \( P^* \) matrix, polynomiality unlikely.
The convex relaxation

Define $\Theta^{(l)} = -M v^{(l)} - h k^{(l)}$. We solve the following LCP

$$
\begin{bmatrix}
M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\
\tilde{\nu}^T & 0 & 0 & 0 & 0 \\
\tilde{n}^T & 0 & 0 & 0 & -\tilde{\mu} \\
\tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\
0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0
\end{bmatrix}
\begin{bmatrix}
v^{(l+1)} \\
\tilde{c}_v \\
\tilde{c}_n \\
\tilde{\beta} \\
\tilde{\lambda}
\end{bmatrix}
+ 
\begin{bmatrix}
\Gamma \\
\Delta \\
0 \\
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\tilde{\rho} \\
\tilde{\sigma} \\
\tilde{\zeta}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\tilde{c}_n \\
\tilde{\beta} \\
\tilde{\lambda}
\end{bmatrix}^T
\begin{bmatrix}
\tilde{\rho} \\
\tilde{\sigma} \\
\tilde{\zeta}
\end{bmatrix}
= 0,
\begin{bmatrix}
\tilde{c}_n \\
\tilde{\beta} \\
\tilde{\lambda}
\end{bmatrix}
\geq 0,
\begin{bmatrix}
\tilde{\rho} \\
\tilde{\sigma} \\
\tilde{\zeta}
\end{bmatrix}
\geq 0.
$$

The LCP is actually equivalent to a strongly convex QP.
The new convergence result with convex subproblems

H1 The functions \( n^{(j)}(q), t_1^{(j)}(q), t_2^{(j)}(q) \) are smooth and globally Lipschitz, and they are bounded in the 2-norm.

H2 The mass matrix \( M \) is positive definite.

H3 The external force increases at most linearly with the velocity and position.

H4 The uniform pointed friction cone assumption holds.

Then there exists a subsequence \( h_k \to 0 \) where

- \( q^{h_k}(\cdot) \to q(\cdot) \) uniformly.
- \( v^{h_k}(\cdot) \to v(\cdot) \) pointwise a.e.
- \( d v^{h_k}(\cdot) \to d v(\cdot) \) weak * as Borel measures in \([0,T]\), and every such subsequence converges to a solution \((q(\cdot), v(\cdot))\) of MDI. Here \( q^{h_k} \) and \( v^{h_k} \) is produced by the relaxed algorithm.
Comparison between methods

**Dropped particle**

\[ h_k = \frac{0.1}{2^k}, \mu = 0.3 \]

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<th>(h_k)</th>
<th>(y_{QP} - y_{LCP})_2</th>
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<tr>
<td>7</td>
<td>3.2649217e-005</td>
<td></td>
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</tbody>
</table>

**Painleve example**

\[ h_k = \frac{0.1}{2^k}, \mu = 0.75 \]

<table>
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<th>(h_k)</th>
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</tbody>
</table>

No convergence, but small absolute error.
Granular matter

- Sand, Powders, Rocks, Pills are examples of granular matter.
- The range of phenomena exhibited by granular matter is tremendous. Size-based segregation, jamming in grain hoppers, but also flow-like behavior.
- There is still no accepted continuum model of granular matter.
- Direct simulation methods (discrete element method) are still the most general analysis tool, but they are also computationally costly.
- The favored approach: the penalty method which works with time-steps of microseconds for moderate size configurations.
Brazil nut effect simulation

- Time step of 100ms, for 50s. 270 bodies.
- Convex Relaxation Method. **One QP/step.  No collision backtrack.**
- Friction is 0.5, restitution coefficient is 0.5.
- Large ball emerges after about 40 shakes. Results in the same order of magnitude as MD simulations (but with 4 orders of magnitude larger time step).
Brazil nut effect simulations performance

- Time spent solving QPs
- Number of active contacts

Graphs showing the performance metrics over time.
Conclusions and remarks

- We have shown that we find solutions to measure differential inclusions by solving quadratic programs, as opposed to LCP with possible nonconvex solution set.

- PATH is very robust for the original formulation when problem and friction is small but fails for larger problems. However, PATH is successful in solving the QP.

- This is a major progress for solving very large scale problems, since it opens the possibility of applying a variety of algorithms, including iterative algorithms.