Solving nonconvex problems of nonsmooth dynamics by convex relaxation

Mihai Anitescu

Argonne National Laboratory

Nonsmooth multi-rigid-body dynamics

Nonsmooth rigid multibody dynamics (NRMD) methods attempt to predict the position and velocity evolution of a group of rigid particles subject to certain constraints and forces.

- non-interpenetration.
- collision.
- joint constraints
- adhesion
- Dry friction – Coulomb model.
- global forces: electrostatic, gravitational.

■ These we cover in our approach.

Areas that use NRMD

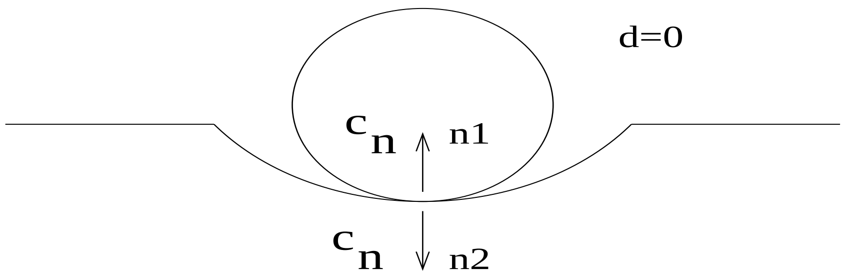
- granular and rock dynamics.
- masonry stability analysis.
- simulation of concrete obstacle response to explosion.
- tumbling mill design (mineral processing industry).
- interactive virtual reality.
- robot simulation and design.

Model Requirements and Notations

- MBD system : generalized positions q and velocities v. Dynamic parameters: mass $M(q)$ (positive definite), external force $k(t,q,v)$.
- Non interpenetration constraints: $\Phi^{(j)}(q) \geq 0,\ 1 \leq j \leq n_{total}$ and compressive contact forces at a contact.
- Joint (bilateral) constraints: $\Theta^{(i)}(q) = 0,\ 1 \leq i \leq m$.
- Frictional Constraints: Coulomb friction, for friction coefficients $\mu^{(j)}$.
- Dynamical Constraints: Newton laws, conservation of impulse at collision.

Normal velocity: v_n

Normal impulse: c_n



Contact Model

- Contact configuration described by the (generalized) distance function $d = \Phi(q)$, which is defined for some values of the interpenetration. Feasible set: $\Phi(q) \geq 0$.
- Contact forces are compressive, $c_n \geq 0$.
- Contact forces act only when the contact constraint is exactly satisfied, or

$$\Phi(q) \text{ is complementary to } c_n \quad \text{or } \Phi(q)c_n = 0, \quad \text{or } \Phi(q) \perp c_n.$$

Coulomb Friction Model

- Tangent space generators: $\widehat{D}(q) = \left[\widehat{d}_1(q), \widehat{d}_2(q)\right]$, tangent force multipliers: $\beta \in R^2$, tangent force $D(q)\beta$.
- Conic constraints: $||\beta|| \leq \mu c_n$, where μ is the friction coefficient.
- Max Dissipation Constraints: $\beta = \text{argmin}_{||\widehat{\beta}|| \leq \mu c_n} v^T \widehat{D}(q)\widehat{\beta}$.
- v_T, the tangential velocity, satisfies $|v_T| = \lambda = -v^T \widehat{D}(q) \frac{\beta}{||\beta||}$. λ is the Lagrange multiplier of the conic constraint.
- Discretized Constraints: The set $\widehat{D}(q)\beta$ where $||\beta|| \leq \mu c_n$ is approximated by a polygonal convex subset: $D(q)\tilde{\beta}$, $\tilde{\beta} \geq 0$, $\left|\left|\tilde{\beta}\right|\right|_1 \leq \mu c_n$. Here $D(q) = [d_1(q), d_2(q), \ldots, d_m(q)]$.

For simplicity, we denote $\tilde{\beta}$ the vector of force multipliers by β.

Defining the friction cone

For one contact:

$$
\begin{aligned}
FC^{(j)}(q) \quad = \quad & \Big\{ c_n^{(j)} n^{(j)} + \beta_1^{(j)} t_1^{(j)} + \beta_2^{(j)} t_2^{(j)} \Big| \\
& c_n^{(j)} \geq 0, \; \sqrt{\left(\beta_1^{(j)}\right)^2 + \left(\beta_2^{(j)}\right)^2} \leq \mu^{(j)} c_n^{(j)} \Big\} .
\end{aligned}
$$

The total friction cone:

$$
\begin{aligned}
FC(q) \quad = \quad & \Big\{ \textstyle\sum_{j=1,2,\ldots,p} c_n^{(j)} n^{(j)} + \beta_1^{(j)} t_1^{(j)} + \beta_2^{(j)} t_2^{(j)} \Big| \\
& \sqrt{\left(\beta_1^{(j)}\right)^2 + \left(\beta_2^{(j)}\right)^2} \leq \mu^{(j)} c_n^{(j)}, \\
& c_n^{(j)} \geq 0 \perp \Phi^{(j)}(q) = 0, j = 1, 2, \ldots, p \Big\} .
\end{aligned}
$$

We have

$$
FC(q) = \sum_{j=1,2,\ldots,p, \; \Phi^{(j)}(q)=0} FC^{(j)}(q).
$$

Acceleration Formulation

$$M(q)\frac{d^2q}{dt^2} - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j=1}^{p} \left(n^{(j)}(q) c_n^{(j)} \quad + D^{(j)}(q)\beta^{(j)} \right) = k(t, q, \frac{dq}{dt})$$

$$\Theta^{(i)}(q) = 0, \qquad i = 1 \ldots m$$

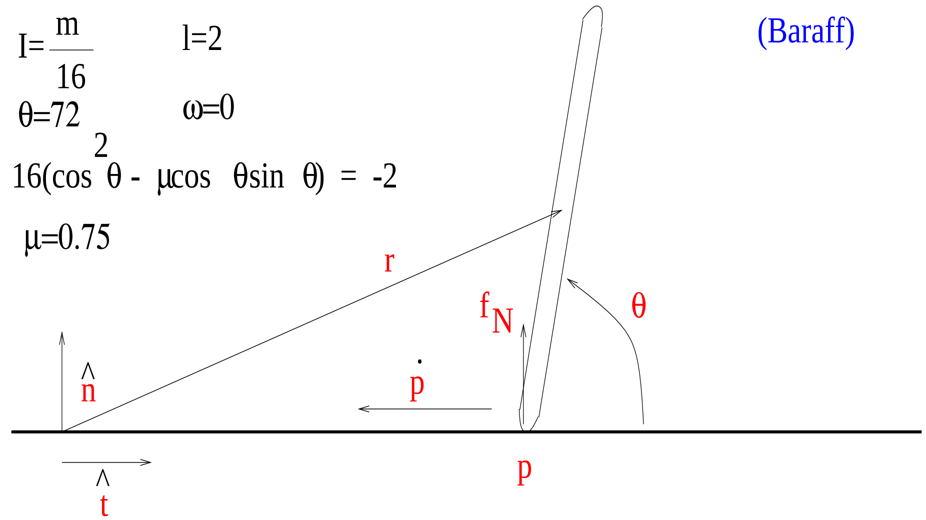
$$\Phi^{(j)}(q) \geq 0, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j = 1 \ldots p$$

$$\beta = \text{argmin}_{\widehat{\beta}(j)} v^T D(q)^{(j)} \widehat{\beta}^{(j)} \quad \textbf{subject to} \quad \left|\left|\widehat{\beta}^{(j)}\right|\right|_1 \leq \mu^{(j)} c_n^{(j)}, \; j = 1 \ldots p$$

Here $\nu^{(\mathbf{i})} = \nabla \mathbf{\Theta}^{(\mathbf{i})}$, $n^{(j)} = \nabla \Phi^{(j)}$.

It is known that these problems do not have a classical solution even in 2 dimensions, where the discretized cone coincides with the total cone.**Painleve's paradox**

A Painleve paradox example



$$p = r - \frac{l}{2}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}$$

Constraint: $\hat{n}p \geq 0$ (defined everywhere).

$$\hat{n}\ddot{p} = -g + f_N\left(\frac{1}{m} + \frac{l}{2I}(\cos^2(\theta) - \mu\sin(\theta)\cos(\theta))\right)$$

$$\hat{n}\ddot{p}_a = -g - \frac{f_N}{m}$$

Painleve Paradox: No classical solutions!

Continuous formulation in terms of friction cone

$$
\begin{aligned}
M\,\tfrac{dv}{dt} &= f_C(q,v) + k(q,v) + \rho \\
\tfrac{dq}{dt} &= v. \\
\rho &= \textstyle\sum_{j=1}^{p} \rho^{(j)}(t). \\
\rho^{(j)}(t) &\in FC^{(j)}(q(t)) \\
\Phi^{(j)}(q) &\geq 0, \\
\left|\left|\rho^{(j)}\right|\right| \Phi^{(j)}(q) &= 0, \quad j = 1, 2, \ldots, p.
\end{aligned}
$$

However, we cannot expect even that the velocity is continuous!. So we must consider a weaker form of differential relationship

Measure Differential Inclusions

We must now assign a meaning to

$$M\frac{dv}{dt} - f_c(q,v) - k(t,q,v) \in FC(q).$$

Definition If ν is a measure and $K(\cdot)$ is a convex-set valued mapping, we say that v satisfies the differential inclusions

$$\frac{dv}{dt} \in K(t)$$

if, for all continuous $\phi \geq 0$ with compact support, not identically 0, we have that

$$\frac{\int \phi(t)\nu(dt)}{\int \phi(t)dt} \in \bigcup_{\tau:\phi(\tau)\neq 0} K(\tau).$$

Weaker formulation for NRMD

Find $q(\cdot), v(\cdot)$ such that

1. $v(0)$ is a function of bounded variation (but may be discontinuous).
2. $q(\cdot)$ is a continuous, locally Lipschits function that satisfies

$$q(t) = q(0) + \int_0^t v(\tau) d\tau$$

3. The measure $dv(t)$, which exists due to v being a bounded variation function, must satisfy, (where $f_c(q, v)$ is the Coriolis and Centripetal Force)

$$\frac{d(Mv)}{dt} - k(t, v) - f_c(q, v) \in FC(q(t))$$

4. $\Phi^{(j)}(q) \geq 0, \forall j = 1, 2, \ldots, p.$

Linearization method

For time-stepping scheme, the geometrical constraints are enforced by linearization.

$$\nabla\Phi(q^{(l)})^T v^{(l+1)} \geq 0 \Longrightarrow \Phi^{(j)}(q^{(l)}) + \gamma h_l \nabla\Phi(q^{(l)})^T v^{(l+1)} \geq 0.$$

$$\nabla\Theta(q^{(l)})^T v^{(l+1)} = 0 \Longrightarrow \Theta^{(j)}(q^{(l)}) + \gamma h_l \nabla\Theta(q^{(l)})^T v^{(l+1)} = 0.$$

Here $\gamma \in (0,1]$. $\gamma = 1$ corresponds to exact linearization.

Time-stepping scheme

Euler method, half-explicit in velocities, linearization for constraints.
Maximum dissipation principle enforced through optimality conditions.

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j\in\mathcal{A}} \left(n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)}\right) = hk$$

$$\nu^{(i)^T} v^{l+1} = -\gamma \frac{\Theta^{(i)}}{h}, \qquad i = 1, 2, \ldots, m$$

$$\rho^{(j)} = n^{(j)^T} v^{l+1} \geq -\gamma \frac{\Phi^{(j)}(q)}{h}, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\sigma^{(j)} = \lambda^{(j)} e^{(j)} + D^{(j)T} v^{l+1} \geq 0, \quad \textbf{compl. to} \quad \beta^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\zeta^{(j)} = \mu^{(j)} c_n^{(j)} - e^{(j)^T} \beta^{(j)} \geq 0, \quad \textbf{compl. to} \quad \lambda^{(j)} \geq 0, \quad j \in \mathcal{A}.$$

Here $\nu^{(\mathbf{i})} = \nabla \boldsymbol{\Theta}^{(\mathbf{i})}$, $n^{(j)} = \nabla \Phi^{(j)}$. h is the time step. The set $\mathcal{A}$ consists of the active constraints. Stewart and Trinkle, 1996, MA and Potra,1997:
Scheme has a solution although the classical formulation doesn't!

Matrix Form of the Integration Step

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & 0 \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} -Mv^{(l)} - hk \\ \Upsilon \\ \Delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

Regularity Conditions: Friction cone assumptions

Define ϵ cone

$$^{\epsilon}\widehat{FC}(q) = \sum_{\Phi^{(j)}(q) \leq \epsilon} FC^{(j)}(q).$$

Pointed friction cone assumption: $\exists\ K_{\epsilon}$, K_{ϵ}^{*}, and $t(q,\epsilon) \in^{\epsilon} \widehat{FC}(q)$ and $v(q,\epsilon) \in^{\epsilon} \widehat{FC}^{*}(q)$, such that, $\forall q \in R^{n}$, and $\forall \epsilon \in [0, \bar{\epsilon}]$, we have that

- $t(q,\epsilon)^{T} w \geq K_{\epsilon}\, ||t(q,\epsilon)||\, ||w||, \forall w \in^{\epsilon} \widehat{FC}(q)$.
- $n^{(j)^{T}} v(q,\epsilon) \geq \mu \sqrt{t_1^{(j)^{T}} v(q,\epsilon) + t_2^{(j)^{T}} v(q,\epsilon)} + K_{\epsilon}^{*}\, ||v(q,\epsilon)||$, $j = 1, 2, \ldots, p$.

Convergence result

(Stewart) Assume

H1 The functions $n^{(j)}(q), t_1^{(j)}(q), t_2^{(j)}(q)$ are smooth and globally Lipschitz, and they are bounded in the 2-norm.

H2 The mass matrix M is positive definite.

H3 The external force increases at most linearly with the velocity and position.

H4 The uniform pointed friction cone assumption holds.

Then there exists a subsequence $h_k \to 0$ where

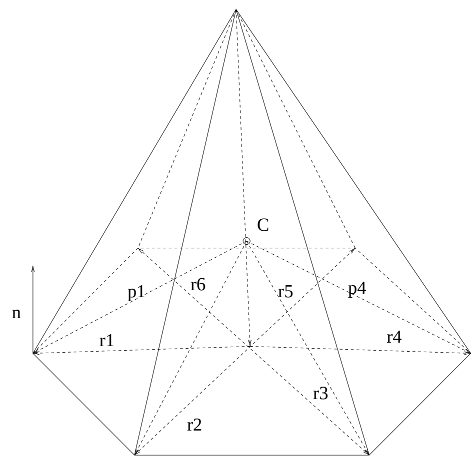
- $q^{h_k}(\cdot) \to q(\cdot)$ uniformly.
- $v^{h_k}(\cdot) \to v(\cdot)$ pointwise a.e.
- $dv^{h_k}(\cdot) \to dv(\cdot)$ weak * as Borel measures. in [0,T], and every such subsequence converges to a solution $(q(\cdot), v(\cdot))$ of MDI.

Solving the LCP

Is it possible to obtain an algorithm that has modest conceptual complexity?

- Lemke's method after reduction to proper LCP works, but for larger scale problems alternatives to it are desirable. Works well for tens of bodies, most of the time.
- Interior Point methods work for the frictionless problem (since matrices are PSD), but their applicability to the problem with friction depends on the convexity of the solution set.
- Is the solution set of the complementarity problem convex?

Nonconvex solution set



Force Balance:

$$\sum_{j=1}^{6} c_n^{(j)} n^{(j)} \quad - \quad hmg \begin{pmatrix} n \\ \mathbf{0}_3 \end{pmatrix} = 0.$$

$$\mu c_n^{(j)} \geq 0 \quad \perp \quad \lambda^{(j)} \geq 0, \quad j = 1, 2, \ldots, 6.$$

Nonconvex solution set

The following solutions

1. $c_n^{(1)} = c_n^{(3)} = c_n^{(5)} = \frac{hmg}{3}$, $c_n^{(2)} = c_n^{(4)} = c_n^{(6)} = 0$,
   $\lambda^{(1)} = \lambda^{(3)} = \lambda^{(5)} = 0$, $\lambda^{(2)} = \lambda^{(4)} = \lambda^{(6)} = 1$,
2. $c_n^{(1)} = c_n^{(3)} = c_n^{(5)} = 0$, $c_n^{(2)} = c_n^{(4)} = c_n^{(6)} = \frac{hmg}{3}$,
   $\lambda^{(1)} = \lambda^{(3)} = \lambda^{(5)} = 1$, $\lambda^{(2)} = \lambda^{(4)} = \lambda^{(6)} = 0$.

The average of these solutions satisfies $c_n^{(j)} = \frac{hmg}{6}$, $\lambda^{(j)} = \frac{1}{2}$, for $j = 1, 2, \ldots, 6$, which violate

$$\mu c_n^{(j)} \geq 0 \ \perp \ \lambda^{(j)} \geq 0, \quad j = 1, 2, \ldots, 6,$$

The average of these solutions, that both induce $v = 0$, violates,

$$\beta_1^{(2)} \geq 0 \ \perp \ \lambda^{(2)} \geq 0.$$

For any $\mu > 0$ the LCP matrix is no $P*$ matrix, polynomiality unlikely.

The convex relaxation

Define $\Theta^{(l)} = -Mv^{(l)} - hk^{(l)}$. We solve the following LCP

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & -\tilde{\mu} \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} \Theta^{(l)} \\ \Upsilon \\ \Delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

The LCP is actually equivalent to a strongly convex QP.

The new convergence result with convex subproblems

H1 The functions $n^{(j)}(q), t_1^{(j)}(q), t_2^{(j)}(q)$ are smooth and globally Lipschitz, and they are bounded in the 2-norm.

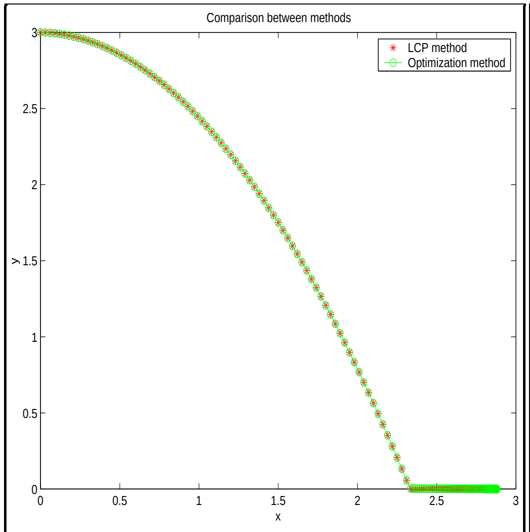
H2 The mass matrix M is positive definite.

H3 The external force increases at most linearly with the velocity and position.

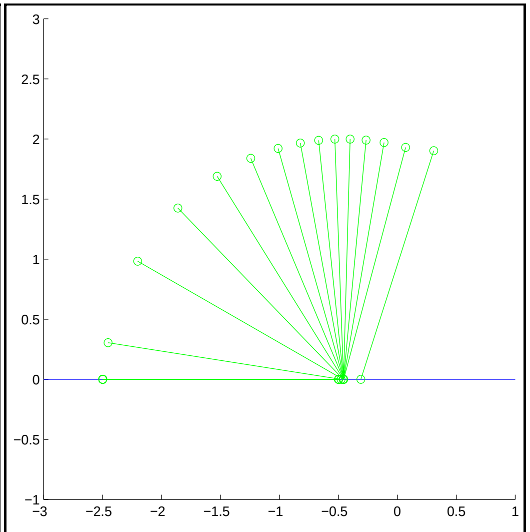
H4 The uniform pointed friction cone assumption holds.

Then there exists a subsequence $h_k \to 0$ where

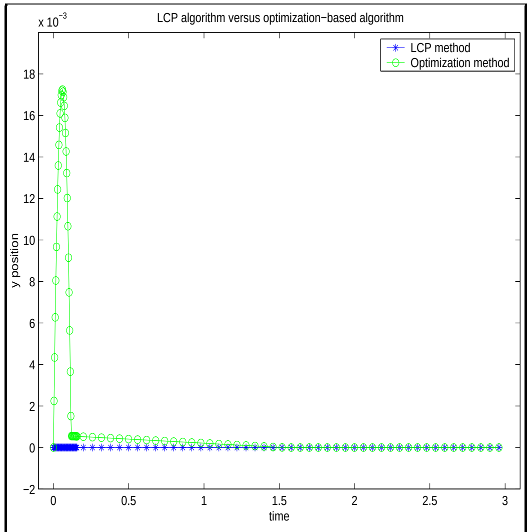
- $q^{h_k}(\cdot) \to q(\cdot)$ uniformly.
- $v^{h_k}(\cdot) \to v(\cdot)$ pointwise a.e.
- $dv^{h_k}(\cdot) \to dv(\cdot)$ weak * as Borel measures. in [0,T], and every such subsequence converges to a solution $(q(\cdot), v(\cdot))$ of MDI. Here q^{h_k} and v^{h_k} is produced by the relaxed algorithm.



Dropped particle



Painleve example



Sliding particle

$h_k = \frac{0.1}{2^k}, \mu = 0.3$

k	h_k $\left\Vert y_{QP} - y_{LCP} \right\Vert_2$
0	5.6314784e-002
1	1.7416198e-002
2	6.7389905e-003
3	2.1011170e-003
4	7.6112319e-004
5	2.6647317e-004
6	9.2498029e-005
7	3.2649217e-005

$h_k = \frac{0.1}{2^k}, \mu = 0.75$

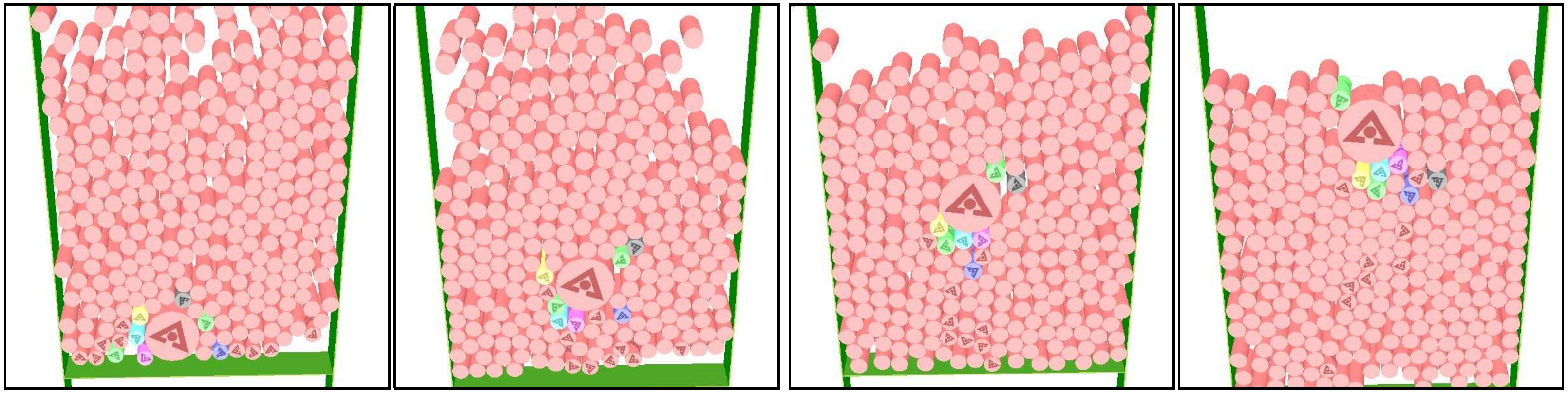
k	h_k $\left\Vert y_{QP} - y_{LCP} \right\Vert_2$
0	1.5736018e+000
1	7.2176724e-001
2	1.4580267e-001
3	9.2969637e-002
4	5.5543025e-003
5	4.3982975e-003
6	3.7537593e-003
7	3.7007014e-004

No convergence, but small absolute error.

Granular matter

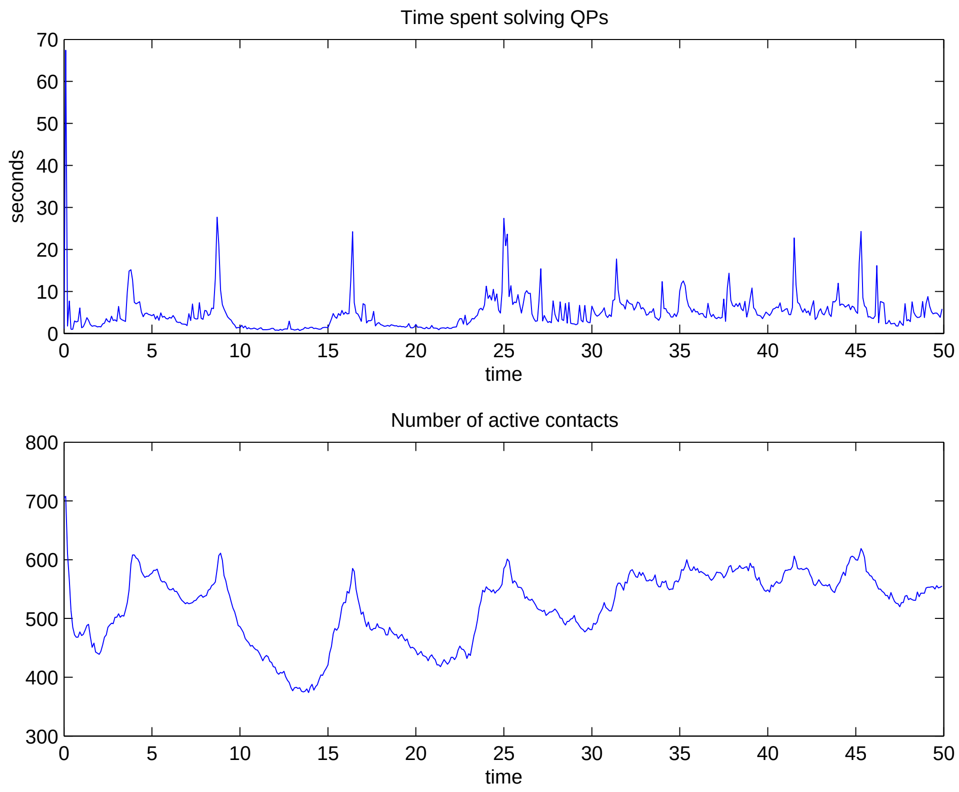
- Sand, Powders, Rocks, Pills are examples of granular matter.
- The range of phenomena exhibited by granular matter is tremendous. Size-based segregation, jamming in grain hoppers, but also flow-like behavior.
- There is still no accepted continuum model of granular matter.
- Direct simulation methods (discrete element method) are still the most general analysis tool, but they are also computationally costly.
- The favored approach: the penalty method which works with time-steps of microseconds for moderate size configurations.

Brazil nut effect simulation



- Time step of 100ms, for 50s. 270 bodies.
- Convex Relaxation Method. One QP/step. No collision backtrack.
- Friction is 0.5, restitution coefficient is 0.5.
- Large ball emerges after about 40 shakes. Results in the same order of magnitude as MD simulations (but with 4 orders of magnitude larger time step).

Brazil nut effect simulations performance



Conclusions and remarks

- We have shown that we find solutions to measure differential inclusions by solving quadratic programs, as opposed to LCP with possible nonconvex solution set.
- PATH is very robust for the original formulation when problem and friction is small but fails for larger problems. However, PATH is successfull in solving the QP.
- This is a major progress for solving very large scale problems, since it opens the possibility of applying a variety of algorithms, including iterative algorithms.