Hard Constraint Methods for Multi Rigid Body Dynamics with Contact and Friction

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Nonsmooth multi-rigid-body dynamics

Nonsmooth rigid multibody dynamics (NRMD) methods attempt to predict the position and velocity evolution of a group of rigid particles subject to certain constraints and forces.

- non-interpenetration, contact.
- collision (mentioned, but not emphasized).
- joint constraints
- adhesion
- Dry friction – Coulomb model.
- global forces: electrostatic, gravitational.

■ These we cover in our approach.

Applications that use NRMD

- Civil and Environmental Engineering Rock dynamics, Masonry stability analysis. Concrete response to earthquake and explosion, Avalanches.
- Materials Processing Tumbling mill design (mineral ore). Drug manufacturing design. Granular materials (TRISO nuclear fuel manufacturing).
- Physically-Based Simulation Gaming. Interactive virtual reality. Robot simulation and design.

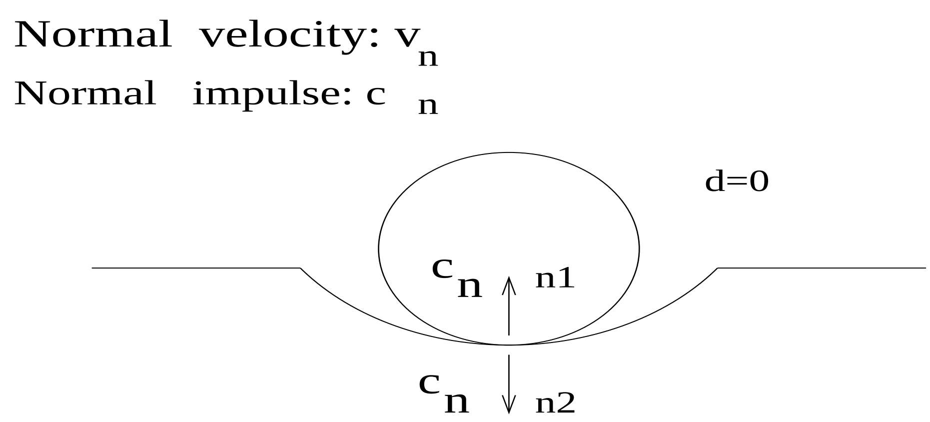
Complementarity

- Definition ($\perp$),

$$a \perp b \Leftrightarrow a, b \geq 0, ab = 0$$

- Two vectors are complementary if they are complementary componentwise.
- The linear complementarity problem (LCP).
$s = \mathcal{M}x + q,\ s \geq 0,\ x \geq 0,\ s^T x = 0.$
- Most familiar example: optimality conditions for quadratic programming, M.

$$\min_{x \geq 0} \ \frac{1}{2} x^T M x + q^T x$$



Contact Model

- Contact configuration described by the (generalized) distance function $d = \Phi(q)$, which is defined for some values of the interpenetration. Feasible set: $\Phi(q) \geq 0$.
- Contact forces are compressive, $c_n \geq 0$.
- Contact forces act only when the contact constraint is exactly satisfied, or

$\Phi(q)$ is complementary to c_n or $\Phi(q)c_n = 0$, or $\Phi(q) \perp c_n$.

Coulomb Friction Model

- Tangent space generators: $\widehat{D}(q) = \left[\widehat{d}_1(q), \widehat{d}_2(q)\right]$, tangent force multipliers: $\beta \in R^2$, tangent force $D(q)\beta$.
- Conic constraints: $||\beta|| \leq \mu c_n$, where μ is the friction coefficient.
- Max Dissipation Constraints: $\beta = \text{argmin}_{||\widehat{\beta}|| \leq \mu c_n} v^T \widehat{D}(q)\widehat{\beta}$.

Polyhedral approximation:

$$\left\{\widehat{D}(q)\beta \mid \ ||\beta|| \leq \mu c_n\right\} \approx \left\{D(q)\tilde{\beta} \mid \tilde{\beta} \geq 0, \ \left|\left|\tilde{\beta}\right|\right|_1 \leq \mu c_n\right\},$$

where $D(q) = [d_1(q), d_2(q), \ldots, d_m(q)]$.

Strong Form

$$M(q)\frac{d^2 q}{dt^2} - \sum_{i=1}^{m} \nu^{(i)} c_{\nu}^{(i)} - \sum_{j=1}^{p} \left(n^{(j)}(q) c_n^{(j)} \quad + D^{(j)}(q)\beta^{(j)} \right) = k(t, q, \frac{dq}{dt})$$

$$\Theta^{(i)}(q) = 0, \qquad\qquad i = 1 \ldots m$$

$$\Phi^{(j)}(q) \geq 0, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j = 1 \ldots p$$

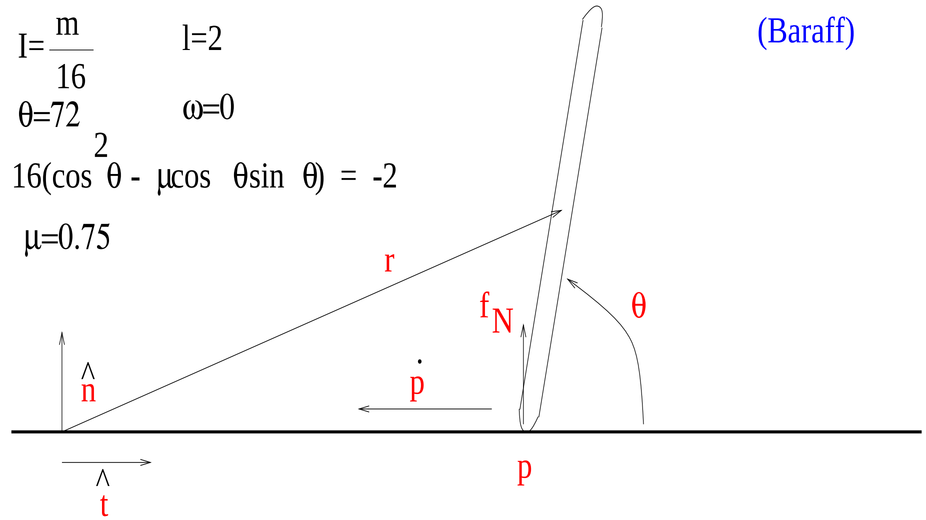
$$\beta = \mathrm{argmin}_{\widehat{\beta}^{(j)}} v^T D(q)^{(j)} \widehat{\beta}^{(j)} \quad \textbf{subject to} \quad \left|\left|\widehat{\beta}^{(j)}\right|\right|_1 \leq \mu^{(j)} c_n^{(j)}, \ j = 1 \ldots p$$

$M(q)$: the PD mass matrix, $k(t, q, v)$: external force, $\Theta^{(i)}(q)$: joint constraints.

It is known that these problems do not have a classical solution even in 2 dimensions, where the discretized cone coincides with the total cone: **Painleve's paradox – no strong solutions**

■: unknowns

A Painleve paradox example



$$p = r - \frac{l}{2}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}$$

Constraint: $\hat{n}p \geq 0$ (defined everywhere).

$$\hat{n}\ddot{p} = -g + f_N\left(\frac{1}{m} + \frac{l}{2I}(\cos^2(\theta) - \mu\sin(\theta)\cos(\theta))\right)$$

$$\hat{n}\ddot{p}_a = -g - \frac{f_N}{m}$$

Painleve Paradox: No classical solutions!

Time-stepping scheme

Euler method, half-explicit in velocities, linearization for constraints. Maximum dissipation principle enforced through optimality conditions.

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j\in\mathcal{A}} (n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)}) = hk$$

$$\nu^{(i)^T} v^{l+1} = -\gamma \frac{\Theta^{(i)}}{h}, \qquad i = 1, 2, \ldots, m$$

$$\rho^{(j)} = n^{(j)^T} v^{l+1} \geq -\gamma \frac{\Phi^{(j)}(q)}{h}, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\sigma^{(j)} = \lambda^{(j)} e^{(j)} + D^{(j)T} v^{l+1} \geq 0, \quad \textbf{compl. to} \quad \beta^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\zeta^{(j)} = \mu^{(j)} c_n^{(j)} - e^{(j)^T} \beta^{(j)} \geq 0, \quad \textbf{compl. to} \quad \lambda^{(j)} \geq 0, \quad j \in \mathcal{A}.$$

Result: The LCP is solvable, the geometrical constraint infeasibility is bounded above by $O(h^2)$ and stabilized, (as opposed to $O(h)$), and the numerical velocities sequence is uniformly bounded.

Significance and comparison with other methods

- The most popular competitors are "spring and dashpot" regularization approaches, a.k.a compliance approaches. One integrates explicitly the regularization with time step in region of stability.
- Compliance approaches are easier to implement, but they can be slow, and the regularization parameter tuning may be very difficult.
- That (and the finite termination) explains the popularity of the LCP approach in gaming applications, where the variety of users does not mesh well with the extra parameters in regularization approaches.
- There is one industrial implementation (KARMA, the Epic Games Unreal Engine subcomponent) and one open source (q12.org), both seemingly with large number of users.

Defining the active set

- Moreau: No backtracking and

$$\mathcal{A} = \left\{ j \in \{1, 2, \ldots, p\} \,|\Phi^{(j)}(q) \leq 0 \right\}$$

- Original LCP method: Same $\mathcal{A}$ with backtracking.
- The stabilized method: No backtracking and

$$\mathcal{A} = \left\{ j \in \{1, 2, \ldots, p\} \,|\Phi^{(j)}(q) \leq \epsilon \right\}$$

- Key: Because velocity is uniformly bounded, the effective active set in the second case, is asymptotically the same with the one before, due to

$$h \nabla \Phi^T v + \Phi \geq 0.$$

- Our method can progress with fixed time step, which is highly desirable for dense groups of bodies.

Solving the LCP, h=0.05, PATH (Lemke)

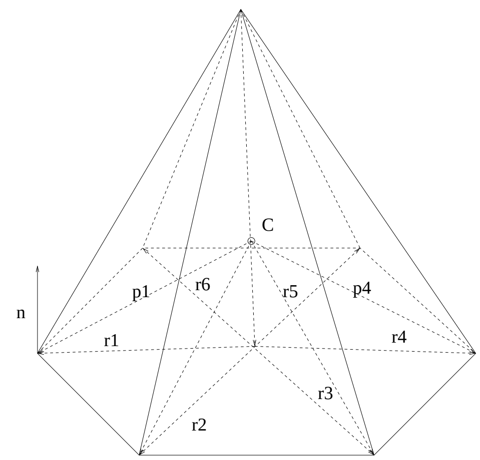
Problem	Bodies	Initial Contacts	μ	Average CPU time (s)
1	10	21	0.2	0.04
2	10	21	0.8	0.03
3	21	52	0.2	0.28
4	21	52	0.8	0.20
5	36	93	0.2	0.81
6	36	93	0.8	0.82
7	55	146	0.2	2.10
8	55	146	0.8	2.07
9	210	574	0.0	0.80
10	210	574	0.2	174.29
11	210	574	0.8	MAXIT

Solving the LCP

Question Is it possible to obtain an algorithm that has modest conceptual complexity and approach large scale problems?

- Lemke's method after reduction to proper LCP works, but for larger scale problems alternatives to it are desirable. PATH Works well for tens of bodies, most of the time, and very well for up to 20 bodies–OK for gaming.
- Interior Point methods work for the frictionless problem (since matrices are PSD), but their applicability to the problem with friction depends on the convexity of the solution set.
- Is the solution set of the complementarity problem convex? From practical experience, this is the key property that separates "hard" problems from "easy" problems.

Nonconvex solution set



Force Balance:

$$\sum_{j=1}^{6} c_n^{(j)} n^{(j)} \quad - \quad hmg \begin{pmatrix} n \\ \mathbf{0}_3 \end{pmatrix} = 0.$$

$$\mu c_n^{(j)} \geq 0 \quad \perp \quad \lambda^{(j)} \geq 0, \quad j = 1, 2, \ldots, 6.$$

Nonconvex solution set

The following solutions

1. $c_n^{(1)} = c_n^{(3)} = c_n^{(5)} = \frac{hmg}{3}$, $c_n^{(2)} = c_n^{(4)} = c_n^{(6)} = 0$,
   $\lambda^{(1)} = \lambda^{(3)} = \lambda^{(5)} = 0$, $\lambda^{(2)} = \lambda^{(4)} = \lambda^{(6)} = 1$,
2. $c_n^{(1)} = c_n^{(3)} = c_n^{(5)} = 0$, $c_n^{(2)} = c_n^{(4)} = c_n^{(6)} = \frac{hmg}{3}$,
   $\lambda^{(1)} = \lambda^{(3)} = \lambda^{(5)} = 1$, $\lambda^{(2)} = \lambda^{(4)} = \lambda^{(6)} = 0$.

The average of these solutions satisfies $c_n^{(j)} = \frac{hmg}{6}$, $\lambda^{(j)} = \frac{1}{2}$, for $j = 1, 2, \ldots, 6$, which violate

$$\mu c_n^{(j)} \geq 0 \;\perp\; \lambda^{(j)} \geq 0, \quad j = 1, 2, \ldots, 6,$$

The average of these solutions, that both induce $v = 0$, violates,

$$\beta_1^{(2)} \geq 0 \;\perp\; \lambda^{(2)} \geq 0.$$

For any $\mu > 0$ the LCP matrix is no $P*$ matrix, polynomiality unlikely.

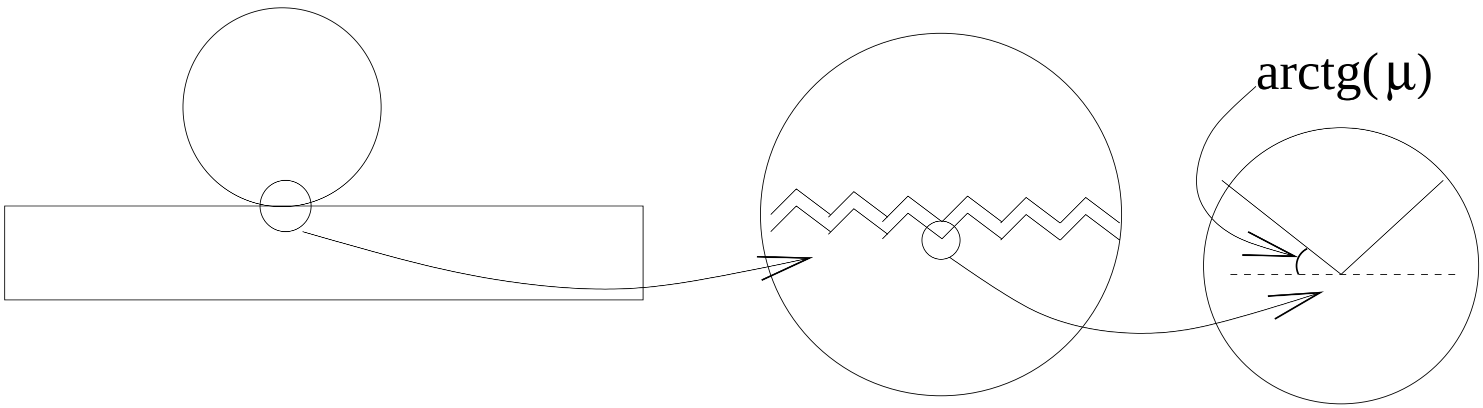
The convex relaxation

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & -\tilde{\mu} \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} \theta^{(l)} \\ \Upsilon \\ \Delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0,$$

The LCP is actually equivalent to a QP–but is the method any good? A fixed point iteration approach based on the above was proved to converge for small μ and pointed friction cone (MA and G.D.Hart, (2004b)).

Microscopic interpretation



It is “almost” as if we integrate with the exact reaction given by frictionless asperities, mitigated by the proximity modification $\frac{\Phi}{h}$

Defining the friction cone (no joints)

The total friction cone:

$$
\begin{aligned}
FC(q) \quad = \quad & \Big\{ \textstyle\sum_{j=1,2,\ldots,p} c_n^{(j)} n^{(j)} + \beta_1^{(j)} t_1^{(j)} + \beta_2^{(j)} t_2^{(j)} \Big| \\
& \sqrt{\left(\beta_1^{(j)}\right)^2 + \left(\beta_2^{(j)}\right)^2} \le \mu^{(j)} c_n^{(j)}, \\
& c_n^{(j)} \ge 0 \perp \Phi^{(j)}(q) = 0, j = 1, 2, \ldots, p \Big\} .
\end{aligned}
$$

We have

$$
FC(q) = \sum_{j=1,2,\ldots,p,\ \Phi^{(j)}(q)=0} FC^{(j)}(q).
$$

Pointed friction cone: if $0 \in FC(q)$ can be realized only by $\tilde{c}_n = \tilde{\beta}_1 = \tilde{\beta}_2 = 0$.

Continuous formulation in terms of friction cone

$$
\begin{aligned}
M\frac{dv}{dt} &= f_C(q,v) + k(q,v) + \rho \\
\frac{dq}{dt} &= v. \\
\rho &= \sum_{j=1}^{p} \rho^{(j)}(t). \\
\rho^{(j)}(t) &\in FC^{(j)}(q(t)) \\
\Phi^{(j)}(q) &\geq 0, \\
\left|\left|\rho^{(j)}\right|\right| \Phi^{(j)}(q) &= 0, \quad j = 1, 2, \ldots, p.
\end{aligned}
$$

However, we cannot expect even that the velocity is continuous!. So we must consider a weaker form of differential relationship

Measure Differential Inclusions

We must now assign a meaning to

$$M\frac{dv}{dt} - f_c(q,v) - k(t,q,v) \in FC(q).$$

Definition If ν is a measure and $K(\cdot)$ is a convex-set valued mapping, we say that v satisfies the differential inclusions

$$\frac{dv}{dt} \in K(t)$$

if, for all continuous $\phi \geq 0$ with compact support, not identically 0, we have that

$$\frac{\int \phi(t)\nu(dt)}{\int \phi(t)dt} \in \bigcup_{\tau:\phi(\tau)\neq 0} K(\tau).$$

Weaker formulation for NRMD

Find $q(\cdot), v(\cdot)$ such that

1. $v(0)$ is a function of bounded variation (but may be discontinuous).

2. $q(\cdot)$ is a continuous, locally Lipschitz function that satisfies

$$q(t) = q(0) + \int_0^t v(\tau)d\tau$$

3. The measure $dv(t)$, which exists due to v being a bounded variation function, must satisfy, (where $f_c(q, v)$ is the Coriolis and Centripetal Force)

$$\frac{d(Mv)}{dt} - k(t, v) - f_c(q, v) \in FC(q(t))$$

4. $\Phi^{(j)}(q) \geq 0, \forall j = 1, 2, \ldots, p.$

Regularity Conditions: Friction cone assumptions

Define ϵ cone

$$^{\epsilon}\widehat{FC}(q) = \sum_{\Phi^{(j)}(q) \leq \epsilon} FC^{(j)}(q).$$

Pointed friction cone assumption: $\exists\ K_\epsilon$, K_ϵ^*, and $t(q,\epsilon) \in^{\epsilon} \widehat{FC}(q)$ and $v(q,\epsilon) \in^{\epsilon} \widehat{FC}^{*}(q)$, such that, $\forall q \in R^n$, and $\forall \epsilon \in [0, \bar{\epsilon}]$, we have that

- $t(q,\epsilon)^T w \geq K_\epsilon \, ||t(q,\epsilon)|| \, ||w||$, $\forall w \in^{\epsilon} \widehat{FC}(q)$.
- $n^{(j)^T} v(q,\epsilon) \geq \mu \sqrt{t_1^{(j)^T} v(q,\epsilon) + t_2^{(j)^T} v(q,\epsilon)} + K_\epsilon^* \, ||v(q,\epsilon)||$, $j = 1, 2, \ldots, p$.

The new convergence result with convex subproblems

H1 The functions $n^{(j)}(q), t_1^{(j)}(q), t_2^{(j)}(q)$ are smooth and globally Lipschitz, and they are bounded in the 2-norm.

H2 The mass matrix M is positive definite.

H3 The external force increases at most linearly with the velocity and position.

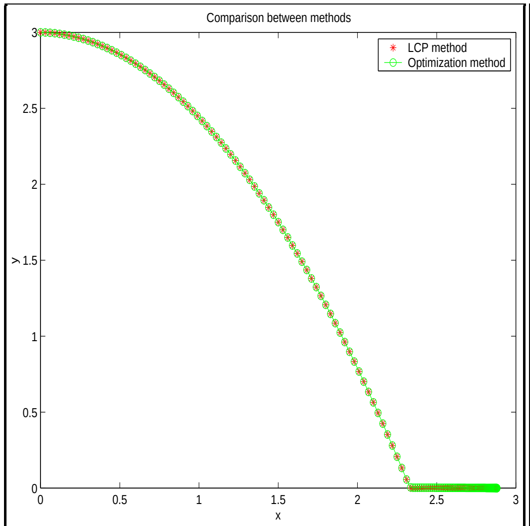
H4 The uniform pointed friction cone assumption holds.

Then there exists a subsequence $h_k \to 0$ where

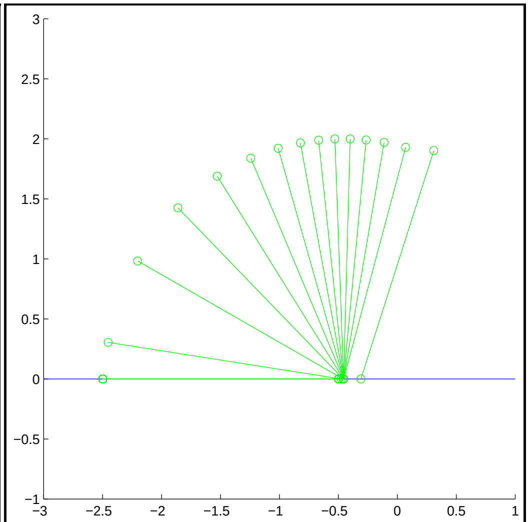
- $q^{h_k}(\cdot) \to q(\cdot)$ uniformly.
- $v^{h_k}(\cdot) \to v(\cdot)$ pointwise a.e.
- $dv^{h_k}(\cdot) \to dv(\cdot)$ weak * as Borel measures. in [0,T], and every such subsequence converges to a solution $(q(\cdot), v(\cdot))$ of MDI. Here q^{h_k} and v^{h_k} is produced by the relaxed algorithm.

The convergence result

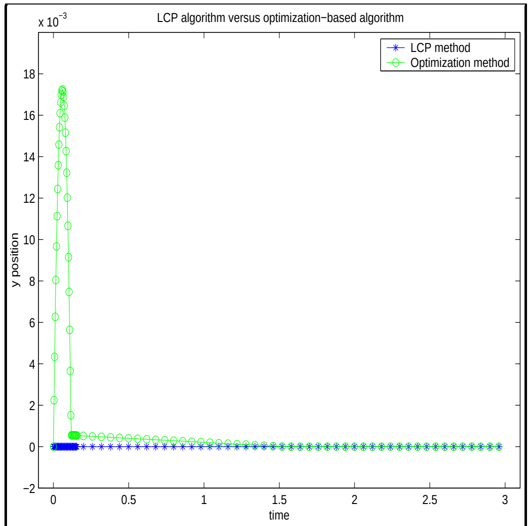
- Mimics the similar result for the original scheme (Stewart,(1998)), including decrease of energy ...
- ... but says nothing of the Coulomb Law.
- In a regime with small tangential velocity it can be show that the difference of the two schemes is small.
- In some sense, it is the natural integration procedure based on the microscopic modeling of friction with a large time step.



Dropped particle



Painleve example



Sliding particle

$\boxed{h_k = \frac{0.1}{2^k}, \mu = 0.3}$

$\boxed{h_k = \frac{0.1}{2^k}, \mu = 0.75}$

k	$h_k \left\| y_{QP} - y_{LCP} \right\|_2$
0	5.6314784e-002
1	1.7416198e-002
2	6.7389905e-003
3	2.1011170e-003
4	7.6112319e-004
5	2.6647317e-004
6	9.2498029e-005
7	3.2649217e-005

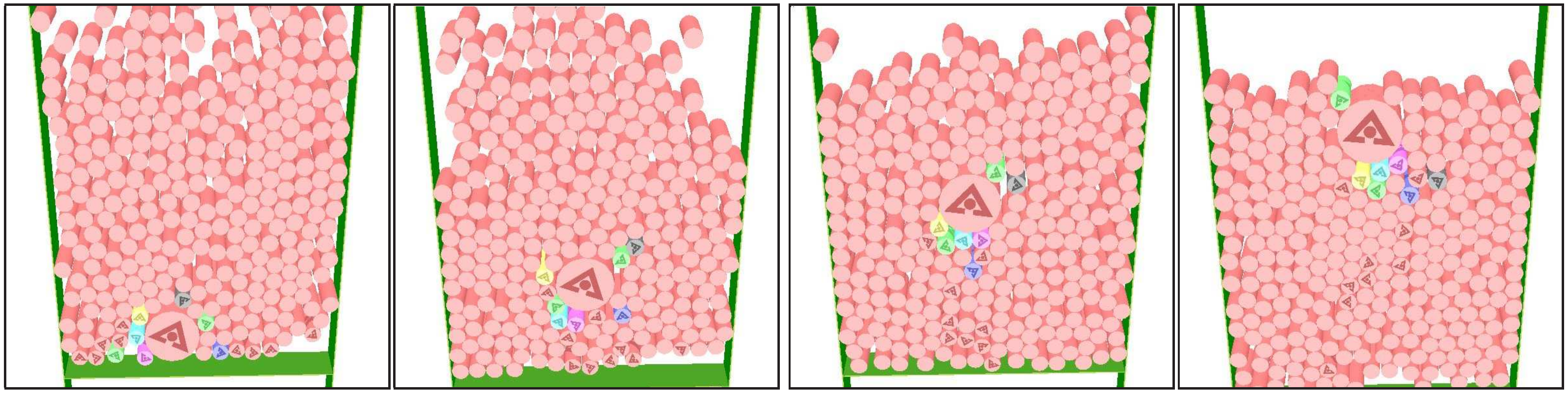
k	$h_k \left\| y_{QP} - y_{LCP} \right\|_2$
0	1.5736018e+000
1	7.2176724e-001
2	1.4580267e-001
3	9.2969637e-002
4	5.5543025e-003
5	4.3982975e-003
6	3.7537593e-003
7	3.7007014e-004

No convergence, but small absolute error.

Granular matter

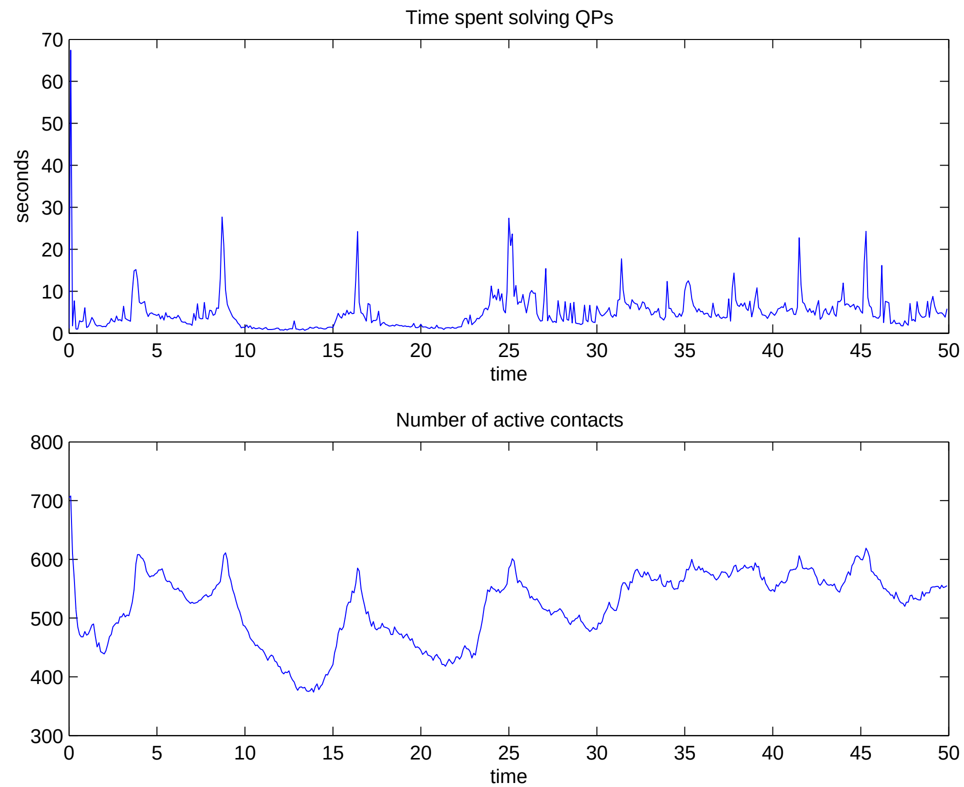
- Sand, Powders, Rocks, Pills are examples of granular matter.
- The range of phenomena exhibited by granular matter is tremendous. Size-based segregation, jamming in grain hoppers, but also flow-like behavior.
- There is still no accepted continuum model of granular matter.
- Direct simulation methods (discrete element method) are still the most general analysis tool, but they are also computationally costly.
- The favored approach: the penalty method which works with time-steps of microseconds for moderate size configurations.

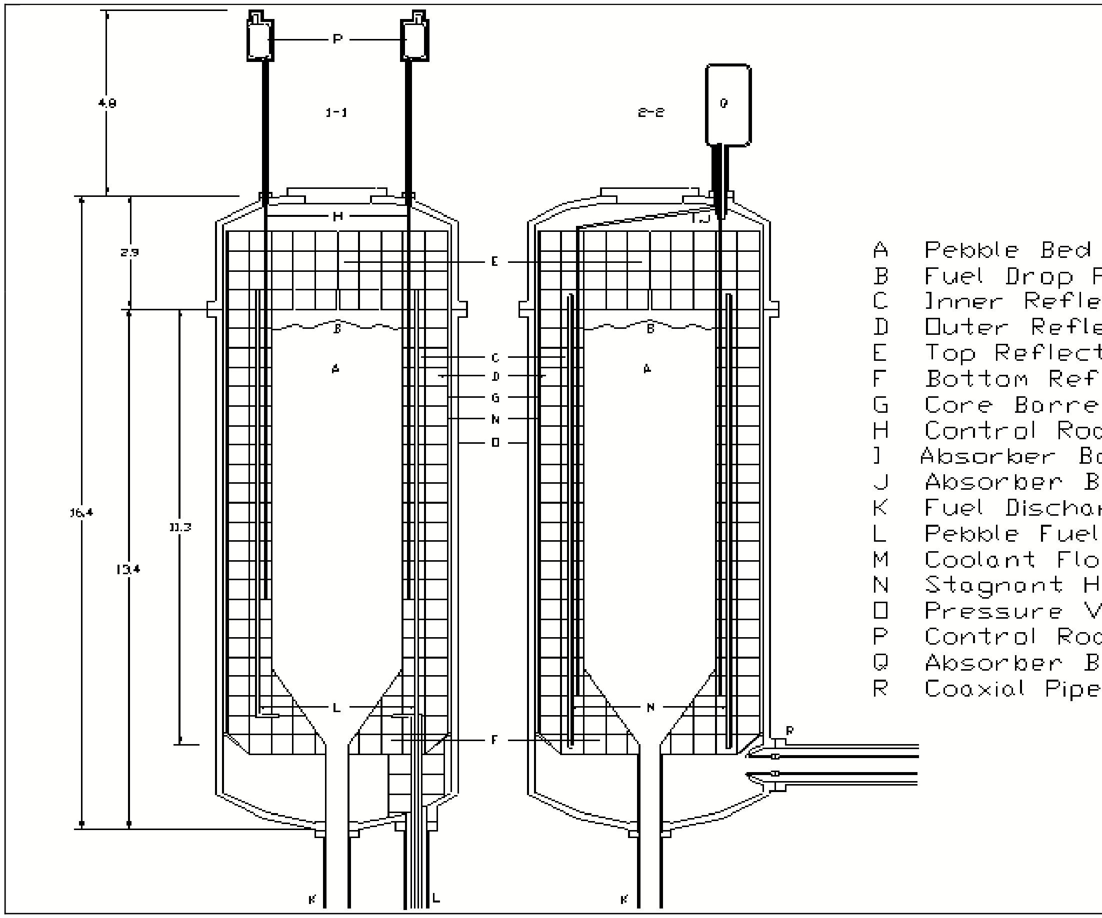
Brazil nut effect simulation

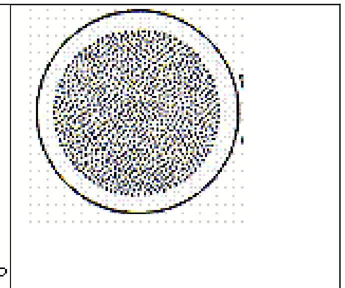


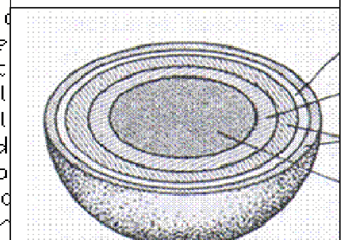
- Time step of 100ms, for 50s. 270 bodies.
- Convex Relaxation Method. One QP/step. No collision backtrack.
- Friction is 0.5, restitution coefficient is 0.5.
- Large ball emerges after about 40 shakes. Results in the same order of magnitude as MD simulations (but with 4 orders of magnitude larger time step).

Brazil nut effect simulations performance









A fuel microsphere. Triple coated with UO2 center.

There are about 400000 pebbles in the reactor at one time

The pebble bed nuclear reactor

- One of the great hopes of achieving low maintenance passively safe reactors.
- The fuel consists of tennis-ball-size pebbles filled with UO_2,
- The fuel is in continuous motion and the fuel pebbles are either recycled or replaced.
- Cooled with helium through the inter-pebble voids.
- Prototype to be completed by 2015 by INL.
- Initial simulation of loading with Bogdan Gavrea, UMBC.

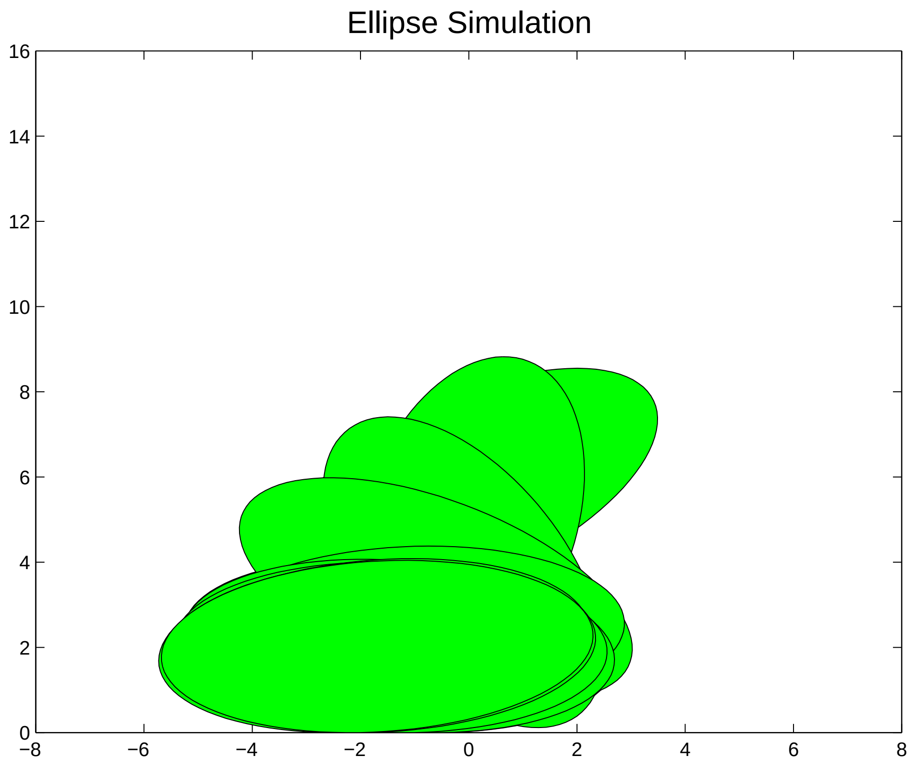
In progress

- Trapezoidal scheme, though fixed time-stepping property is lost.
- Nonsmooth bodies with fixed time step.
- Using projected gradient type approaches to accelerate the solution of the quadratic program.

Conclusions and remarks

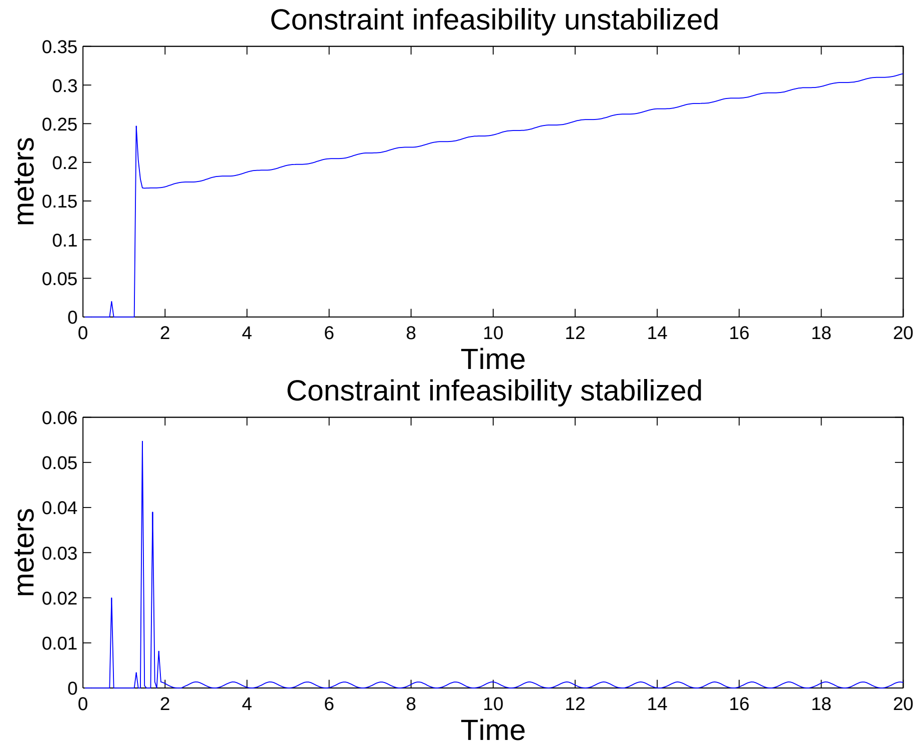
- We have described recent progress in the use of hard constraint time stepping schemes for multi-rigid body dynamics with contact and friction (NRMD).
- We have shown that we find solutions to measure differential inclusions by solving quadratic programs, as opposed to LCP with possible nonconvex solution set.
- There remain quite a few challenges (the most important of which is computational efficiency in solving the subproblem), but the large number of applications that can be impacted are worth the investigation in these areas.

Elliptic body simulation



We present ten frames of the simulation of an elliptic body that is dropped on the table. There is an initial angular velocity of 3, the body has axes 4 and 8 and is dropped from a height of 8.

Infeasibility behavior unstabilized versus stabilized metod



We see that drift becomes catastrophic for the unstabilized method, whereas remains in a narrow range for the stabilized method.

Constraint stabilization is accomplished!

Constraint Stabilization

- Despite the fact that we have the term $\frac{1}{h}$ the scheme is still stable (for h fixed but arbitrary).
- For solvability, we need a stronger condition, pointed friction cone assumption, though weaker than linear independence of constraints.
- Note that in the case of DAE, even the postprocessing method (Ascher, 1998) needs one additional linear system (with same matrix).
- The method was implemented in GraspIt!, a dynamical grasp simulation tool by Andrew Miller at Columbia.
- The scheme can be modified to include partial elasticity and seems to work fine, though we did not prove the same stability results (MA, (2003)).

Related Research

- Time stepping methods of this type originate with the work of Moreau, early 70's, though most (all?) of those developments are NLCPs, not guaranteed to be solvable, expressed in language of projections. The key here: work with optimality conditions (S & T 96).
- Other LCP approaches use accelerations as primary variables (Glocker and Pfeiffer, (1992), Baraff(1993), Pang and Trinkle, (1996)). They need the existence of a strong solution, and an extra derivative of the data, but work well in many applications.
- Piecewise differential algebraic equation approaches (DAE) (Haug et al., 1988),create difficult nonlinear systems and can get stuck at points of inconsistency.
- Differential variational inequalities (DAVINCI).

About convergence of the scheme

- For this class of time stepping methods, Stewart (1998) proved convergence to a Measure Differential Inclusion MDI as $h \to 0$, and satisfaction of the Coulomb Friction law for one contact, or several contacts at points of continuity of the velocity.
- Note that one has to accommodate discontinuous velocity due to Painleve paradoxes and collisions, though the strong form contains $\frac{dv}{dt}$.
- We use a similar technique for proving convergence of our convex relaxation method.

Can the LCP approach be extended for

- Stiff systems ?
- Constraint stabilization?
- Fixed time step ?
- Efficient computation of the subproblems?

while preserving the linearity, the solvability and the stability?

The “numerical analysis” of LCP time-stepping schemes is done by exploiting the the stability of the solution of LCP with respect to perturbations, as an extension to DAE approaches. We describe the results.

Acommodating stiffness

Define

$$
\begin{aligned}
\widehat{M} &= \left[M\left(q^{(n)}\right) - h^2 \nabla_q k\left(q^{(n)}, v^{(n)}\right) - h \nabla_v k\left(q^{(n)}, v^{(n)}\right) \right], \\
\widehat{k} &= k\left(q^{(n)}, v^{(n)}\right) - \nabla_v k\left(q^{(n)}, v^{(n)}\right) v^{(n)}
\end{aligned}
$$

and replace $\widehat{M} \to M$, in the LCP matrix and $\widehat{k} \to k$ in the right hand side (linear implicit approach). Then

- If the external force is linear spring and damper, resulting problem is solvable LCP and the scheme is unconditionally stable. MA & FP, 2002,
- Can extend to nonlinear spring and damper with small modifications.

Constraint stabilization: Linearization method

Projection methods are expensive. Our solution: enforce geometrical constraints by linearization.

$$\nabla\Phi(q^{(l)})^T v^{(l+1)} \geq 0 \Longrightarrow \Phi^{(j)}(q^{(l)}) + \gamma h_l \nabla\Phi(q^{(l)})^T v^{(l+1)} \geq 0.$$

$$\nabla\Theta(q^{(l)})^T v^{(l+1)} = 0 \Longrightarrow \Theta^{(j)}(q^{(l)}) + \gamma h_l \nabla\Theta(q^{(l)})^T v^{(l+1)} = 0.$$

Here $\gamma \in (0, 1]$. $\gamma = 1$ corresponds to exact linearization.

Is the LCP solvable?

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & 0 \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} -Mv^{(l)} - hk \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

Yes, with Lemke, if M is positive definite, MA & FP, 1997. In addition collision with compression-decompression can be modeled by LCP with the same matrix and are also solvable.

Energy Properties (Stability)

Assumptions

- The Mass matrix M is constant.
- The collisions do not increase the kinetic energy.
- The number of collisions is finite.
- The external force is inertial + at most linear growth:
  $k(t, v, q) = f_c(q, v) + k_1(t, v, q)$, where $v^T f_c(q, v) = 0$,
  $||k_1(t, q, v)|| \leq A(1 + ||q|| + ||v||)$.

Then $v^{(l),h}$ is uniformly bounded.

Time-stepping, the linear complementarity problem (LCP)

Euler method, half-explicit in velocities, linearization for constraints.

Maximum dissipation principle enforced through optimality conditions.

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^{m} \nu^{(i)} c_{\nu}^{(i)} - \sum_{j \in \mathcal{A}} \left(n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)} \right) = hk$$

$$\begin{aligned} \nu^{(i)^T} v^{l+1} &= 0, & & & i &= 1, 2, \ldots, m \\ \rho^{(j)} = n^{(j)^T} v^{l+1} &\geq 0, & \textbf{compl. to} & \quad c_n^{(j)} \geq 0, & j &\in \mathcal{A} \\ \sigma^{(j)} = \lambda^{(j)} e^{(j)} + D^{(j)T} v^{l+1} &\geq 0, & \textbf{compl. to} & \quad \beta^{(j)} \geq 0, & j &\in \mathcal{A} \\ \zeta^{(j)} = \mu^{(j)} c_n^{(j)} - e^{(j)^T} \beta^{(j)} &\geq 0, & \textbf{compl. to} & \quad \lambda^{(j)} \geq 0, & j &\in \mathcal{A}. \end{aligned}$$

$\nu^{(\mathbf{i})} = \nabla \mathbf{\Theta}^{(\mathbf{i})}$, $n^{(j)} = \nabla \Phi^{(j)}$, h: time step, $\mathcal{A}$: active constraints.

Stewart and Trinkle, 1996 (LCP) MA and Potra,1997 (solvable LCP).

We use the same notation for impulses that replace forces. ■: unknowns