Convergence Framework for Trapezoidal-Like Semi-Implicit Time-Stepping Schemes for Multi Body Dynamics with Contacts and Joints

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(joint work with Bogdan Gavrea (Penn-U Cluj), Jeff Trinkle (RPI), and Florian Potra (UMBC))
Problem Description

- Rigid bodies. Generalized positions and velocities: \((q, v)\).

- Joint constraints:
  \[
  \Theta^{(i)}(q) = 0, \quad i = 1, 2, \ldots, m.
  \]

- No inter-penetration (Hard-Constraints):
  \[
  \Phi^{(j)}(q) \geq 0, \quad j = 1, 2, \ldots, p.
  \]

- Compressive normal forces at contact.

- Impact resolution (Poisson hypothesis).

- Coulomb friction.
Applications

- Robot simulation and design.

- Granular and rock dynamics.

- Other areas: interactive virtual reality, masonry stability analysis, pedestrian and evacuation dynamics, e.t.c.
Contact and Frictional Constraints

- **Contact Constraints:**
  
  \[ 0 \leq c_n^{(j)} \perp \Phi^{(j)}(q) \geq 0, \quad j = 1, \ldots, p. \]

- **Coulomb Friction. Maximal Dissipation Principle.**
  
  \[
  \min_{\beta^{(j)} \in \mathbb{R}^d} \; v^T \bar{D}^{(j)}(q) \bar{\beta}^{(j)} \quad \text{subject to} \quad \|\beta^{(j)}\|_2 \leq \mu c_n^{(j)}
  \]

- **Optimality Conditions with Polyhedral Approximation.**
  
  \[
  0 \leq D^{(j)}(q)^T v + \lambda^{(j)} e^{(j)} \perp \beta^{(j)} \geq 0, \quad j = 1, 2, \ldots, p
  \]

  \[
  0 \leq \mu^{(j)} c_n^{(j)} - e^{(j)^T} \beta^{(j)} \perp \lambda^{(j)} \geq 0, \quad j = 1, 2, \ldots, p,
  \]

  where \( e^{(j)} = (1, 1, \ldots, 1)^T \in \mathbb{R}^{m_{j_C}} \) and \( m_{j_C} \) is the number of facets used in the approximation.
The Classical Mechanics Framework

\[ M(q) \frac{d^2 q}{dt^2} - \sum_{i=1}^{m} \nu^{(i)} c^{(i)}_\nu - \sum_{j=1}^{p} \left( n^{(j)}(q) c^{(j)}_n + D^{(j)} \beta^{(j)} \right) = k \left( t, q, \frac{dq}{dt} \right) \]

\[ \Theta^{(i)}(q) = 0, \quad i = 1, 2, \ldots, m \]

\[ \Phi^{(j)}(q) \geq 0, \quad \perp c^{(j)}_n \geq 0, \quad j = 1, \ldots, p \]

\[ D^{(j)}(q)^T v + \lambda^{(j)} e^{(j)} \geq 0, \quad \perp \beta^{(j)} \geq 0 \]

\[ \mu^{(j)} c^{(j)}_n - e^{(j)^T} \beta^{(j)} \geq 0, \quad \perp \lambda^{(j)} \geq 0, \quad j = 1, \ldots, p \]

- \( M(q) \): generalized mass matrix (symmetric positive definite)
- \( k(t, q, dq/dt) \): external and inertial forces

- The above DCP doesn’t always have a solution in the classical sense (which cases we call Painleve paradoxes).
- We discretize it as if it does, multiply through with time step (move to velocity-impulse) and investigate weak convergence.
Trapezoidal-like LCP Time-Stepping -(Potra et al.),

- Position update, $\gamma \in [0, 1]$:
  \[ q^{l+1} = q^{(l)} + h \left( (1 - \gamma)v^{l} + \gamma v^{l+1} \right) \]

- Velocity $v^{l+1}$ is found from the solution of the MLCP ($\alpha \in [0, 1]$):
  \[ \tilde{M}v^{l+1} - \sum_{i=1}^{m} \nu^{(i)} c^{(i)}_{\nu} - \sum_{j \in A} (n^{(j)} c^{(j)}_{n} + D^{(j)} \beta^{(j)}) = \tilde{M}v^{l} + \tilde{k} \]
  \[ \nu^{(i)}^T \left( \alpha v^{(l+1)} + (1 - \alpha)v^{(l)} \right) = 0, \ i = 1, 2, \ldots m \]
  \[ 0 \leq \rho^{(j)} := n^{(j)}^T \left( \alpha v^{(l+1)} + (1 - \alpha)v^{(l)} \right) \perp c^{(j)}_{n} \geq 0, \ j \in A \]
  \[ 0 \leq \sigma^{(j)} := \lambda^{(j)} e^{(j)} + D^{(j)}^T \left( \alpha v^{(l+1)} + (1 - \alpha)v^{(l)} \right) \perp \beta^{(j)} \geq 0, \ j \in A \]
  \[ 0 \leq \zeta^{(j)} := \mu^{(j)} c^{(j)}_{n} - e^{(j)}^T \beta^{(j)} \perp \lambda^{(j)} \geq 0, \ j \in A \]

where $\tilde{M} = \left( M - \alpha h\tilde{k}_{v}^{l} - \alpha \gamma h^{2}\tilde{k}_{q}^{l} \right)$ and

$\tilde{k} = h \left( (1 - \alpha)k \left( t_{l}, q^{l}, v^{l} \right) + \alpha k \left( t_{l+1}, q^{l}, v^{l} \right) \right) + \alpha h^{2}\tilde{k}_{q}^{l}v^{l}$
Historical and Computational Considerations for Time-Stepping

- First described by Stewart and Trinkle, 1996, Anitescu and Potra (1997-LCP only), with some similarities with past work of Moreau.


- Is not affected by Painleve paradoxes.

- Does not suffer from the stiffness of the compliant contact or spring and dashpot - or its parameter tuning.

- The solution set of the LCP may be nonconvex (though it is always solvable) but optimization-based relaxation may provide efficient approximations especially for slow granular flow simulations (Alessandro Tasora’s talk this morning), which are also quite accurate if the flow is slow.
Solvability of the Integration Step

Matrix Form of the Integration Step.

\[
\begin{bmatrix}
\tilde{M} & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\
\tilde{\nu}^T & 0 & 0 & 0 & 0 \\
\tilde{n}^T & 0 & 0 & 0 & 0 \\
\tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\
0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0
\end{bmatrix}
\begin{bmatrix}
v^{l+1,\alpha} \\
\tilde{c}_\nu \\
\tilde{c}_n \\
\tilde{\beta} \\
\tilde{\lambda}
\end{bmatrix}
= \begin{bmatrix}
\tilde{M}v^l + \alpha\tilde{k} \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
\tilde{\rho} \\
\tilde{\sigma} \\
\tilde{\zeta}
\end{bmatrix}
\]

\[0 \leq \begin{bmatrix} \tilde{c}_n, \tilde{\beta}, \tilde{\lambda} \end{bmatrix}, \perp \begin{bmatrix} \tilde{\rho}, \tilde{\sigma}, \tilde{\zeta} \end{bmatrix} \geq 0\]

where \(v^{l+1,\alpha} = \alpha v^{l+1} + (1 - \alpha)v^l\).

Solvability.

Theorem (Anitescu, Potra 97, applied for \(\alpha = 0\) but also here) If the matrix \(\tilde{M}\) is positive definite, then the solution set \(\mathcal{L}(q^l, v^l, h, \tilde{k}, \alpha)\) of the above mixed LCP is nonempty. A solution can be found by Lemke’s algorithm.
Simulations-UMBRA

- The trapezoidal method was implemented in UMBRA, an industrial strength package for multibody dynamics simulations.

- Umbra software package built at Sandia National Laboratories by Eric Gottlieb, Fred Oppel, and Patrick Xavier. The software is now distributed and maintained by Orion, Inc, in Albuquerque, NM.

- Simulations at webpage of Jeff Trinkle.
What is the big deal with convergence for joint constraints?

- The original proof by Stewart was predicated heavily on a friction cone being properly pointed, and its extension to the joint case is non-trivial.

- Of course one could reduce the joints by changing parameterization. This is an incredibly costly process at times, since it involves a nonlinear solve at every step. It is the reason we study DAEs as opposed to pure ODEs.

- We prove convergence of time-stepping schemes, as actually implemented.
Strong Formulation in Terms of Friction Cone

\[
\frac{dq}{dt} = \nu \\
M\frac{d\nu}{dt} = k(q, \nu) + \rho \\
\Theta^{(i)}(q) = 0, \quad i = 1, 2, \ldots, m \\
\Phi^{(j)}(q) \geq 0, \quad j = 1, \ldots, p \\
\rho(t) = \bar{\rho}(t) + \sum_{j=1}^{p} \rho^{(j)}(t) \in FC(q) \\
\bar{\rho}(t) \in \text{span}\{\nu^{(i)}(q(t)) : i = 1, \ldots, m\} \\
\|\rho^{(j)}\| \Phi^{(j)}(q) = 0, \quad j = 1, 2, \ldots, p
\]

We cannot expect that the velocity is even continuous. Therefore, we must consider a weak form of the differential inclusion above.
Measure Differential Inclusions

We must assign a meaning to inclusions of the form

\[ M \frac{dv}{dt} - k(q, v) \in FC(q). \]

**Definition (Stewart,1998).** If \( v \) is a function of bounded variation and \( K(\cdot) \) is a convex–set valued mapping we say that \( v \) satisfies the differential inclusion

\[ \frac{dv}{dt} \in K(t), \]

if, for all continuous \( \Phi \geq 0 \) with compact support, not identically 0, we have that

\[ \frac{\int \Phi(t)dv(t)}{\int \Phi(t)dt} \in \bigcup_{\tau: \Phi(\tau) \neq 0} K(\tau). \]
Pointed Friction Cone

- closed cone $K \in IR^n$ is pointed if it does not contain any proper linear subspace (has truly a point at 0).

- Main convergence tool in all MDI convergence proofs (Stewart 98, 00, Anitescu 06) Theorem 4 (Stewart 1998b) Suppose that $q_\hat{n}(\cdot)$ are continuous, $v_\hat{n}(\cdot)$ have uniformly bounded variation and $k_\hat{n}(\cdot)$ are uniformly bounded, all on $[0, T]$, and $q_\hat{n}(\cdot) \to q(\cdot)$ uniformly, $v_\hat{n}(\cdot) \to v(\cdot)$ pointwise a.e. and $k_\hat{n}(\cdot) \to k(\cdot)$ pointwise a.e. Suppose also that $K : IR^n \Rightarrow C(IR^n)$ has closed graph, $\min \{||z|| \mid z \in K(w)\}$ is uniformly bounded and $K(w)$ is pointed for all $w \in IR^n$. Then if

$$\frac{dv_\hat{n}}{dt}(t) \in K(q_\hat{n}(t)) - k_\hat{n}(t)$$

for all $\hat{n}$, the limit satisfies

$$\frac{dv}{dt}(t) \in K(q(t)) - k(t).$$

Here $C(IR^n)$ denotes all the closed and convex subsets of $IR^n$. 

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The Joint Constrained Friction Cone

- Corresponding to each contact define

\[ \mathcal{FC}^{(j)}(q) = \left\{ z = \tilde{\nu} \tilde{c}_\nu + n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)} \left| c_n^{(j)} \geq 0, \| \beta^{(j)} \|_2 \leq \mu^{(j)} c_n^{(j)} \right. \right\} \]

and the total friction cone

\[ \mathcal{FC}(q) = \sum_{\Phi^{(j)}(q) = 0} \mathcal{FC}^{(j)}(q). \]

- In a similar fashion define the polyhedral approximation

\[ \widehat{\mathcal{FC}}(q) = \sum_{\Phi^{(j)}(q) = 0} \widehat{\mathcal{FC}}^{(j)}(q). \]

where \( \widehat{\mathcal{FC}}^{(j)}(q) \)

\[ \widehat{\mathcal{FC}}^{(j)}(q) = \left\{ z = \tilde{\nu} \tilde{c}_\nu + n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)} \left| c_n^{(j)}, \beta^{(j)} \geq 0, \| \beta^{(j)} \|_1 \leq \mu^{(j)} c_n^{(j)} \right. \right\}. \]

- None can be pointed – they contain the range of \( \tilde{\nu} \).
How does a mathematician solve a problem?

- A physicist and a mathematician sit in a faculty lounge. Suddenly, the coffee machine catches on fire.
- The physicist grabs a bucket and leaps towards the sink, fills the bucket with water and puts out the fire.
- Second day, the same two sit in the same lounge. Again, the coffee machine catches on fire.
- This time, the mathematician grabs the bucket and hand it to the physicist, and reduces the problem to a previously solved one.
- This is our approach: We hand the bucket to Theorem 4.
Reduced Friction Cone. Regularity Assumption.

- The Reduced Friction Cone (Gavrea, Ph.D. thesis). where $\tilde{\nu}(q)$ is the orthogonal complement of $\nu(q)$.

$$\mathcal{FC}_r(q) := (\tilde{\nu}(q))^T \mathcal{FC}(q) = \left\{ (\tilde{\nu}(q))^T z : z \in \mathcal{FC}(q) \right\}$$

- We assume that the friction cone is uniformly pointed: $\tilde{\nu}(q)$ has a lowest singular value bounded away from 0, and the angle between any two vectors in $\mathcal{FC}_r(q)$ is bounded away from $\pi$.

- Equivalent to

$$\exists c_{\mathcal{FC}} > 0, \text{ such that } \tilde{c}_n, \tilde{\beta} \geq 0, \text{ and } z = \tilde{\nu}\tilde{c}_\nu + \tilde{n}\tilde{c}_n + \tilde{D}\tilde{\beta} \in \mathcal{FC}(q).$$

$$\Rightarrow \| (\tilde{c}_\nu, \tilde{c}_n, \tilde{\beta}) \| \leq c_{\mathcal{FC}} \| z \|$$

with $c_{\mathcal{FC}}$ uniformly bounded.
Solution of MDI reduced cone problem

We say that \( q(t), w(t) \) is a reduced weak solution on \([0, T]\) if

1. \( w(\cdot) \) is a function of bounded variation on \([0, T]\).
2. \( q(\cdot) \) is an absolutely continuous function that satisfies

\[
q(t) = q(0) + \int_0^t \tilde{\nu}_\perp(q(\tau))w(\tau)\,d\tau, \quad \text{for } t \in [0, T].
\]

3. The measure \( dw(t) \) must satisfy

\[
\begin{align*}
&\left((\tilde{\nu}_\perp(q))^T M \tilde{\nu}_\perp(q)\right) \frac{dw}{dt} - k_{w, \perp}(t, q, w) \in FC_r(q), \\
&k_{w, \perp}(t, q, w) \triangleq (\tilde{\nu}_\perp(q))^T k_w(t, q, w) \\
k_w(t, q, w) \triangleq k(t, q, \tilde{\nu}_\perp(q)w) - M \left((\frac{\partial}{\partial q} (\tilde{\nu}_\perp(q)w)) \tilde{\nu}_\perp(q)w\right)
\end{align*}
\]

4. \( \Phi^{(j)}(q) \geq 0, \; j = 1, \ldots, p. \)

5. Lemma: \( q, w \) solve MDI reduced problem \( \iff \) \( q, w \) solve MDI original problem.
Uniform Boundedness of the Numerical Velocities

Assumptions

- The generalized mass matrix $M$ is constant.
- The number of collisions is uniformly upper bounded as $h \to 0$.
- The friction cone $\mathcal{FC}(q)$ is uniformly pointed.
- The applied forces are inertial $+$ at most linear growth:

\[
k(t, q, v) = F(v)v + k_1(t, q, v), \text{ where } F(v) \text{ is skew-symmetric},
\]
\[
k_1(t, q, v) \leq c_1 + c_2||q|| + c_3||v||.
\]

- We use $\tilde{k}_q^l = \tilde{k}_1^l q, \tilde{k}_v^l = F(v^l) + \tilde{k}_1^l v$ with $\tilde{k}_1^l q, \tilde{k}_1^l v$ uniformly bounded.
- The parameter $\alpha$ satisfies

\[
\frac{1}{2} \leq \alpha \leq 1.
\]

Conclusion

The numerical velocities $v^l, h$ are uniformly bounded as $h \to 0$. 
Numerical Positions and Velocities

We define:

- **Numerical velocity sequence**, \( v^{h,\alpha}(.), \)

\[
v^{h,\alpha}(t) = \begin{cases} (1 - \alpha)v^{l,h} + \alpha v^{l+1,h} & \text{if no collision in } (t_l, t_{l+1}] \text{ and } t \in (t_l, t_{l+1}] \\ v^{l+1,h} & \text{if collision in } (t_l, t_{l+1}] \text{ and } t \in (t_l, t_{l+1}] \end{cases}
\]

- **Numerical position sequence** \( q^{h,\gamma}(\cdot), \)

\[
q^{h,\gamma}(t) \big|_{t \in [t_l, t_{l+1}]} = \text{linear interpolant of } (t_l, q^{l,h}), (t_{l+1}, q^{l+1,h})
\]

**Obtaining convergent subsequences**

- Uniform bound for the numerical velocities \( \Rightarrow \) existence of a uniform convergent position subsequence.

- Uniform pointedness of the reduced friction cone \( FC_r \) \( \Rightarrow \) existence of a pointwise convergent velocity subsequence.
What is the big deal with the definition of velocities?

- We have not been able to prove or disprove bounded variation of \( v^h \). However, behavior of the numerical velocities suggests the latter (see-saw velocity profile). If amplified numerically by Coriolis ... But we could not find any such example at the moment.
- On the positive, we have been able to prove bounded variation of \( v^{h,\alpha} \), and numerical results also suggest that its bounded variation is \( O(1) \) smaller than the one of \( v^{h,l} \).
The Convergence Result

Assumptions \((1/2 \leq \alpha \leq 1, \gamma \in [0, 1])\).

- The mass matrix \(M\) is a constant, symmetric positive definite matrix.
- The constraint data \(\tilde{\nu}(q), \tilde{n}(q), \tilde{D}(q)\) is sufficiently smooth.
- The number of collisions is uniformly upper bounded as \(h \to 0\). All collision are plastic.
- The external forces \(k_1(t, q, v)\) increase at most linearly with the velocity and position. The approximations \(\tilde{k}_1q, \tilde{k}_1v\) are bounded.
- The friction cone \(FC(\cdot)\) is uniformly pointed.

Conclusion. There exists a subsequence \(h_k \to 0\), such that:

- \(q^{\gamma, h_k}(\cdot) \to q(\cdot)\) uniformly in \([0, T]\),
- \(v^{\alpha, h_k}(\cdot) \to v(\cdot)\) pointwise a.e. in \([0, T]\),
- \(dv^{\alpha, h_k}(\cdot) \to dv(\cdot)\) weak* as Borel measures in \([0, T]\), and every \((q(\cdot), v(\cdot))\) is a solution of the corresponding MDI.
Double Pendulum with a Long Wall

\( m_1 = m_2 = 1, \ L_1 = L_2 = 1 \)

\( (x_1^0, y_1^0) = (\sin(\pi/3), -\cos(\pi/3)) \)

\( (x_2^0, y_2^0) = (x_1^0, y_1^0) + (\sin(\pi/5), -\sin(\pi/5)) \)

\( (\dot{x}_1^0, \dot{y}_1^0) = (0, 0), \ (\dot{x}_2^0, \dot{y}_2^0) = (0, 0) \)

Rest. Coeff: \( e_1 = e_2 = 0.1 \). Final Time: \( T_f = 2.5(s) \)
Double Pendulum: Numerical Results

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Conclusions

- We have shown convergence in the MDI sense for a family of linearly implicit time-stepping schemes.
- For the first time, the convergence analysis includes joint constraints.
- Innovative choice of the velocity sequence.
- Future work: Full cones (non-approximated), iterative schemes, extending the convergence theorem.
Painleve’s Paradox (1895)

\[
\begin{align*}
1 &= m/16 \quad 1 = 2 \\
\dot{\theta} &= \frac{G}{2} \quad \omega = 0 \\
16 (\cos^2 \theta - \mu \cos \theta \sin \theta) &= -2 \\
\mu &= 0.75
\end{align*}
\]

Contact constraint: \( y_P \geq 0 \)

\[
\ddot{y}_P = \left( -g + \frac{1}{2} \dot{\theta}^2 \sin \theta \right) + N \left( \frac{1}{m} + \frac{l^2}{4l} (\cos^2 \theta - \mu \sin \theta \cos \theta) \right)
\]

Baraff: \( \ddot{y}_P = -g - \frac{N}{m} \)

Painleve Paradox: No classical solutions!
Modelling Contacts. Compliant Contact Models -Stiff!

- The state is augmented with local deformations and their derivatives, \( \delta = (\delta_N, \delta_T) \), \( \dot{\delta} = (\dot{\delta}_N, \dot{\delta}_T) \).

- Normal \( (\lambda_N) \) and tangential \( (\lambda_T) \) forces depend on \( \delta \) and \( \dot{\delta} \). For a linear viscoelastic model (the Kelvin–Voigt model),

\[
\begin{align*}
\lambda_N &= K_N \delta_N + C_N \dot{\delta}_N \\
\lambda_T &= K_T \delta_T + C_T \dot{\delta}_T
\end{align*}
\]

- Compliance in the tangential direction will result in continuity of \( \lambda_T \) in situations like "stick–slip" transitions.