Recent Results in the Conformal Bootstrap

David Poland

Yale & IAS

April 22, 2016

Lattice for BSM Physics 2016, ANL
Why Study CFTs?

There are many interesting applications of conformal field theories:

- **2D**: String Theory
- **2D/3D**: Statistical and Condensed Matter Systems
- **4D**: Scenarios for Physics Beyond the Standard Model
- **6D**: Mysterious $(2,0)$ Theory and Dualities
- **Holography and AdS/CFT**: Study Quantum Gravity with CFTs
Main Goal

We would like to map out the space of CFTs and predict their observables.
How far can we get using mathematical consistency alone?
The **conformal bootstrap** aims to use **mathematical consistency** conditions to map out and solve the space of CFTs

- Conformal Symmetry
- Crossing Symmetry
- Unitarity / Reflection Positivity
The **conformal bootstrap** aims to use mathematical consistency conditions to map out and solve the space of CFTs:

- Conformal Symmetry
- Crossing Symmetry
- Unitarity / Reflection Positivity

**Beautiful success story in 2D**

[Ferrara, Gatto, Grillo '73; Polyakov '74; Belavin, Polyakov, Zamolodchikov '83]

**Exciting progress in** $D > 2$ **starting in 2008**

[Rattazzi, Rychkov, Tonni, Vichi '08; ...]
Conformal Block Expansion

Can probe spectrum by expanding 4-point functions in \textit{conformal blocks}:

\[
\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle = \sum_{\Delta,\ell} \lambda_\mathcal{O}^2 g_{\Delta,\ell}(x_1, x_2, x_3, x_4)
\]

- \( g_{\Delta,\ell}(x_1, x_2, x_3, x_4) = g_{\Delta,\ell}(u, v)/x_{12}^{2\Delta} x_{34}^{2\Delta} \) known functions capturing contribution of an operator \( \mathcal{O} \in \sigma \times \sigma \) with dimension \( \Delta \) and spin \( \ell \)

- Similar to expansion in spherical harmonics \( Y_{\ell}^m \), but for CFTs
\[ \langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle \] is symmetric under permutations of \( x_i \):

- Switching \( x_1 \leftrightarrow x_3 \) gives the crossing symmetry condition:

\[
\sum \begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 
\end{array} = \sum \begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 
\end{array}
\]

\[
\sum_{\Delta, \ell} \lambda_{\mathcal{O}}^2 g_{\Delta, \ell}(x_1, x_2, x_3, x_4) = \sum_{\Delta, \ell} \lambda_{\mathcal{O}}^2 g_{\Delta, \ell}(x_3, x_2, x_1, x_4)
\]

- Only unknowns are set of scaling dimensions and coefficients: \( \{\Delta, \lambda_{\mathcal{O}}\} \)
By applying clever linear functionals $\alpha$ one can prove that some assumptions on $\{\Delta, \lambda_\mathcal{O}\}$ are incompatible with crossing + unitarity:

$$0 = \sum_{\Delta, \ell} \lambda_\mathcal{O}^2 \alpha [g_{\Delta, \ell}(x_1, x_2, x_3, x_4) - g_{\Delta, \ell}(x_3, x_2, x_1, x_4)] > 0$$
Numerical Approach

- By applying clever linear functionals $\alpha$ one can prove that some assumptions on $\{\Delta, \lambda_\mathcal{O}\}$ are incompatible with crossing + unitarity:

  \[ 0 = \sum_{\Delta, \ell} \lambda_\mathcal{O}^2 \alpha \left[ g_{\Delta, \ell}(x_1, x_2, x_3, x_4) - g_{\Delta, \ell}(x_3, x_2, x_1, x_4) \right] > 0 \]

- Find $\alpha \sim \sum_n a_n \partial^n$ numerically using linear/semidefinite programming [Rattazzi, Rychkov, Tonni, Vichi '08; DP, Simmons-Duffin, Vichi '11]
  - Functional search space ranges from $\sim 20$ to $\sim 1200$ components
  - Each plot $\leftrightarrow$ Solve $\mathcal{O}(1000)$ optimization problems on HPC clusters
  - State of the art: SDPB [Simmons-Duffin '15]
Bound on leading scalar in $\sigma \times \sigma \sim 1 + \epsilon + \ldots$

3D Ising (Lattice): $\Delta_\sigma \simeq 0.51813(5)$, $\Delta_\epsilon \simeq 1.41275(25)$ [Hasenbusch '10]
We can get more powerful constraints by considering the system \( \{ \langle \sigma \sigma \sigma \sigma \rangle, \langle \sigma \sigma \epsilon \epsilon \rangle, \langle \epsilon \epsilon \epsilon \epsilon \rangle \} \), which leads to 5 sum rules:

\[
\sum_{\mathcal{O}^+} \left( \lambda_{\sigma \sigma \mathcal{O}} \lambda_{\epsilon \epsilon \mathcal{O}} \right) \vec{V}^+_{\pm, \Delta, \ell}(u, v) \left( \begin{array}{c} \lambda_{\sigma \sigma \mathcal{O}} \\ \lambda_{\epsilon \epsilon \mathcal{O}} \end{array} \right) + \sum_{\mathcal{O}^-} \lambda_{\sigma \epsilon \mathcal{O}}^2 \vec{V}^-_{\pm, \Delta, \ell}(u, v) = 0,
\]

where \( \vec{V}^\pm_{\pm, \Delta, \ell}(u, v) \) are 5-vectors and \( \vec{V}^+_{\pm, \Delta, \ell}(u, v) \) is a 2 \( \times \) 2 matrix.
Mixed Correlators

- We can get more powerful constraints by considering the system \( \{\langle \sigma \sigma \sigma \sigma \rangle, \langle \sigma \sigma \epsilon \epsilon \rangle, \langle \epsilon \epsilon \epsilon \epsilon \rangle \} \), which leads to 5 sum rules:

\[
\sum_{\mathcal{O}^+} \left( \lambda_{\sigma \sigma \mathcal{O}} \lambda_{\epsilon \epsilon \mathcal{O}} \right) \vec{V}_{+,\Delta,\ell}(u,v) \begin{pmatrix} \lambda_{\sigma \sigma \mathcal{O}} \\ \lambda_{\epsilon \epsilon \mathcal{O}} \end{pmatrix} + \sum_{\mathcal{O}^-} \lambda_{\sigma \epsilon \mathcal{O}}^2 \vec{V}_{-,\Delta,\ell}(u,v) = 0,
\]

where \( \vec{V}_{\pm,\Delta,\ell}(u,v) \) are 5-vectors and \( \vec{V}_{+,\Delta,\ell}(u,v) \) is a \( 2 \times 2 \) matrix.

- Bounds follow from applying a 5-vector of functionals \( \vec{\alpha} \) such that

\[
\begin{pmatrix} 1 & 1 \end{pmatrix} \vec{\alpha} \cdot \vec{V}_{+,0,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1,
\]

\[
\vec{\alpha} \cdot \vec{V}_{+,\Delta,\ell} \geq 0, \quad \text{for all } \mathbb{Z}_2\text{-even operators } \mathcal{O}^+,
\]

\[
\vec{\alpha} \cdot \vec{V}_{-,\Delta,\ell} \geq 0, \quad \text{for all } \mathbb{Z}_2\text{-odd operators } \mathcal{O}^-.
\]
Mixed Correlator Islands

Imposing that $\sigma$ and $\epsilon$ are the only relevant scalars ($\Delta_{\sigma',\epsilon'} \geq 3$), we obtain a rigorous island isolated from the rest of the allowed region.

Kos, DP, Simmons-Duffin '14
Mixed Correlator Islands (Last Year)

[Δσ, Δε] = {0.518151(6), 1.41264(6)} (10× more precise than lattice)

Pushing to 1265 components using SDPB, region keeps shrinking!
Mixed Correlator Island (This Year)

Best bounds: first map out a 3d Island in $\{\Delta_\sigma, \Delta_\epsilon, \lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}\}$

Since the functional can be different for each choice of $\lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}$, the $\{\Delta_\sigma, \Delta_\epsilon\}$ projection is smaller than having no assumption on $\lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}$
Mixed Correlator Island (This Year)

Monte Carlo

Bootstrap

[Kos, DP, Simmons-Duffin, Vichi '16]

\[
\{ \Delta_\sigma, \Delta_\epsilon \} = \{ 0.518149(1), 1.412625(10) \}
\]

\[
\{ \lambda_{\sigma\epsilon\epsilon}, \lambda_{\epsilon\epsilon\epsilon} \} = \{ 1.0518537(41), 1.532435(19) \}
\]
3D $O(N)$ Bounds

- Extension to $\langle \phi_i \phi_j \phi_k \phi_l \rangle$, where $\phi_i$ is $O(N)$ vector
- Large $N$: matches $1/N$ expansion, Small $N$: matches experiment!

[Kos, DP, Simmons-Duffin '13]
**O(\(N\)) Archipelago from Mixed Correlators**

The \(O(\(N\))\) archipelago

- Mixed \(\{\phi_i, s\}\) system with one relevant \(O(\(N\))\) vector \(\phi_i\) and singlet \(s\)

\[\text{[Kos, DP, Simmons-Duffin, Vichi '15; '16]}\]
\[ \{ \Delta_\phi, \Delta_s, \lambda_{\phi \phi s}, \lambda_{sss} \} = \{0.51926(32), 1.5117(25), .68726(65), .8286(60)\} \]

- Close to resolving 8\(\sigma\) discrepancy between lattice and \(^4\text{He}\) expt
\(\{\Delta_\phi, \Delta_s, \lambda_{\phi\phi s}, \lambda_{ss s}\} = \{0.51928(62), 1.5957(55), 0.5244(11), 0.499(12)\}\)
O(2) Conductivity (1197 comp.)

- Rigorous determination of $\langle JJ \rangle \propto C_J \propto \sigma_\infty$, giving high-frequency conductivity in (2 + 1)D superconductors: $2\pi\sigma_\infty = 0.3554(6)$
Rigorous determination of $\langle JJ \rangle \propto C_J \propto \sigma_\infty$, giving high-frequency conductivity in $(2 + 1)$D superconductors: $2\pi \sigma_\infty = 0.3554(6)$

Quantum Monte Carlo: $0.355(5)$ [Gazit, Podolsky, Auerbach '14] (statistical errors only) $0.3605(3)$ [Katz, Sachdev, Sørensen, Witczak-Krempa '14]
Mysteries in the Bootstrap: 3D

Bootstrap for fermions $\langle \psi \psi \psi \psi \rangle$ in 3D CFT with parity

- Bound on leading parity-even scalar in $\psi \times \psi \sim 1 + \epsilon + \ldots$

[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15]
Mysteries in the Bootstrap: 3D

- Bound on leading parity-odd scalar in $\psi \times \psi \sim \sigma + \ldots$
- Jump does not coincide with known CFTs
  (e.g., Large N Gross-Neveu: $\{\Delta_\psi, \Delta_\sigma\} = 1 + \frac{4}{3\pi^2 N}, 1 - \frac{32}{3\pi^2 N}$)

[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15]
Mysteries in the Bootstrap: 4D $\mathcal{N} = 1$

- Chiral operator $\phi$ in 4D $\mathcal{N} = 1$ SCFT
- Bound on leading scalar in $\overline{\phi} \times \phi \sim 1 + \overline{\phi} \phi + \ldots$
- Kink: Minimal value $\Delta_\phi \gtrsim 1.415$ consistent with imposing $\phi^2 = 0$

[DP, Simmons-Duffin, Vichi '11; DP, Stergiou '15]
Can place bounds on central charge $\langle TT \rangle \propto c$ assuming $\phi^2 = 0$

Upper bounds depend on gap until second spin 1 operator (here $\Delta_{V'} \geq 3.1, \ldots, 4.1$), but value is unique at minimal $\Delta_{\phi}$
4D $\mathcal{N} = 1$ Minimal Model?

- Computed minimal point in $\{\Delta \phi, c\}$ space increasing derivative cutoff $\Lambda$
- Naïve extrapolation points to $\{\Delta \phi, c\} \sim \{1.428, 0.111\} \sim \{10/7, 1/9\}$
- Has the bootstrap discovered a new non-Lagrangian SCFT?

[DP, Stergiou '15]
Where do we go from here?

- Make $O(N)$ predictions more **precise** (resolve $8\sigma$ discrepancy!)
  - Extend mixed correlator bootstrap to include external $t_{ij}$?
  - Higher spectrum ($\{\phi', s', t'\}$, higher $O(N)$ reps, leading twist trajectory)
Bootstrap Future

Where do we go from here?

- **Make O(N) predictions more precise** (resolve $8\sigma$ discrepancy!)
  - Extend mixed correlator bootstrap to include external $t^{ij}$?
  - Higher spectrum ($\{\phi', s', t'\}$, higher $O(N)$ reps, leading twist trajectory)

- **Find rigorous islands** for fermionic/gauge/mystery CFTs
  - 3D fermions: Study mixed $\{\psi, \sigma\}$ system and add global symmetries
  - 3D QED: Bootstrap monopole operators [Chester, Pufu '16]
  - Conformal window of 4D QCD: $? < N_f/N_c < 11/2$
Bootstrap Future

Where do we go from here?

- Make $O(N)$ predictions more **precise** (resolve $8\sigma$ discrepancy!)
  - Extend mixed correlator bootstrap to include external $t_{ij}$?
  - Higher spectrum ($\{\phi', s', t'\}$, higher $O(N)$ reps, leading twist trajectory)

- Find **rigorous islands** for fermionic/gauge/mystery CFTs
  - 3D fermions: Study mixed $\{\psi, \sigma\}$ system and add global symmetries
  - 3D QED: Bootstrap monopole operators [Chester, Pufu ’16]
  - Conformal window of 4D QCD: $? < \frac{N_f}{N_c} < \frac{11}{2}$

- Bootstrap currents/stress tensor and develop analytical methods
Bootstrap Future

Where do we go from here?

- Make $O(N)$ predictions more **precise** (resolve $8\sigma$ discrepancy!)
  - Extend mixed correlator bootstrap to include external $t^{ij}$?
  - Higher spectrum ($\{\phi', s', t'\}$, higher $O(N)$ reps, leading twist trajectory)

- Find **rigorous islands** for fermionic/gauge/mystery CFTs
  - 3D fermions: Study mixed $\{\psi, \sigma\}$ system and add global symmetries
  - 3D QED: Bootstrap monopole operators [Chester, Pufu ’16]
  - Conformal window of 4D QCD: $? < \frac{N_f}{N_c} < \frac{11}{2}$

- Bootstrap currents/stress tensor and develop analytical methods
  - Analytic Bootstrap for $\langle TT\phi\phi \rangle \rightarrow$ Sum rules for coefficients in $\langle TTT \rangle$
    $\rightarrow$ Proof of Hofman-Maldacena bound $\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}$ + generalizations
    [Hartman, Jain, Kundu ’15; ’16; Hofman, Li, Meltzer, DP, Rejon-Barrera ’16]
  - Can it be strengthened? Interplay with numerical studies?
With more work I believe we can create a detailed map of the space of conformal field theories...we may even discover a new world!