Recent Results in the Conformal Bootstrap

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Why Study CFTs?

There are many interesting applications of conformal field theories:

- 2D: String Theory
- 2D/3D: Statistical and Condensed Matter Systems
- 4D: Scenarios for Physics Beyond the Standard Model
- 6D: Mysterious $(2,0)$ Theory and Dualities
- Holography and AdS/CFT: Study Quantum Gravity with CFTs

Main Goal

We would like to map out the space of CFTs and predict their observables

Conformal Bootstrap

How far can we get using mathematical consistency alone?

Conformal Bootstrap

- The conformal bootstrap aims to use mathematical consistency conditions to map out and solve the space of CFTs
  - Conformal Symmetry
  - Crossing Symmetry
  - Unitarity / Reflection Positivity

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- Beautiful success story in 2D

  [Ferrara, Gatto, Grillo '73; Polyakov '74; Belavin, Polyakov, Zamolodchikov '83]
- Exciting progress in $D > 2$ starting in 2008

  [Rattazzi, Rychkov, Tonni, Vichi '08; ...]

Conformal Block Expansion

Can probe spectrum by expanding 4-point functions in conformal blocks:

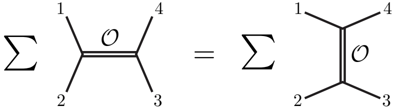
$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle = \sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2\, g_{\Delta,\ell}(x_1,x_2,x_3,x_4)$$

- $g_{\Delta,\ell}(x_1,x_2,x_3,x_4) = g_{\Delta,\ell}(u,v)/x_{12}^{2\Delta_\sigma}x_{34}^{2\Delta_\sigma}$ known functions capturing contribution of an operator $\mathcal{O} \in \sigma \times \sigma$ with dimension Δ and spin ℓ
- Similar to expansion in spherical harmonics Y_ℓ^m, but for CFTs

Crossing Symmetry

$\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$ is symmetric under permutations of x_i:

- Switching $x_1 \leftrightarrow x_3$ gives the crossing symmetry condition:



$$\sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(x_1,x_2,x_3,x_4) \;=\; \sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(x_3,x_2,x_1,x_4)$$

- Only unknowns are set of scaling dimensions and coefficents: $\{\Delta, \lambda_{\mathcal{O}}\}$

Numerical Approach

- By applying clever linear functionals α one can prove that some assumptions on $\{\Delta, \lambda_{\mathcal{O}}\}$ are incompatible with crossing + unitarity:

$$0 = \sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2 \alpha[g_{\Delta,\ell}(x_1,x_2,x_3,x_4) - g_{\Delta,\ell}(x_3,x_2,x_1,x_4)] > 0$$

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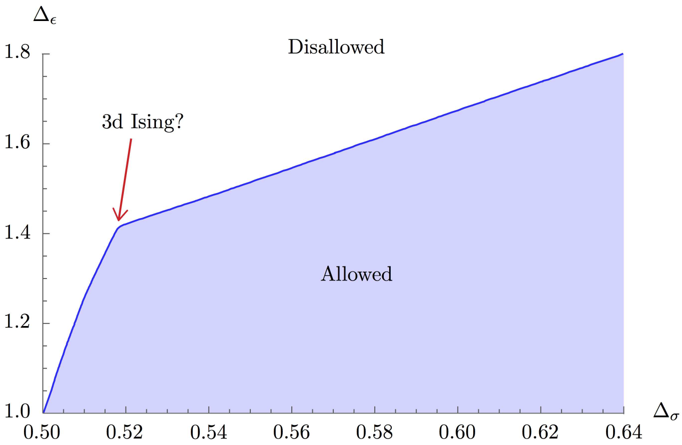
- Find $\alpha \sim \sum_n a_n \partial^n$ numerically using linear/semidefinite programming [Rattazzi, Rychkov, Tonni, Vichi '08; DP, Simmons-Duffin, Vichi '11]
  - Functional search space ranges from ~ 20 to ~ 1200 components
  - Each plot $\leftrightarrow$ Solve $\mathcal{O}(1000)$ optimization problems on HPC clusters
  - State of the art: SDPB [Simmons-Duffin '15]

3D Dimension Bounds



[El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, '12; '14]

- Bound on leading scalar in $\sigma \times \sigma \sim \mathbb{1} + \epsilon + \ldots$
- 3D Ising (Lattice): $\Delta_\sigma \simeq 0.51813(5)$, $\Delta_\epsilon \simeq 1.41275(25)$ [Hasenbusch '10]

Mixed Correlators

- We can get more powerful constraints by considering the system $\{\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle\}$, which leads to 5 sum rules:

$$\sum_{\mathcal{O}^+} \begin{pmatrix} \lambda_{\sigma\sigma\mathcal{O}} & \lambda_{\epsilon\epsilon\mathcal{O}} \end{pmatrix} \vec{V}_{+,\Delta,\ell}(u,v) \begin{pmatrix} \lambda_{\sigma\sigma\mathcal{O}} \\ \lambda_{\epsilon\epsilon\mathcal{O}} \end{pmatrix} + \sum_{\mathcal{O}^-} \lambda^2_{\sigma\epsilon\mathcal{O}} \vec{V}_{-,\Delta,\ell}(u,v) = 0,$$

where $\vec{V}_{\pm,\Delta,\ell}(u,v)$ are 5-vectors and $\vec{V}_{+,\Delta,\ell}(u,v)$ is a 2×2 matrix

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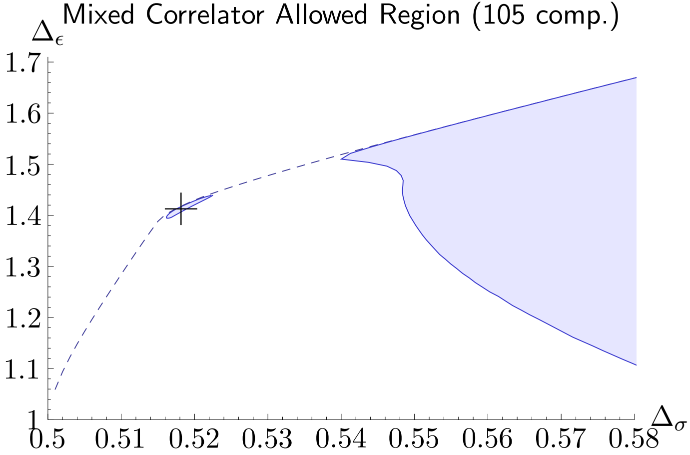
- Bounds follow from applying a 5-vector of functionals $\vec{\alpha}$ such that

$$\begin{aligned} \begin{pmatrix} 1 & 1 \end{pmatrix} \vec{\alpha} \cdot \vec{V}_{+,0,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= 1, \\ \vec{\alpha} \cdot \vec{V}_{+,\Delta,\ell} &\succeq 0, \quad \text{for all } \mathbb{Z}_2\text{-even operators } \mathcal{O}^+, \\ \vec{\alpha} \cdot \vec{V}_{-,\Delta,\ell} &\geq 0, \quad \text{for all } \mathbb{Z}_2\text{-odd operators } \mathcal{O}^-. \end{aligned}$$

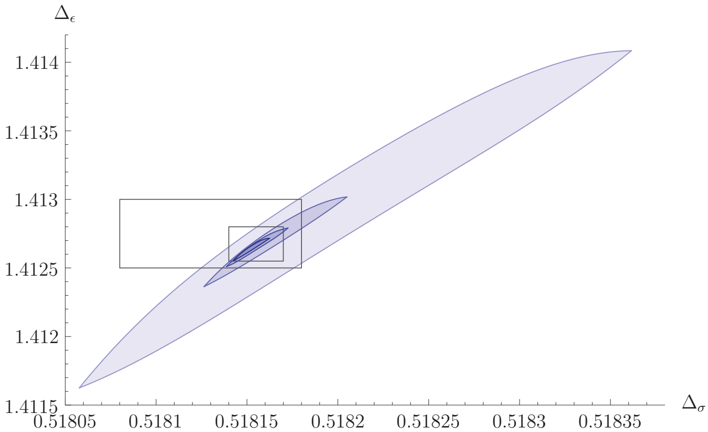
Mixed Correlator Islands



[Kos, DP, Simmons-Duffin '14]

- Imposing that σ and ϵ are the only relevant scalars ($\Delta_{\sigma',\epsilon'} \geq 3$), we obtain a rigorous island isolated from the rest of the allowed region.

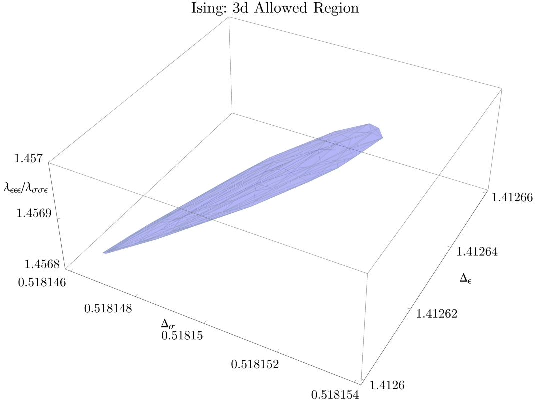
Mixed Correlator Islands (Last Year)



[Kos, DP, Simmons-Duffin '14; Simmons-Duffin '15]

- Pushing to 1265 components using SDPB, region keeps shrinking! $\{\Delta_\sigma, \Delta_\epsilon\} = \{0.518151(6), 1.41264(6)\}$ ($10\times$ more precise than lattice)

Mixed Correlator Island (This Year)



[Kos, DP, Simmons-Duffin, Vichi '16]

- Best bounds: first map out a 3d Island in $\{\Delta_\sigma, \Delta_\epsilon, \lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}\}$
- Since the functional can be different for each choice of $\lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}$, the $\{\Delta_\sigma, \Delta_\epsilon\}$ projection is smaller than having no assumption on $\lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}$

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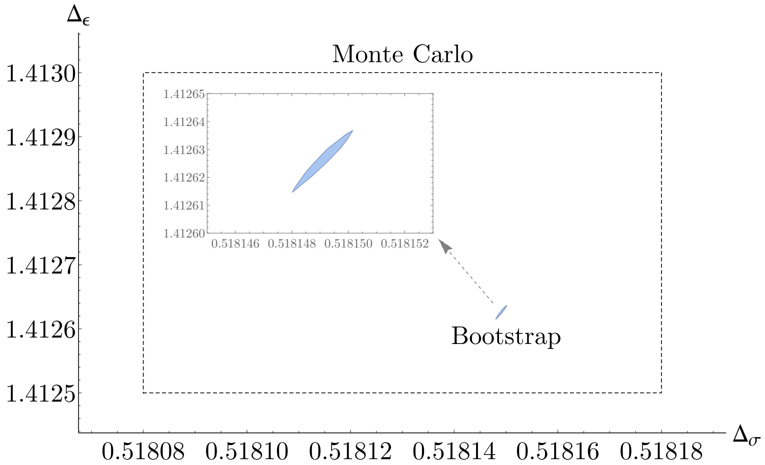
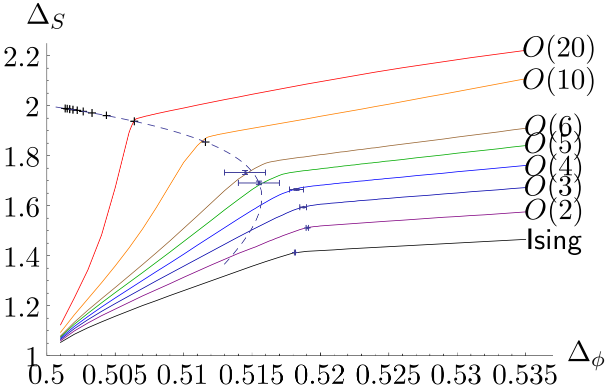


[Kos, DP, Simmons-Duffin, Vichi '16]

$$\{\Delta_\sigma, \Delta_\epsilon\} = \{0.518149(1), 1.412625(10)\}$$
$$\{\lambda_{\sigma\sigma\epsilon}, \lambda_{\epsilon\epsilon\epsilon}\} = \{1.0518537(41), 1.532435(19)\}$$

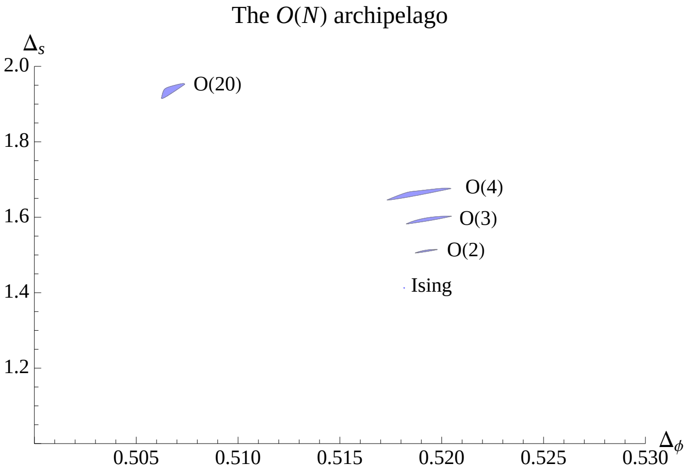
3D $O(N)$ Bounds



[Kos, DP, Simmons-Duffin '13]

- Extension to $\langle \phi_i \phi_j \phi_k \phi_l \rangle$, where ϕ_i is $O(N)$ vector
- Large N: matches $1/N$ expansion, Small N: matches experiment!

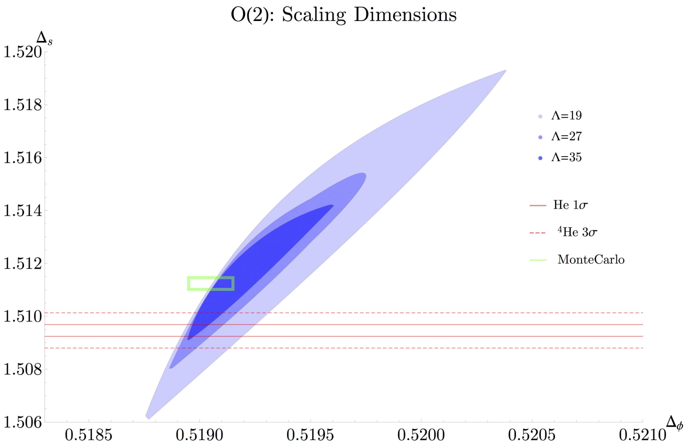
$O(N)$ Archipelago from Mixed Correlators



[Kos, DP, Simmons-Duffin, Vichi '15; '16]

- Mixed $\{\phi_i, s\}$ system with one relevant $O(N)$ vector ϕ_i and singlet s

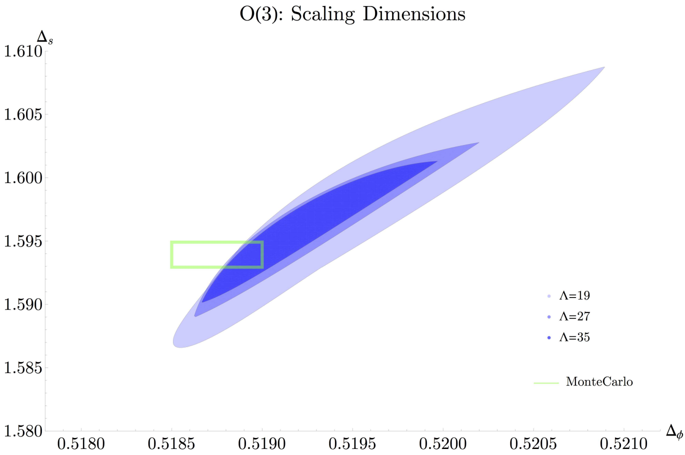
$O(2)$ Zoom



[Kos, DP, Simmons-Duffin, Vichi '16]

- $\{\Delta_\phi, \Delta_s, \lambda_{\phi\phi s}, \lambda_{sss}\} = \{.51926(32), 1.5117(25), .68726(65), .8286(60)\}$
- Close to resolving 8σ discrepancy between lattice and ^{4}He expt

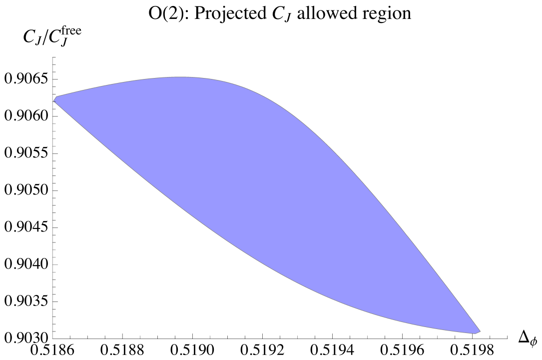
$O(3)$ Zoom



[Kos, DP, Simmons-Duffin, Vichi '16]

- $\{\Delta_\phi, \Delta_s, \lambda_{\phi\phi s}, \lambda_{sss}\} = \{.51928(62), 1.5957(55), .5244(11), .499(12)\}$

$O(2)$ Conductivity (1197 comp.)



[Kos, DP, Simmons-Duffin, Vichi '15]

- Rigorous determination of $\langle JJ \rangle \propto C_J \propto \sigma_\infty$, giving high-frequency conductivity in $(2+1)$D superconductors: $2\pi\sigma_\infty = 0.3554(6)$

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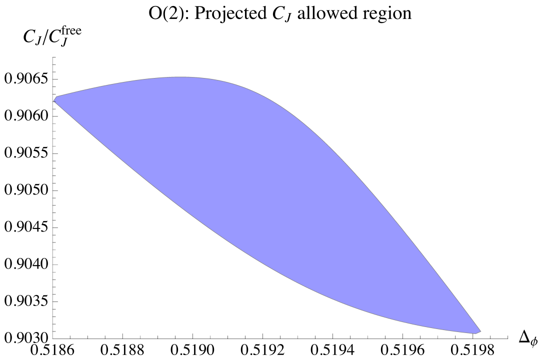

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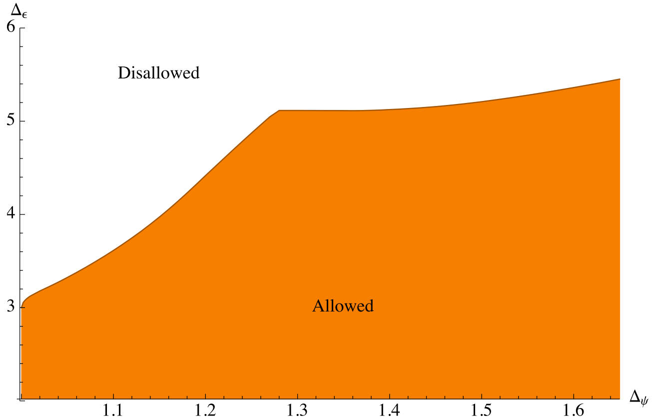
- Quantum Monte Carlo: $0.355(5)$ [Gazit, Podolsky, Auerbach '14]
  (statistical errors only) $0.3605(3)$ [Katz, Sachdev, Sørensen, Witczak-Krempa '14]

Mysteries in the Bootstrap: 3D



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15]

- Bootstrap for fermions $\langle\psi\psi\psi\psi\rangle$ in 3D CFT w/ parity
- Bound on leading parity-even scalar in $\psi \times \psi \sim \mathbb{1} + \epsilon + \ldots$

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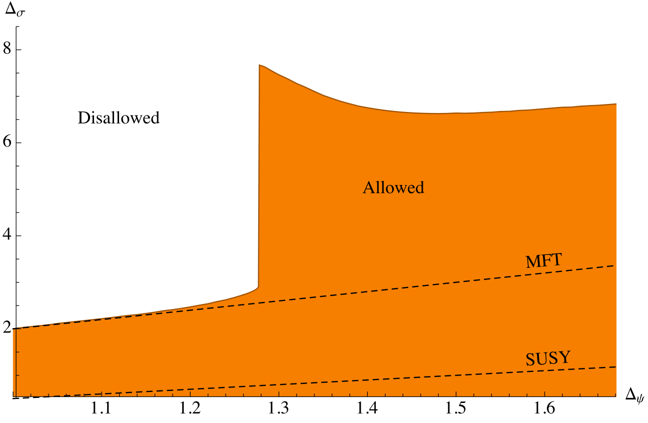
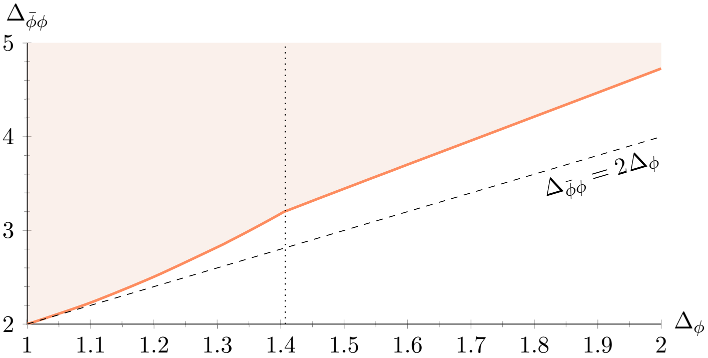


[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15]

- Bound on leading parity-odd scalar in $\psi \times \psi \sim \sigma + \ldots$
- Jump does not coincide with known CFTs
  (e.g., Large N Gross-Neveu: $\{\Delta_\psi, \Delta_\sigma\} = 1 + \frac{4}{3\pi^2 N}, 1 - \frac{32}{3\pi^2 N}$)

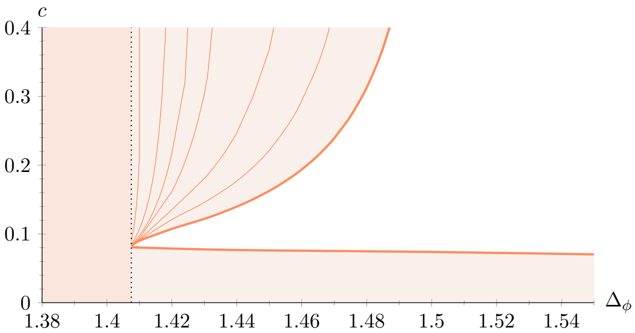
Mysteries in the Bootstrap: 4D $\mathcal{N} = 1$



[DP, Simmons-Duffin, Vichi '11; DP, Stergiou '15]

- Chiral operator ϕ in 4D $\mathcal{N} = 1$ SCFT
- Bound on leading scalar in $\overline{\phi} \times \phi \sim \mathbb{1} + \overline{\phi}\phi + \ldots$
- Kink: Minimal value $\Delta_\phi \gtrsim 1.415$ consistent with imposing $\phi^2 = 0$

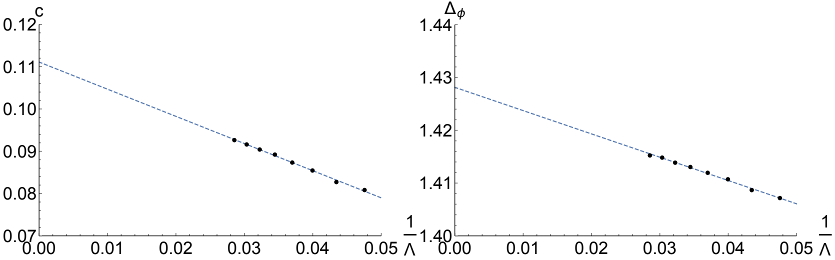
4D $\mathcal{N} = 1$ Central Charge Bound



[DP, Stergiou '15]

- Can place bounds on central charge $\langle TT \rangle \propto c$ assuming $\phi^2 = 0$
- Upper bounds depend on gap until second spin 1 operator (here $\Delta_{V'} \geq 3.1, \ldots, 4.1$), but value is unique at minimal Δ_ϕ

4D $\mathcal{N} = 1$ Minimal Model?



[DP, Stergiou '15]

- Computed minimal point in $\{\Delta_\phi, c\}$ space increasing derivative cutoff Λ
- Naïve extrapolation points to $\{\Delta_\phi, c\} \sim \{1.428, 0.111\} \sim \{10/7, 1/9\}$
- Has the bootstrap discovered a new non-Lagrangian SCFT?

Bootstrap Future

Where do we go from here?

- Make O(N) predictions more precise (resolve 8σ discrepancy!)
  - Extend mixed correlator bootstrap to include external t^{ij}?
  - Higher spectrum ($\{\phi', s', t'\}$, higher $O(N)$ reps, leading twist trajectory)

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- Find rigorous islands for fermionic/gauge/mystery CFTs
  - 3D fermions: Study mixed $\{\psi, \sigma\}$ system and add global symmetries
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  - Conformal window of 4D QCD: $? < N_f/N_c < 11/2$

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- Bootstrap currents/stress tensor and develop analytical methods
  - Analytic Bootstrap for $\langle TT\phi\phi\rangle \rightarrow$ Sum rules for coefficients in $\langle TTT\rangle$ $\rightarrow$ Proof of Hofman-Maldacena bound $\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}$ + generalizations [Hartman, Jain, Kundu '15; '16; Hofman, Li, Meltzer, DP, Rejon-Barrera '16]
  - Can it be strengthened? Interplay with numerical studies?

Bootstrap Future

- With more work I believe we can create a detailed map of the space of conformal field theories...we may even discover a new world!